Mon Wed Thurs Fri.	(C 14) 4.2 Field of Polarized Object (C14) 4.3 Electric Displacement Washington 3-2 Rep 7pm in AHon 116 (C14) 4.4.1 Linear Dielectrics (read rest at your discretion)	 HW 5

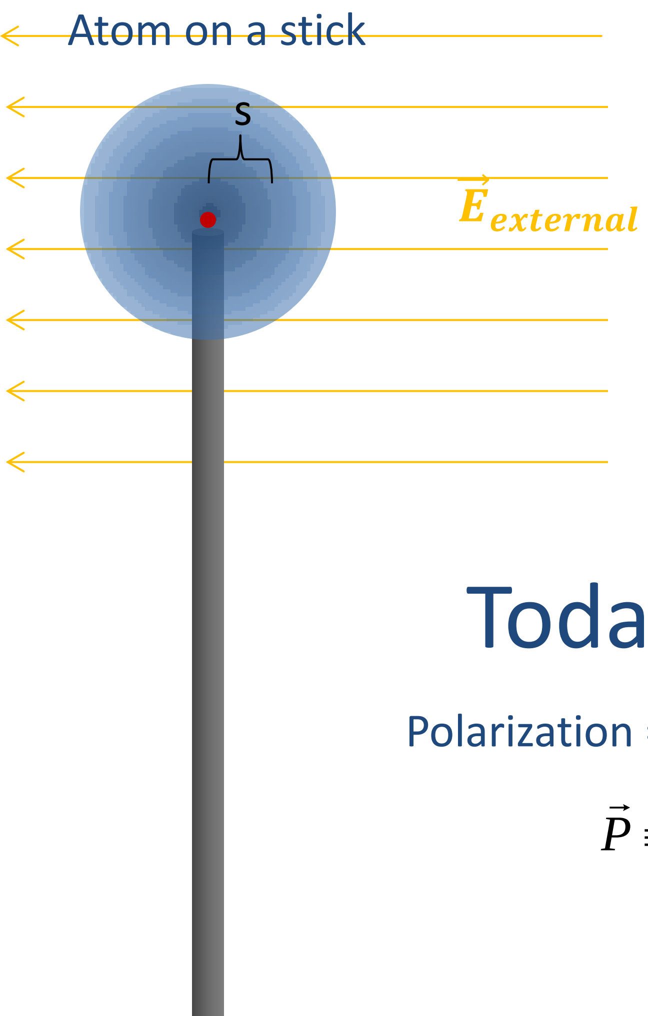
Useful relations From the Past

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\hat{r}\cdot\vec{p}}{r^2}$$

Potential *due to* Dipole (term)

$$\vec{E}_{dip}(r,\theta) = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

Field *due to* Dipole (term)



$$\vec{p} \approx \overleftrightarrow{\alpha}\vec{E}_{ext}$$

Dipole Moment and Polarizability Tensor

$$\vec{N} = \vec{p}\times\vec{E}$$

Torque on Dipole

$$\Delta U(\vec{r}) = -\Delta\left(\vec{p}\cdot\vec{E}\right)$$

Energy of rotating or changing dipole

$$\vec{F} = \left(\vec{p}\cdot\vec{\nabla}\right)\vec{E}$$

Force on dipole

Today's Starting Point

Polarization = volume density of dipole moments

$$\vec{P} \equiv \frac{d\vec{p}}{d\tau}$$

akin to charge density

$$\rho \equiv \frac{dq}{d\tau}$$

Polarization & “Bound Charge”

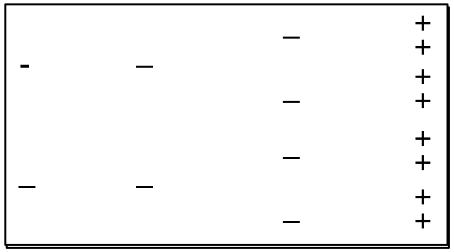
Conceptual

Uniform density of dipoles (Uniform Polarization)

Produces surface-charge density

$$V_{surf}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\oint \frac{\sigma(\vec{r}')da'}{\mathscr{r}}$$

Varying density of dipoles (Varying Polarization)



