

| | | |
|------------|---|------|
| Fri., 12/6 | 6.1 Magnetization, 6.2 Field of a Magnetized Object | HW10 |
| Mon., 12/9 | 6.3, 6.4 Auxiliary Field & Linear Media | HW11 |
| Wed. 12/11 | 12 noon Exam 3 (Ch 7, 10, 4, 6) | |

Transition.

So, we've just finished talking about a model for electric fields interacting with non-conductors – microscopic electric dipoles may be induced in response to an externally imposed field, and those will naturally generate a field of their own. We developed some specialized tools for dealing with this situation.

Now we'll consider *magnetic* fields interacting with matter. We will again develop some specialized tools for describing how the matter responds. As when considering the electric field, we'll start microscopic, and then scale up.

Summary

Before we get started, recall from Chapter 5 that the simplest possible current source is a current loop. If you're observing such a loop from a great distance (relative to its radius), then a multi-pole expansion seems a reasonable way to approximate its appearance. For the potential, first non-zero term in that expansion is

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_o}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}$$

where

$$\vec{m} = I \int d\vec{a}_{enclosed}$$

The corresponding field is just the curl of this,

$$\vec{B}_{dip}(\vec{r}) = \frac{\mu_o}{4\pi} \frac{3(\hat{n} \cdot \hat{r})\hat{r} - \vec{m}}{r^3}$$

or, if we take m be at the origin and pointing up z ,

$$\vec{B}_{dip}(\vec{r}) = \frac{\mu_o}{4\pi} \frac{m}{r^3} (\cos\theta\hat{r} + \sin\theta\hat{\theta})$$

Now, while a single atom is electrically neutral, most single atoms are *not* 'current neutral', that is, they have microscopic currents running around them. While we can't legitimately apply a semi-classical 'solar system' model to the particles that make up the atoms, if there's angular momentum, then there's circulation, and if the things with angular momentum are *charged*, then that spells *charge circulation = current*.

Electronic magnetic moment

We can get a little more specific about the relationship between angular momentum and current, or rather magnetic dipole moment. Now, the electron cloud about an atom necessarily gives proportional mass and charge densities, that is, at any location

$$\frac{\rho_{mass}}{\rho_{charge}} = \frac{m_{tot}}{q_{tot}}$$

Let's compare the magnetic dipole moment associated with an electron cloud and its angular momentum.

Recall that the approach we take in determining a real current distribution's dipole moment is to break it up into current loops, find *their* dipole moments, and then sum over all loops. With that in mind, we'll imagine we can do the same to an electron cloud and just bother ourselves with looking at *one* differential current loop and compare the dipole moment and angular momentum for it.

$$\vec{m} = I_{loop} \vec{a}_{encl} = \rho_q dA_{\perp} v \vec{a}_{encl}$$

$$\vec{L}_{loop} = \sum \vec{s}' \times m \vec{v} = \oint \vec{s}' \times \vec{v} \rho_m dA_{\perp} dl'$$

Now, the way we define the loop is that it's a path of constant current, where the current runs along the path, so $v\rho$ is constant along it and dl/v (note that you can't always define such paths without the cross-sectional areas needing to be parameterized to varying along the length.) Taking advantage of that,

$$\vec{L}_{loop} = v\rho_m dA_{\perp} \oint \vec{s}' \times d\vec{l}' = v\rho_m dA_{\perp} 2\frac{1}{2} \oint \vec{s}' \times d\vec{l}' = v\rho_m dA_{\perp} 2\vec{a}_{encl}$$

Comparing the two,

Apparently,

$$\vec{m} = \rho_q dA_{\perp} v \vec{a}_{encl} = \frac{q}{2m_{mass}} 2\rho_m dA_{\perp} v \vec{a}_{encl} = \frac{q}{2m_{mass}} \vec{L}$$

Now, this reasoning isn't perfect, but it gives us the general idea: *if a charged particle has an angular momentum, it has a corresponding magnetic dipole moment that is proportional to it.*

Armed with that knowledge, let's consider some simple atoms, starting with the simplest – Hydrogen. The proton has rotational angular momentum $\vec{s}_{p,1}$ which, about any axis, you'd measure to be $s_{p,1} = \pm \frac{1}{2} \hbar$. Ditto for the electron. Except at very cold temperatures, the fields that these two produce aren't enough to create significant interaction; in general, the nuclear angular momentum doesn't get pinned down except at very low temperatures, so, it's usually a blurry wash – we'll ignore it.

Now, let's add one more proton and one more electron – we've made Helium. The two protons tend to anti-align and so do the two electrons, so you've got no net angular momentum and no dipole moment.

Now, let's add one more proton and one more electron – Lithium. As with Hydrogen, we'll have one unbalanced spin electron and one unbalanced spin proton, so we'll have dipole moments. Also, there's orbital angular momentum.

Let's add one more protons and electrons – Be. The two electrons will be unpaired and there will likely be a net orbital angular momentum. If we add one more though, there will be no net orbital angular momentum but there will be spin.

From here, you get the picture. Almost every single atom, except those with closed shells, will have some net angular momentum, and thus a magnetic moment.

That said, when atoms bond up, they often do so in a way that eliminates angular momentum. Even if they don't, it's quite rare that the interaction between neighboring atoms strongly encourages alignment.

