Fri.,	7.2.3-7.2.5 Inductance and Energy of B	
Mon., Tues. Wed.	7.3.1-.3.3 Maxwell's Equations 10.1 - .2.1 Potential Formulation Lunch with UCR Engr – 12:20 – 1:00	HW10

Where we've been

Stationary Charges – producing and interacting via Electric Fields

Steady Currents – producing and interacting via Magnetic Fields

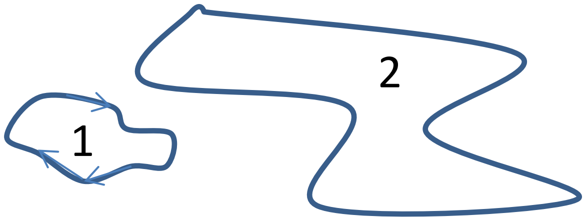
Where we're going

Varying currents and charge distributions – producing and interacting with varying Electric and Magnetic Fields

A step closer to

Inductance

What is flux through path 2 due to current following path 1?



$$\Phi_2 = \int \vec{B}_1 \cdot d\vec{a}_2$$

$$\vec{B}_1 = \frac{\mu_o}{4\pi} I_1 \oint \frac{d\vec{l}_1 \times \vec{\mathscr{r}}}{\mathscr{r}^2}$$

$$\Phi_2 = \int \frac{\mu_o}{4\pi} I_1 \oint \frac{d\vec{l}_1 \times \vec{\mathscr{r}}}{\mathscr{r}^2} \cdot d\vec{a}_2 = I_1 \left(\frac{\mu_o}{4\pi} \int \oint \frac{d\vec{l}_1 \times \vec{\mathscr{r}}}{\mathscr{r}^2} \cdot d\vec{a}_2 \right) = I_1 M_{1,2}$$

Purely geometric factor

Equivilantly, can rephrase using product rules, or use A to get same result

$$\Phi_2 = \int \vec{B}_1 \cdot d\vec{a}_2$$

$$\vec{B}_1 = \vec{\nabla} \times \vec{A}_1$$

$$\Phi_2 = \int \left(\vec{\nabla} \times \vec{A}_1 \right) \cdot d\vec{a}_2 = \oint \vec{A}_1 \cdot d\vec{l}_2$$

$$\vec{A}_1 = \frac{\mu_o}{4\pi} \oint \frac{I_1 d\vec{l}_1}{\mathscr{r}}$$

$$\Phi_2 = I_1 \left(\frac{\mu_o}{4\pi} \oint \oint \frac{d\vec{l}_1 \cdot d\vec{l}_2}{\mathscr{r}} \right)$$

$$M_{1,2} \equiv \left(\frac{\mu_o}{4\pi} \int \oint \frac{d\vec{l}_1 \times \vec{\mathscr{r}}}{\mathscr{r}^2} \cdot d\vec{a}_2 \right) = \left(\frac{\mu_o}{4\pi} \oint \oint \frac{d\vec{l}_1 \cdot d\vec{l}_2}{\mathscr{r}} \right)$$