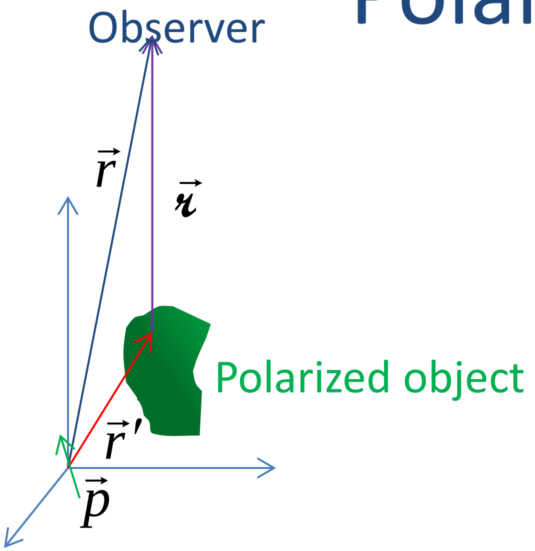
Produces volume-charge density too

$$V_{vol}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\oint \frac{\rho(\vec{r}')d\tau'}{\mathscr{r}}$$

b for “bound” charge densities – charges bound to their atoms, can’t translate through material

$$V(\vec{r}) = V_{surf}(\vec{r}) + V_{vol}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\oint \frac{\sigma_b(\vec{r}')da'}{\mathscr{r}} + \frac{1}{4\pi\varepsilon_o}\oint \frac{\rho_b(\vec{r}')d\tau'}{\mathscr{r}}$$

Polarization & "Bound Charge"



mathematical

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\hat{r}\cdot\vec{p}(0)}{r^2}$$

Being explicit that this equation was derived in expansion about origin; works best if dipole at origin

Generalized to dipole *off* origin

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\hat{\mathscr{r}}\cdot\vec{p}(\vec{r}\,')}{\mathscr{r}^2}$$

Sum over all dipoles in object

$$V_{dips}(\vec{r}) = \sum_{dipoles}\frac{1}{4\pi\varepsilon_0}\frac{\hat{\mathscr{r}}\cdot\vec{p}(\vec{r}\,')}{\mathscr{r}^2}$$

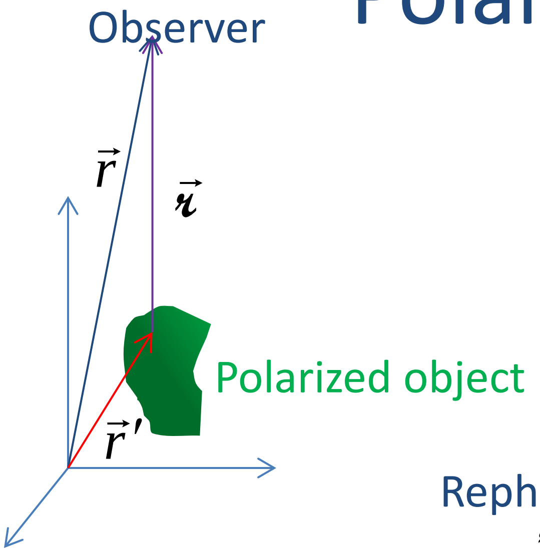
In limit of differentially-small dipoles

$$V_{dips}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{volume}\frac{\hat{\mathscr{r}}\cdot d\vec{p}(\vec{r}\,')}{\mathscr{r}^2} = \frac{1}{4\pi\varepsilon_0}\int_{volume}\frac{\hat{\mathscr{r}}\cdot\vec{P}(\vec{r}\,')d\tau'}{\mathscr{r}^2}$$

since $\dfrac{d\vec{p}}{d\tau} = \vec{P}$ or $d\vec{p} = \vec{P}d\tau$

Polarization & "Bound Charge"

mathematical



$$V_{dips}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{volume}\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\cdot\vec{P}(\vec{r}\,')d\tau'$$

Recall conceptual-based expectation that uniform polarization looks like surface charge and spatially-varying looks like volume charge

Rephrase expression to force that appearance

$$\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = -\vec{\nabla}_r\left(\frac{1}{\mathscr{r}}\right) = \vec{\nabla}_{r'}\left(\frac{1}{\mathscr{r}}\right) \qquad \text{since } \vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$$

so

$$V_{dips}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{volume}\left(\vec{\nabla}_{r'}\left(\frac{1}{\mathscr{r}}\right)\right)\cdot\vec{P}(\vec{r}\,')d\tau'$$