Types of Magnetic Dipoles

Electrons in atoms/molecules produce magnetic dipole moments for what we can think of as two kinds of motions:

- (1) “Orbital” – We can think of the electrons semi-classically as orbiting around in an atom. This “current loop” (in the opposite direction b/c the charge is negative) results in a magnetic dipole.
- (2) “Spin” – We can think of an electron as spinning around, which also means a current loop. However, electrons seem to be point particles, so this may be an intrinsic property. The size of the spin magnetic dipole moment of an electron is $\approx e\hbar/2m_e = 9.285 \times 10^{-24} \text{ A} \cdot \text{m}^2$. (The dipole moment for heavier particle is much less.)

We'll consider the things that can affect these two types of magnetic dipoles, then discuss the types of magnetism.

Torques on Magnetic Dipoles

We can find the torque on a magnetic dipole by considering a physical dipole which is a rectangular loop (sides a and b) carrying a current I . We actually did this back in chapter 5.

(Fig 6.2)

In the orientation shown above, the size of the forces on the sides of length b is

$$F = IbB.$$

in the + or – y direction. The torque, which tends to align \vec{m} with \vec{B} , is

$$\vec{N} = 2 \left[\left(\frac{a}{2} \right) (IbB) \sin \theta \right] \hat{x},$$

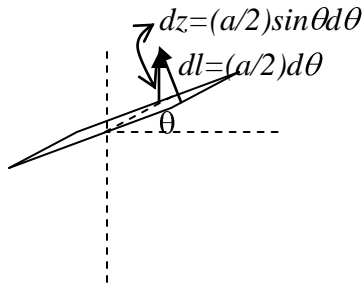
$$\vec{N} = \vec{m} \times \vec{B}.$$

Potential Energy and Force on Magnetic Dipoles

Since the torque is the same form as for electric dipoles, the potential energy is

$$U = -\vec{m} \cdot \vec{B},$$

You can easily see that this is the potential for *rotating* the dipole



$$\Delta U = - \int \vec{F}_{right} \cdot d\vec{l}_{right} + - \int \vec{F}_{left} \cdot d\vec{l}_{left} = -2 \int \vec{F}_{right} \cdot d\vec{l}_{right} = -2 \int (Ib) \vec{B} \cdot \frac{a}{2} d\vec{\theta} = -2 \int (Ib) B \sin \theta \frac{a}{2} d\theta$$

$$- (Iba) B \cos \theta \Big|_{\theta_i}^{\theta_f} = -m B \cos \theta \Big|_{\theta_i}^{\theta_f} = -\vec{m} \cdot \vec{B} \Big|_{\theta_i}^{\theta_f}$$

It's a little tricky, but one can argue that this leads to

$$\Delta U = -\vec{m} \cdot \vec{B} \Big|_{B_i}^{B_f}$$

being the correct expression for the change in energy when you change the magnetic field or *translate* the dipole between regions with different fields. The argument goes like this: rotate the dipole perpendicular to the field – thus this term is 0 *no matter what the field strength is*, then vary the field, then rotate the dipole *back*. For each rotation, the above expression is correct, it just so happens that the field is different for each.

and the force on a *given* dipole is

$$\vec{F} = -\vec{\nabla} U \Big|_m = \vec{\nabla} (\vec{n} \cdot \vec{B}) \Big|_m = m \times (\vec{\nabla} \times \vec{B}) + (\vec{n} \cdot \vec{\nabla}) \vec{B}. \tag{6.3}$$

(Jeffimeko makes the point that we're looking for the force on a *given* current distribution, so *m* must be held constant.)

Example: say that $\vec{m} = m\hat{z}$ and $\vec{B} = kx\hat{z}$, so the field points up but gets stronger as you move in the x direction.

$$\vec{F} = \vec{\nabla} (\vec{n} \cdot \vec{B}) \Big|_m$$

$$\vec{\nabla} (mkx) \Big|_m = mk \frac{\partial x}{\partial x} \hat{x} = mk\hat{x}$$

Alternatively,

$$\vec{F} = m \times (\vec{\nabla} \times \vec{B}) + (\vec{n} \cdot \vec{\nabla}) \vec{B}$$

$$m\hat{z} \times (\vec{\nabla} \times kx\hat{z}) + m \frac{\partial}{\partial z} kx\hat{z}$$

$$m\hat{z} \times (k\hat{y}) + 0$$

$$\vec{F} = mk\hat{x}$$

Take for example m and B parallel to each other in the z -direction, but B varies with location, say get's stronger in the x direction. Then you'

Magnetic Field of a Dipole

This is needed if we want to calculate how dipoles affect each other. If a magnetic dipole points in the z direction, $\vec{m} = m\hat{z}$, then the magnetic field that it produces is

$$\vec{B}_{dip} = \frac{\mu_0 m}{4\pi r^3} (2 \cos\theta \hat{r} + \sin\theta \hat{\theta}). \quad (5.86)$$

Effect of a Magnetic Field on an Atomic Orbit

Make a simplified (semiclassical) model how a magnetic field affects an electron's orbit. Suppose an electron is circling a nucleus with the opposite charge at a radius R and speed v . The time that it takes for one orbit is $T = 2\pi R/v$. The size of the "current" is

$$I = \frac{e}{T} = \frac{ev}{2\pi R},$$

so the orbital dipole moment (with the z axis as shown below) is

$$\vec{m} = -\frac{1}{2} evR \hat{z}.$$

Fig 6.9, 6.10

According to Newton's second law, we have

$$\frac{1}{4\pi\epsilon_0} \frac{e^2}{R^2} = m_e \frac{v^2}{R}.$$

Suppose we subject the atom to a magnetic field in the z direction (as shown above). There is an additional force toward the center (charge is negative).