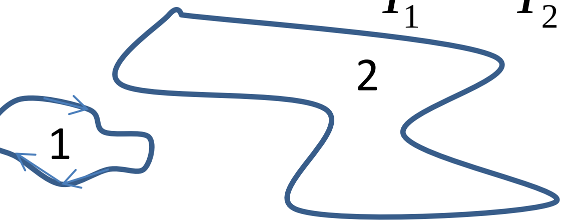
$$\Phi_2 = M_{1,2} I_1$$

$$\Phi_1 = M_{1,2} I_2$$

Symmetric between two loops

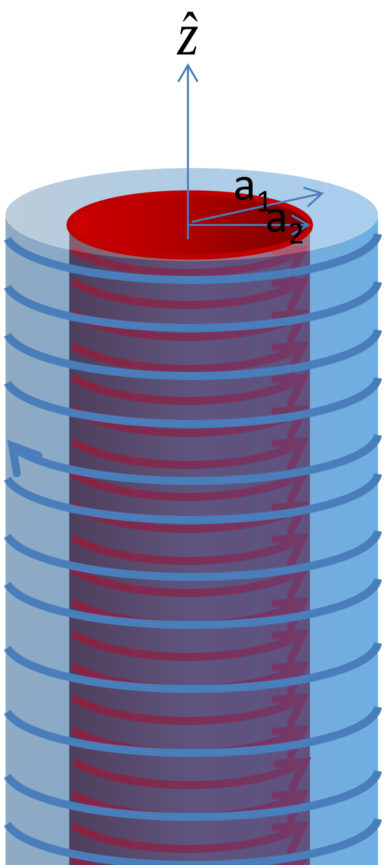
Inductance

$$\frac{\Phi_2}{I_1} = \frac{\Phi_1}{I_2} = M_{1,2} = \left(\frac{\mu_o}{4\pi} \int \oint \frac{d\vec{l}_1 \times \hat{\mathscr{r}}}{\mathscr{r}^2} \cdot d\vec{a}_2 \right) = \left(\frac{\mu_o}{4\pi} \oint \oint \frac{d\vec{l}_1 \cdot d\vec{l}_2}{\mathscr{r}} \right)$$



As with Resistance, sometimes it's easiest to do the geometric integral, sometimes it's easiest to find flux, factor out current, and thus find M.

Example: Coaxial solenoids of radii $a_1 > a_2$ and windings per length n_1 and n_2.



$$\Phi_1 = B_2 \left(\pi a_1^2\right) N_1$$

$$N_1 = n_1 l_1$$

$$B_2 = \mu_0 n_2 I_2$$

$$\Phi_1 = \mu_o n_2 I_2 \left(\pi a_1^2\right) n_1 l_1 = \underbrace{\left(\mu_o n_2 n_1 \left(\pi a_1^2\right) l_1\right)}_{M_{1,2}} I_2$$

M1,2

Overlapping volumes

Faraday's Law: time varying current in one solenoid induces $\mathcal{E}mf$ and drives current in other

$$\mathcal{E}mf_1 = -\frac{d\Phi_{B.1}}{dt} = -\frac{d}{dt}\left(M_{1,2} I_2\right)$$

Demo!

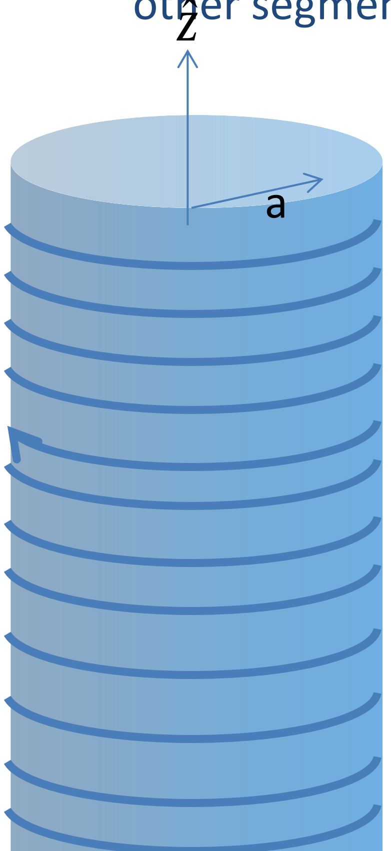
Self Inductance

$$\frac{\Phi_1}{I_1} = L = \left(\frac{\mu_o}{4\pi} \int \oint \frac{d\vec{l}_1 \times \vec{\mathscr{r}}}{\mathscr{r}^2} \cdot d\vec{a}_1 \right) = \left(\frac{\mu_o}{4\pi} \oint \oint \frac{d\vec{l}_1 \cdot d\vec{l}_1}{\mathscr{r}} \right)$$

Current passing along the loop is itself responsible for flux through the loop

$$\mathcal{E}mf = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}(LI)$$

Time varying current along one segment of the loop produces field and $\mathcal{E}mf$ felt by other segments of the same loop..