If the radius stays the same, then the new speed $\bar{v} = v + \Delta v$ is determined by

$$\frac{1}{4\pi\epsilon_0} \frac{e^2}{R^2} + e\bar{v}B = m_e \frac{\bar{v}^2}{R}.$$

Combining the two previous equations gives

$$e\bar{v}B = m_e \frac{\bar{v}^2}{R} - \frac{1}{4\pi\epsilon_0} \frac{e^2}{R^2} = \frac{m_e}{R} (\bar{v}^2 - v^2) = \frac{m_e}{R} (\bar{v} + v)(\bar{v} - v).$$

If the change in speed, $\Delta v = \bar{v} - v$, is small, then

$$e\bar{v}B \approx \frac{m_e}{R} (2\bar{v})\Delta v.$$

This gives

$$\Delta v \approx \frac{eRB}{2m_e},$$

which means a change in the dipole moment of

$$\Delta \vec{m} = -\frac{1}{2} e(\Delta v)R \hat{z} = -\frac{e^2 R^2}{4m_e} \vec{B},$$

in the opposite direction from the magnetic field.

Note: this argument doesn't really hold water – B exerts a radial force on the charge loop, and thus can't be responsible for a torque about the loop's axis, thus can't be responsible for changing L or thus changing m. What *can* change m is the curled electric field that accompanies a change in B over time.

This change in magnetic dipole moment of course changes the field that the dipole generates. This is an example of Lenz's Law: Introduce a magnetic field and induce a change in current that will itself be responsible for a counter field.

Quantum Approach

Recall that when we'd met the vector potential, I'd noted Newton's 2nd law could be rephrased as

$$\begin{aligned} \frac{d}{dt} (m\vec{v}_1) &= q\vec{v} \times \vec{B} \\ \frac{d}{dt} (m\vec{v}_1 + q\vec{A}) &= \vec{\nabla} q \cdot \vec{A} \end{aligned}$$

So it's as if we had a 'system' with momentum $\vec{p}_{system} \equiv m\vec{v}_1 + q\vec{A}$.

Or, rephrasing, $m\vec{v}_1 = \vec{p}_{system} - q\vec{A}$

From here, one can build the Lagrangian and then the Hamiltonian of the system.

The Hamiltonian of this system is

$$\begin{aligned} H &= \frac{1}{2M} (m\vec{v}_1 - q\vec{A})^2 = \frac{1}{2m_e} (m\vec{v}_1 + e\vec{A})^2 = \frac{1}{2m_e} (p^2 + e\vec{p} \cdot \vec{A} + e\vec{A} \cdot \vec{p} + e^2 A^2) \\ H &= \frac{p^2}{2m_e} + \frac{e}{2m_e} (\vec{A} \cdot \vec{p}) + \frac{e^2}{2m_e} A^2 \end{aligned}$$

If the field is uniform in the z direction, then we can relate the potential to it through

$$\vec{A} = \frac{B_z s}{2} \hat{\phi} = \frac{\vec{B} \times \vec{r}}{2} \quad (\text{applying Stoke's Law to the relation between A and B})$$

$$H = \frac{p^2}{2m_e} + \frac{e}{2m_e} \left(2 \left(\frac{\vec{B} \times \vec{r}}{2} \right) \cdot \vec{p} \right) + \frac{e^2}{2m_e} \left(\frac{\vec{B} \times \vec{r}}{2} \right)^2$$

$$H = \frac{p^2}{2m_e} + \frac{e}{2m_e} \left(\vec{B} \times \vec{r} \right) \cdot \vec{p} + \frac{e^2}{8m_e} \left(\vec{B} \times \vec{r} \right)^2$$

Then by Product Rule 2,

$$H = \frac{p^2}{2m_e} + \frac{e}{2m_e} \vec{B} \cdot \left(\vec{r} \times \vec{p} \right) + \frac{e^2}{8m_e} \left(\vec{B} \times \vec{r} \right)^2$$

$$H = \frac{p^2}{2m_e} + \frac{e}{2m_e} \vec{B} \cdot \vec{L} + \frac{e^2}{8m_e} \left(\vec{B} \times \vec{r} \right)^2$$

The second term could be rephrased since $\left(\vec{B} \times \vec{r} \right)^2 = B_z^2 s^2$

$$H = \frac{p^2}{2m_e} + \frac{e}{2m_e} \vec{B} \cdot \vec{L} + \frac{e^2 B_z^2}{8m_e} s^2$$

So, the second and third terms represent the energy for interacting with the field.

Now, if we *insist* on modeling the electron as a magnetic dipole, then we have

$$U = -\vec{m} \cdot \vec{B}$$

Then we can read off what the magnetic dipole moment would have to be more-or-less by taking the negative derivative with respect to B.

$$m = \frac{-e}{2m_e} L - \frac{e^2 B_z}{4m_e} s^2$$

Note that the first term depends upon the existence of angular momentum while the second depends upon the *existence of a field*. So the magnetic moment changes as the field changes. That's the diamagnetic effect.

Types of Magnetism

This is more complicated than polarization, because the magnetization is not always in the same direction as an applied magnetic field.

(1) Paramagnetism

- Results from the tendency of dipoles to line up with the applied magnetic field. That's easiest to do to unpaired spins (as opposed to orbital angular momentum).
- It leads to a magnetization in the same direction as the applied field and attraction.
- It occurs in materials with unpaired electrons (odd numbers) with spins that can align with an external field. The alignment is not complete because of thermal motion.

(2) Diamagnetism

- Results from the changes in the orbits of electrons.
- It leads to a magnetization in the opposite direction from the applied field and repulsion.
- Compared to aligning spins (and full on *flipping them*) it's a small effect (just *tweaking* orbits), so it dominates in materials with no unpaired electrons (even numbers).

(3) Ferromagnetism – will be discussed in more detail later

- This is the effect responsible for common “permanent” magnets, but it is the most complicated to describe.
- Results from the interactions of the spins of unpaired electrons. They tend to align with their neighbors in regions called *domains*. When a magnetic field is applied, two things happen: (a) the direction of the domains shift toward the direction of the field (but don't perfectly align) and (b) domains in about the same direction grow.
- Magnetization depends on the “history” of the material, not just the current applied magnetic field. More about this later...
- Iron and nickel are examples of ferromagnetic materials.

Magnetization (6.1.4)

Suppose there are a lot of little dipoles pointing in the same direction. Define the magnetization (a vector):

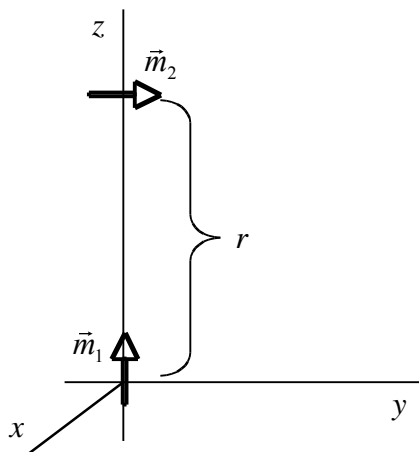
$$\vec{M} = \text{magnetic dipole moment per volume} = \frac{d\vec{m}}{d\tau},$$

which can be induced by an external magnetic field or “frozen in.”

Examples/Exercises:

Exercise – Torque between Dipoles

What is the torque on \vec{m}_2 ?



The magnetic field due to \vec{m}_1 at the location of \vec{m}_2 is (at $\theta = 0$, $\hat{r} = \hat{z}$)

$$\vec{B}_1 = \frac{\mu_0 m}{4\pi r^3} (2 \cos\theta \hat{r} + \sin\theta \hat{\theta}) = \frac{\mu_0 m_1}{2\pi r^3} \hat{z}.$$

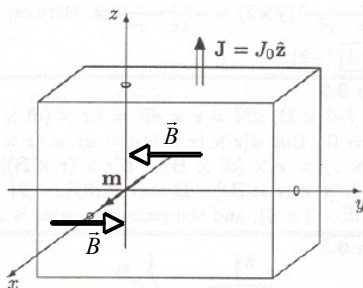
The torque on \vec{m}_2 is

$$\vec{N}_2 = \vec{m}_2 \times \vec{B}_1 = (m_2 \hat{z}) \times \left(\frac{\mu_0 m_1}{2\pi r^3} \hat{y} \right) = \frac{\mu_0 m_1 m_2}{2\pi r^3} \hat{x},$$

which would cause \vec{m}_2 to rotate so that it points more vertically.

Problem 6.5(a)(b) – Force on a Dipole in a Current-Carrying Slab

A uniform current density $\vec{J} = J_0 \hat{z}$ fills a slab straddling the yz plane, from $x = -a$ to $x = +a$. A magnetic dipole $\vec{m} = m_0 \hat{x}$ is situated at the origin (inside the slab).



- a. Find the force on the dipole, using Eq. 6.3.

From Prob. 5.14 (HW 7), but with directions different, the magnetic field inside the slab is $\vec{B} = \mu_0 J_0 x \hat{y}$. For the dipole in the x direction, $\vec{m} \cdot \vec{B} = 0$, so

$$\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B}) = 0.$$

- b. Do the same for a dipole point in the y direction, $\vec{m} = m_0 \hat{y}$.

For the dipole in the x direction, $\vec{m} \cdot \vec{B} = m_0 \mu_0 J_0 x$, so

$$\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B}) = \frac{\partial}{\partial x} (m_0 \mu_0 J_0 x) \hat{x} = m_0 \mu_0 J_0 \hat{x}.$$

- d. Suppose the dipole points in the x direction, $\vec{m} = m_0 \hat{x}$, as in part (b), and is located at $x = +2a$ and $y = z = 0$. Find the torque on the dipole.

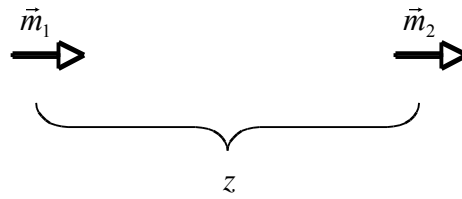
The magnetic field outside of the slab for $x > +a$ is $\vec{B} = \mu_0 J_0 a \hat{y}$, so the torque on the dipole is

$$\vec{N} = \vec{m} \times \vec{B} = (m_0 \hat{x}) \times (\mu_0 J_0 a \hat{y}) = m_0 \mu_0 J_0 a \hat{z}.$$

This torque will rotate the dipole counterclockwise when viewed from the $+z$ direction.