Example: single solenoid

$$\Phi = B\left(\pi a^2\right)N$$

$$N = nL$$

$$B = \mu_0 nI$$

$$\Phi_1 = \mu_o nI\left(\pi a^2\right)nl = \left(\overbrace{\mu_o n^2 \underbrace{\left(\pi a^2\right)l}_{\text{volume}}}^{L}\right)I$$

Energy to Generate Current

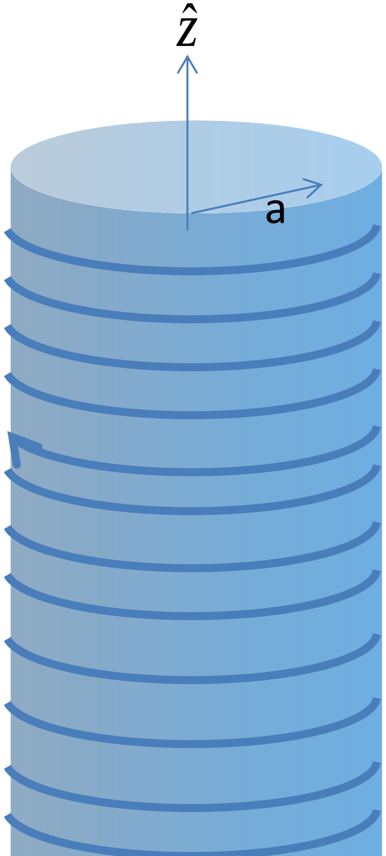
Consider driving charges around a solenoid. How much work would you have to do to get it going?

As you accelerate it up to speed, self inductance means a counter force is generated, so you must at least provide equal and opposite force.

$$\mathcal{E}mf = \frac{\int \vec{F} \cdot d\vec{l}}{q} = \frac{W}{q}$$

So per unit charge,

$$q\mathcal{E}mf = W$$



Then the rate at which work is done by the inductance's emf is

$$\frac{dq}{dt}\mathcal{E}mf = P$$

$$I\mathcal{E}mf = P$$

Or using the self-inductance relationship

$$-I\frac{d}{dt}(LI) = P$$

So bringing the current up to speed, you must oppose this, and invest energy at

$$-\frac{d}{dt}\left(\tfrac{1}{2}LI^2\right) = P \qquad P_{you} = \frac{d}{dt}\left(\tfrac{1}{2}LI^2\right) \qquad W_{you} = \tfrac{1}{2}LI^2$$

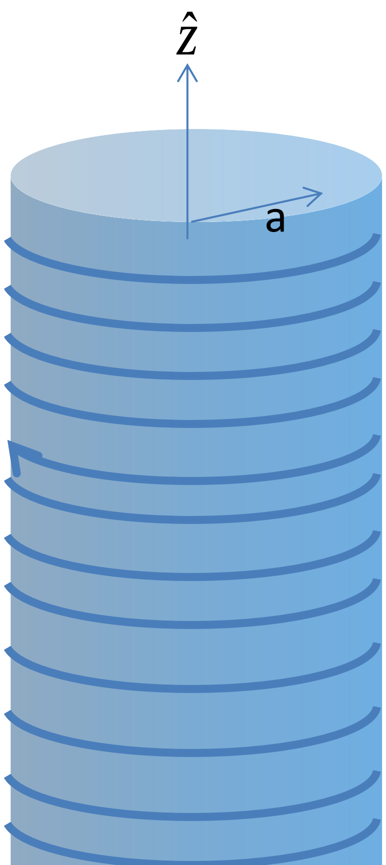
Energy to Generate Current

Consider driving charges around a solenoid. How much work would you have to do to get it going?

$$W_{you} = \tfrac{1}{2} L I^2$$

$$L = \left(\mu_o n^2 \tau\right) \qquad I = \frac{B}{\mu_0 n}$$

Rephrasing in terms of the corresponding field that's generated,



$$W_{you} = \tfrac{1}{2}\left(\mu_o n^2 \tau\right)\left(\frac{B}{\mu_o n}\right)^2$$

$$W_{you} = \tfrac{1}{2\mu_o} \tau B^2$$

Extrapolating to more general cases,

$$W = \tfrac{1}{2\mu_o} \int B^2 d\tau$$

"Where is the energy stored, field or current?" Neither / both – energy isn't a substance (no "caloric fluid") to be stored some where.
It's kinetic and potential energy, it's "stored in" the motion of charges and their interactions situation of current flowing and field generated.