Problem 6.3 – Force between two dipoles

Two magnetic dipoles \vec{m}_1 and \vec{m}_2 are oriented as shown below and a distance z apart.



- (b) Use Eq. 6.3, $\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B})$, to find the size of the force on \vec{m}_2 .
- (c) Treating \vec{m}_2 as a physical dipole, make a sketch showing that it is attracted by \vec{m}_1 .
- a. Call the location of \vec{m}_1 the origin and its direction the z direction. The magnetic field of \vec{m}_1 at the location of \vec{m}_2 is

$$\vec{B}_1 = \frac{\mu_0 m_1}{2\pi z^3} \hat{z}.$$

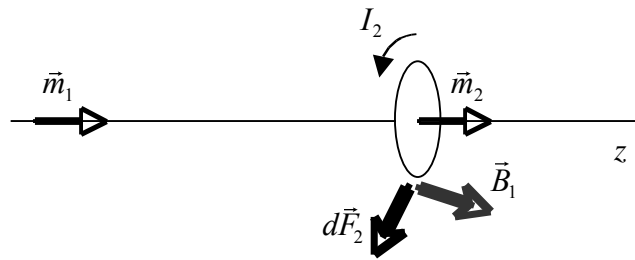
The second dipole can be written as $\vec{m}_2 = m_2 \hat{z}$, so

$$\vec{m}_2 \cdot \vec{B}_1 = \frac{\mu_0 m_1 m_2}{2\pi z^3}$$

and the force on \vec{m}_2 is

$$\vec{F}_2 = \vec{\nabla}(\vec{m}_2 \cdot \vec{B}_1) = \frac{\partial}{\partial z} \left(\frac{\mu_0 m_1 m_2}{2\pi z^3} \right) \hat{z} = - \left(\frac{3\mu_0 m_1 m_2}{2\pi z^4} \right) \hat{z}.$$

- b. If \vec{m}_2 is a physical dipole, the current I_2 flows in the direction shown in the diagram below. Slightly off the z , the magnetic field due to \vec{m}_1 has a small θ component. The force $d\vec{F}_2$ on a small segment of the loop is mostly radial, but also has small component in the $-z$ direction. This is true for every segment of the loop, so it is attracted to the other dipole.



Preview

On Monday, we'll talk about magnetized material, which contains many small magnetic dipoles.

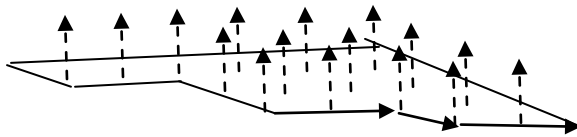
Summary

Magnetization (6.1.4)

Running parallel to our work with *electric* dipoles, suppose you have a chunk of material with a bunch of dipoles distributed all over it.



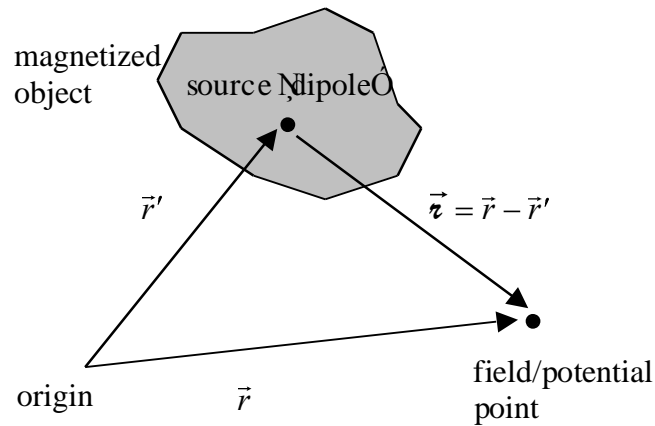
Starting simple, let's say that it's full of magnetic dipoles, evenly distributed, same size, same orientation...



If the vector potential due to just one of them is

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_o}{4\pi} \frac{\vec{m} \times \hat{u}}{r^2}, \text{ then that due to a whole collection of them is}$$

$$\vec{A}_{dip}(\vec{r}) = \sum_i \frac{\mu_o}{4\pi} \frac{\vec{m}_i \times \hat{u}_i}{r_i^2} = \int \frac{\mu_o}{4\pi} \frac{\left(\frac{d\vec{m}}{d\tau}\right) d\tau \times \hat{u}}{r^2}$$



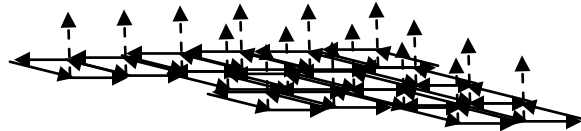
Or, defining the density of magnetic dipoles as

$$\vec{M} \equiv \frac{d\vec{m}}{d\tau}$$

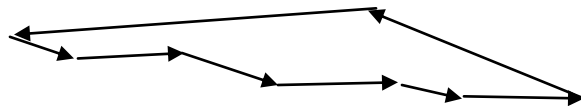
And dubbing it the “Magnetization,”

$$\vec{A}_{dip}(\vec{r}) = \int \frac{\mu_0}{4\pi} \frac{\vec{M} \times \hat{r}}{r^2} d\tau$$

Then again, think about what this mathematical entity “dipole” is here to quantify for us – charge circulation. So illustrating *that*,



Obviously, where two neighboring dipoles meet, the current along those legs cancels. The only place that that cancellation *doesn't* happen is along the edge, where there is no neighbor. So there's a macroscopic surface current.



As you can see, if there's a magnetic dipole pointing z , then there's a current running in the θ direction. Sounds like a cross-product relation.

So, this gives rise to a “bound” surface current density.

$$\vec{K}_b = \frac{dI}{dl_{side}} = \frac{dI a_{\perp side}}{dl_{side} a_{\perp side}} = \frac{dI a_{loop} \times \hat{n}}{d\tau} = \frac{d\vec{m} \times \hat{n}}{d\tau} = \vec{M} \times \hat{n}$$

Now, what if the magnetization *changes* from one location to another, the adjacent legs of two neighboring loops *don't* perfectly cancel. Then you get a volume current density:

$$\vec{J}_b = \vec{\nabla} \times \vec{M}$$

This one takes a little more work to get. Figuring out the bound volume current inside a magnetized object is a little more subtle. Suppose that the magnetization of two adjacent chunks is different. Two examples are shown below:

- (1) The z component of \vec{M} increases in the y direction. The current around the chunk on the right is larger, so there is net current in the x direction between them.

QuickTime™ and a
TIFF (Uncompressed) decompressor
are needed to see this picture.

- (2) The y component of \vec{M} increases in the z direction. The current around ~~the chunk on the right is larger, so there is net current in the $-x$ direction~~ between them.

QuickTime™ and a
TIFF (Uncompressed) decompressor
are needed to see this picture.

The *bound volume current* \vec{J}_b depends on how the magnetization changes inside the material. You might be able to see that it depends on the curl of \vec{M} .

Field of a Magnetized Object & Bound Currents (Mathematically)

Now, once we have translated from the language of dipole density (magnetization) to that of current density, we can use the old familiar relations

$$\vec{A} = \frac{\mu_0}{4\pi} \left\{ \int_{\text{volume}} \frac{\vec{J}_b(\vec{r}')}{r} d\tau' + \oint_{\text{surface}} \frac{\vec{K}_b(\vec{r}')}{r} da' \right\}$$

As with Polarization and bound charge densities, this relation can be derived from

$$\vec{A}_{dip}(\vec{r}) = \int \frac{\mu_0}{4\pi} \frac{\vec{M} \times \hat{r}}{r^2} d\tau$$

By again observing that

$$\vec{\nabla}' \left(\frac{1}{r} \right) = \frac{\hat{r}}{r^2},$$

so

$$\vec{A} = \frac{\mu_0}{4\pi} \int_{\text{volume}} \vec{M} \times \left(\vec{\nabla}' \frac{1}{r} \right) d\tau'.$$

Use a product rule to rewrite the integrand:

$$\vec{\nabla}' \times \left(\frac{1}{r} \vec{M} \right) = \frac{1}{r} (\vec{\nabla}' \times \vec{M}) - \vec{M} \times \left(\vec{\nabla}' \frac{1}{r} \right),$$

where the minus comes from switching the order of the cross product. This gives

$$\vec{M} \times \left(\vec{\nabla}' \frac{1}{r} \right) = \frac{1}{r} (\vec{\nabla}' \times \vec{M}) - \vec{\nabla}' \times \left(\frac{1}{r} \vec{M} \right),$$

so

$$\vec{A} = \frac{\mu_0}{4\pi} \left\{ \int_{\text{volume}} \frac{1}{r} (\vec{\nabla}' \times \vec{M}) d\tau' - \int_{\text{volume}} \vec{\nabla}' \times \left(\frac{1}{r} \vec{M} \right) d\tau' \right\}.$$

We have to use the result from Prob. 1.60(b),

$$\int_{\text{volume}} (\vec{\nabla}' \times \vec{v}) d\tau = - \oint_{\text{surface}} \vec{v} \times d\vec{a},$$

(the solution based on the divergence theorem is on the last page of the notes)

to get (also using $d\vec{a} = \hat{n} da$)

$$\vec{A} = \frac{\mu_0}{4\pi} \left\{ \int_{\text{volume}} \frac{1}{r} (\vec{\nabla}' \times \vec{M}) d\tau' + \oint_{\text{surface}} \frac{1}{r} (\vec{M} \times \hat{n}) da' \right\}, \quad (6.12)$$

$$\vec{A} = \frac{\mu_0}{4\pi} \left\{ \int_{\text{volume}} \frac{\vec{J}_b(\vec{r}')}{r} d\tau' + \oint_{\text{surface}} \frac{\vec{K}_b(\vec{r}')}{r} da' \right\}, \quad (6.15)$$

where the bound currents are defined as

$$\vec{J}_b = \vec{\nabla} \times \vec{M} \quad \text{and} \quad \vec{K}_b = \vec{M} \times \hat{n}. \quad (6.13 \ \& \ 6.14)$$

So, if you want to find the magnetic field from the magnetization \vec{M} , there are two options:

1. Use \vec{M} to find \vec{A} using Equations 6.13-15, then use $\vec{B} = \nabla \times \vec{A}$.
2. Find the bound currents \vec{J}_b and \vec{K}_b from \vec{M} , then use Ampere's law or the Biot-Savart law to find \vec{B} .