(Griffiths does a more general derivation much like he did for the work of generating E field.)

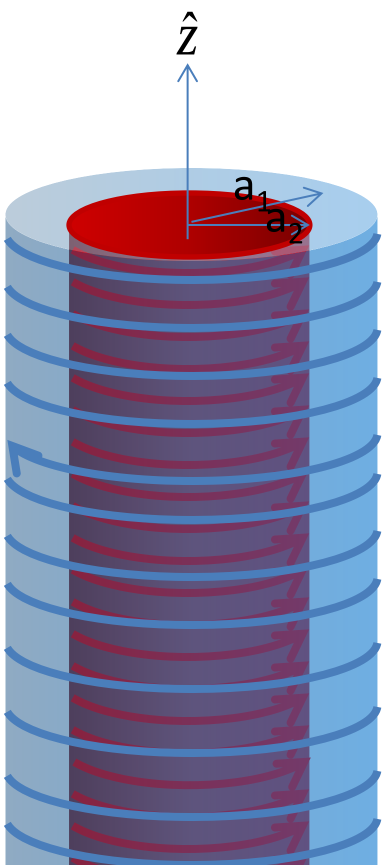
Energy to Generate Current

$$W = \frac{1}{2\mu_o}\int B^2 d\tau$$

Exercise: Work to turn on co-axial solenoids of different wire density, n, and opposite current, I.

For an individual solenoid

$$B_1 = \begin{cases} \mu_0 n_1 I_1 & \text{inside} \\ 0 & \text{outside} \end{cases}$$



Energy to Generate Current

(mathematical and general case derivation)

Self-inductance should be a real phenomenon for *any* current path; the work to establish a current along *any* path should be

$$W = \int P\,dt$$

$$P = -I\mathcal{E}mf$$

$$\mathcal{E}mf = -\frac{d\Phi}{dt} = -\frac{d}{dt}(LI)$$

$$P = I\frac{d}{dt}(LI) = \tfrac{1}{2}\frac{d}{dt}\left(LI^2\right)$$

$$W = \int \tfrac{1}{2}\frac{d}{dt}\left(LI^2\right)dt$$

$$W = \tfrac{1}{2}LI^2$$

$$LI = \Phi \;= \int \vec{B}\cdot d\vec{a}$$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

$$LI = \Phi = \int\left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{a} \;= \oint \vec{A}\cdot d\vec{l}$$

$$W = \tfrac{1}{2}I\oint \vec{A}\cdot d\vec{l} \;= \tfrac{1}{2}\oint \vec{A}\cdot\underbrace{\left(I d\vec{l}\right)}_{\vec{v}dq}$$

Rephrasing as sum over a volume containing current

$$W = \tfrac{1}{2}\int \vec{A}\cdot\vec{J}\,d\tau$$

$$\vec{J} = \tfrac{1}{\mu_o}\vec{\nabla}\times\vec{B}$$

$$W = \frac{1}{2\mu_o}\int \vec{A}\cdot\left(\vec{\nabla}\times\vec{B}\right)d\tau$$

$$\vec{A}\cdot\left(\vec{\nabla}\times\vec{B}\right) = \vec{B}\cdot\left(\vec{\nabla}\times\vec{A}\right) - \vec{\nabla}\cdot\left(\vec{A}\times\vec{B}\right)$$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

$$W = \frac{1}{2\mu_o}\int\left(\vec{B}\cdot\vec{B} - \vec{\nabla}\cdot\left(\vec{A}\times\vec{B}\right)\right)d\tau$$

$$W = \frac{1}{2\mu_o}\int B^2 d\tau - \int \vec{\nabla}\cdot\left(\vec{A}\times\vec{B}\right)d\tau$$

$$W = \frac{1}{2\mu_o}\int B^2 d\tau - \cancel{\oint\left(\vec{A}\times\vec{B}\right)\cdot d\vec{a}}$$

Sending volume to cover all space, surface to infinity, where B is presumed to be 0

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