Examples/Exercises:

Problem 6.7

An infinitely long circular cylinder carries a uniform magnetization \vec{M} parallel to its axis. Find the magnetic field (due to \vec{M}) inside and outside the cylinder.

If we label the axis of the cylinder as z , then $\vec{M} = M\hat{z}$. The bound currents (volume and surface) are

$$\vec{J}_b = \nabla \times \vec{M} = 0 \quad \text{and} \quad \vec{K}_b = \vec{M} \times \hat{n} = (M\hat{z}) \times \hat{s} = M\hat{\phi}.$$

This is like the current for a solenoid, but replace the current per length nI (recall that n is turns per length) by $K_b = M$. The magnetic field is

$$\vec{B} = \begin{cases} \mu_0 M\hat{z} = \mu_0 \vec{M} & s < R, \\ 0 & s > R. \end{cases}$$

(actually work through this using an Amperian Loop)

The traditional Amperian Loop analysis for a solenoid is that

- a) The field is translationally symmetric, thus whatever contributions there may be to the legs of the loop that go radially in and out cancel
- b) You're just left with the axial legs, one inside and outside, now, note that their contributions sum to the same enclosed field regardless of how far in or out you make those legs, which means that the field has to be radially independent (constant) in these two regions (otherwise, moving the outer leg further out would change it's contribution to the sum and yet the other contribution would remain constant and sum to a constant. Outside, the only constant field that makes sense is 0 (since the field has to die off as you get far away.)
- c) So you're just left with the question of whether or not the field parallel to the inner leg is everything – mightn't there be a radial or angular component? Imagine drawing a coaxial amperian loop inside – symmetry says that the angular component of the field must be constant all the way around the loop, yet there's no piercing current – so that constant is 0. As for a radial component, that would constitute a divergence, and B doesn't diverge.

Problem 6.10

An iron rod of length L and square cross section (side a) is given a uniform longitudinal magnetization \vec{M} , then is bent around into a circle with a narrow gap (width w). Find the magnetic field at the center of the gap, assuming $w \ll a \ll L$. [Hint: treat it as the superposition of a *complete* torus plus a square loop with reversed magnetization or current.]

QuickTime™ and a
TIFF (Uncompressed) decompressor
are needed to see this picture.

A bound surface current $K_b = M$ will flow around the square cross section in the direction given by the RHR ($\vec{K}_b = \vec{M} \times \hat{n}$). This is like a torus and a square loop with a current $I = K_b w = Mw$ flowing the opposite way. Inside a torus, the magnetic field is (Ex. 5.10)

$$\vec{B} = \frac{\mu_0 NI}{2\pi s} \hat{\phi},$$

where s is the distance from the center of the “donut hole.” Since $w \ll L$, the circumference of a circle of radius s is $2\pi s \approx L$. Replace the total current NI (recall that N is the number of turns) by $K_b L = ML$. For a complete torus, the field would be

$$B_{\text{torus}} = \frac{\mu_0 (ML)}{L} = \mu_0 M,$$

in the same direction as \vec{M} (by the RHR). The magnetic field at the center of a square loop with side a and current $I = Mw$ is (Prob. 5.8 on HW 6):

$$B_{\text{square}} = \frac{\sqrt{2}\mu_0 I}{\pi R} = \frac{\sqrt{2}\mu_0 (Mw)}{\pi(a/2)} = \frac{2\sqrt{2}\mu_0 Mw}{\pi a},$$

in the opposite direction. The size of the total magnetic field is

$$B = B_{\text{torus}} - B_{\text{square}} = \mu_0 M \left(1 - \frac{2\sqrt{2}w}{\pi a} \right),$$

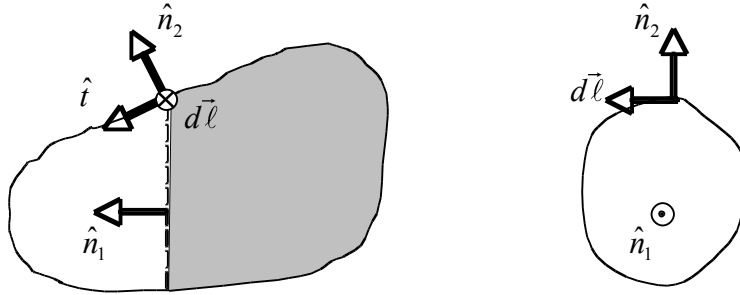
in the direction of \vec{M} . Since $w \ll a$, the magnetic field is $B \approx \mu_0 M$.

Example – Total Bound Current Through a Surface (based on Wangsness pp. 356-7)

Show that the total bound current through any loop surrounding a magnetized object is zero.

Suppose we want to find how much bound current flows from the right (shaded) to the left (unshaded) side of the magnetized object pictured below. Define two unit

normal vectors: \hat{n}_1 is perpendicular to the (imaginary) surface through which positive bound volume current flows and \hat{n}_2 is perpendicular to the outer surface of the object. Let $d\vec{\ell}$ be a differential element of length on the loop around the material in the direction by the RHR. Also, define the unit tangent vector \hat{t} , which is the direction in which positive surface current flows. Note that \hat{n}_2 , $d\vec{\ell}$, and \hat{t} are perpendicular to each other.



The bound current from left to right has two parts: the bound volume current in the direction of \hat{n}_1 and the bound surface current in the direction of \hat{t} . Write it as

$$I_b = \int \vec{J}_b \cdot d\vec{a} + \oint (\vec{K}_b \cdot \hat{t}) d\ell,$$

where $d\vec{a} = \hat{n}_1 da$. Substitute in $\vec{J}_b = \vec{\nabla} \cdot \vec{M}$ and $\vec{K}_b = \vec{M} \times \hat{n}_2$ to get

$$I_b = \int (\vec{\nabla} \times \vec{M}) \cdot d\vec{a} + \oint (\vec{M} \times \hat{n}_2) \cdot \hat{t} d\ell.$$

Use the curl (Stoke's) theorem on the first term and the identity $(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c})$ from the front cover on the second term to get

$$I_b = \oint \vec{M} \cdot d\vec{\ell} + \oint \vec{M} \cdot (\hat{n}_2 \times \hat{t}) d\ell.$$

Finally, notice that $\hat{n}_2 \times \hat{t}$ is in the opposite direction of $d\vec{\ell}$, so $(\hat{n}_2 \times \hat{t}) d\ell = -d\vec{\ell}$.

This gives the desired result of

$$I_b = \oint \vec{M} \cdot d\vec{\ell} - \oint \vec{M} \cdot d\vec{\ell} = 0.$$

Preview

Next time, we'll talk about the auxiliary field \vec{H} which is related to the magnetic field \vec{B} and the magnetization \vec{M} . This will allow us to formulate an "Ampere's law" for \vec{H} in terms of just bound currents.

Problem 1.60 (b)

Show that $\int_{\text{volume}} (\vec{\nabla} \times \vec{v}) d\tau = - \int_{\text{surface}} \vec{v} \times d\vec{a}$. [Hint: Replace \vec{v} by $(\vec{v} \times \vec{c})$ in the divergence theorem, where \vec{c} is a constant vector.]

The divergence theorem is

$$\int_{\text{volume}} (\vec{\nabla} \cdot \vec{v}) d\tau = \int_{\text{surface}} \vec{v} \cdot d\vec{a}.$$

Replacing \vec{v} by $(\vec{v} \times \vec{c})$ gives

$$\int_{\text{volume}} [\vec{\nabla} \cdot (\vec{v} \times \vec{c})] d\tau = \int_{\text{surface}} (\vec{v} \times \vec{c}) \cdot d\vec{a}.$$

On the left-hand side, the integrand can be written using a product rule as

$$\vec{\nabla} \cdot (\vec{v} \times \vec{c}) = \vec{v} \cdot (\vec{\nabla} \times \vec{c}) + (\vec{\nabla} \times \vec{v}) \cdot \vec{c},$$

where the first curl is zero because \vec{c} is constant.

On the right-hand side, the triple product in the integral can be written as

$$(\vec{v} \times \vec{c}) \cdot d\vec{a} = (d\vec{a} \times \vec{v}) \cdot \vec{c} = -(\vec{v} \times d\vec{a}) \cdot \vec{c},$$

where the minus is from switching the order of the cross product.

On both sides, we can take \vec{c} out of the integrals because it is constant. This gives

$$\left[\int_{\text{volume}} (\vec{\nabla} \times \vec{v}) d\tau \right] \cdot \vec{c} = \left[- \int_{\text{surface}} \vec{v} \times d\vec{a} \right] \cdot \vec{c}.$$

Since \vec{c} can be anything, the integrals must be equal.

"I'm still confused about the derivation of 6.8, Can we go over that?" [Jessica](#)

I was also a bit iffy on the steps from 6.6 to 6.8. [Casey McGrath](#)

"If magnetic fields do no work, then why is it that you can pick up objects with ferro-magnets, and doesn't this break conservation of energy since ferro-magnets are permanent?" [Freeman](#),

"Can we talk about how the orbital speed of the electron can cause an atom to be repelled away with the field? Wouldn't increased v just increase the magnitude of the dipole moment and wouldn't it need to change direction to be repelled?" [Ben Kid](#)

"can we talk about the math on page 271?" [Connor W.](#)

"Now, we know that a charge undergoing acceleration will radiate energy. This lead to the important question of why an electron wouldn't just collapse to the center of the nucleus it orbited, but quantum mechanics fortunately solved that problem." [Casey McGrath](#)

I'm curious, when Griffiths says that turning on a magnetic field will increase the orbital speed of the electron, would this manifest as a continuous acceleration (as B increased over time), or would it occur only in discrete velocity increments? Doesn't quantum mechanics solve the above problem precisely by only allowing certain discrete energy orbitals/states, so rather than speeding up the velocity gradually, wouldn't it happen in quantized jumps? [Casey McGrath](#)

"Griffiths says that diamagnetism is typically much weaker than paramagnetism. Why is this the case?" [Spencer](#)
Paramagnetism is a little easier to understand because of the similarity to polarization in electrostatics, but I'm still unclear about diamagnetism. [Spencer](#)

"Do you think we could draw a few more useful parallels about how M the magnetization and P (polarization) are similar? Either mathematically, conceptually or both ." [Rae](#)

"Why isn't paramagnetism "frozen in"?" [Casey P,](#)

"Can we do an example problem like Ex6.1 except with a non uniform magnetization?"
[Jessica](#)