Product Rule 5! $\vec{\nabla}\cdot\left(f\vec{A}\right) = f\left(\vec{\nabla}\cdot\vec{A}\right) + \vec{A}\cdot\left(\vec{\nabla}f\right)$ or $\vec{A}\cdot\left(\vec{\nabla}f\right) = \vec{\nabla}\cdot\left(f\vec{A}\right) - f\left(\vec{\nabla}\cdot\vec{A}\right)$

so

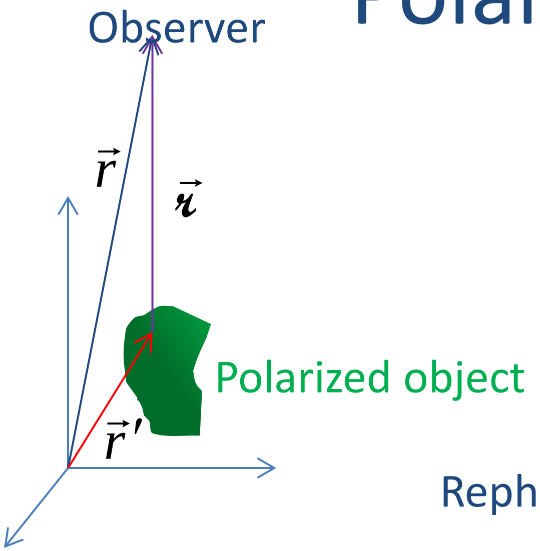
$$V_{dips}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{volume}\vec{\nabla}_{r'}\left(\frac{\vec{P}(\vec{r}\,')}{\mathscr{r}}\right) - \frac{1}{\mathscr{r}}\left(\vec{\nabla}_{r'}\cdot\vec{P}(\vec{r}\,')\right)d\tau'$$

$$V_{dips}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\int_{volume}\vec{\nabla}_{r'}\left(\frac{\vec{P}(\vec{r}\,')}{\mathscr{r}}\right)d\tau' + \int_{volume}\frac{\left(-\vec{\nabla}_{r'}\cdot\vec{P}(\vec{r}\,')\right)}{\mathscr{r}}d\tau'\right]$$

$$V_{dips}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\int_{surface}\frac{\vec{P}(\vec{r}\,')}{\mathscr{r}}\cdot d\vec{a}\,' + \int_{volume}\frac{\left(-\vec{\nabla}_{r'}\cdot\vec{P}(\vec{r}\,')\right)}{\mathscr{r}}d\tau'\right]$$

Polarization & “Bound Charge”

mathematical



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$$V_{dips}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\int_{surface}\frac{\vec{P}(\vec{r}\,')}{\mathscr{r}}\cdot d\vec{a}\,' + \int_{volume}\frac{\left(-\vec{\nabla}_{r'}\cdot\vec{P}(\vec{r}\,')\right)}{\mathscr{r}}d\tau'\right]$$

Form of

$$V_{dips}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\int_{surface}\frac{\sigma_b}{\mathscr{r}}da' + \int_{volume}\frac{\rho_b}{\mathscr{r}}d\tau'\right]$$

where

$$\sigma_b = \vec{P}\cdot\hat{a} \quad \text{and} \quad \rho_b = -\vec{\nabla}\cdot\vec{P}$$

Polarization & “Bound Charge”

$$V_{dips}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\int_{surface} \frac{\sigma_b}{\mathscr{r}}\,da' + \int_{volume} \frac{\rho_b}{\mathscr{r}}\,d\tau'\right]$$

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$$\sigma_b = \vec{P}\cdot\hat{a} \quad \text{and} \quad \rho_b = -\vec{\nabla}\cdot\vec{P}$$

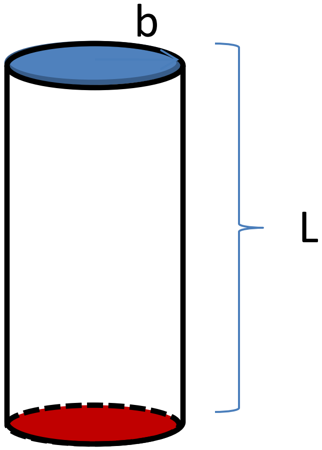
When you polarize a neutral dielectric, charges move a bit, but the *total* remains zero.

$$Q_b = \int_{volume} \rho_b\,d\tau' + \oint_{surface} \sigma_b\,da$$

Exercise: use math like we did to derive the V_{dips} expression to now show that the total bound charge is 0.

Polarization & “Bound Charge”

Bound charges are *real* charges



Conceptual (mathematical) Example: Cylinder of aligned dipoles

Total dipole moment: $$p_{total} = (P \cdot Vol) = qL$$

Surface charge – polarization relation:

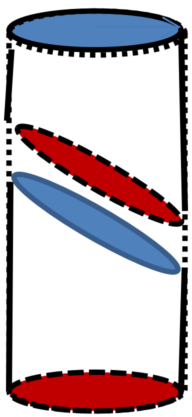
Charge resides on ends

$$PA_{cross}L = qL$$

$$PA_{cross} = q$$

or $$P = \frac{q}{A_{cross}} = \sigma_b$$

Cut obliquely



$$PA_{cross} = q$$

(the half must be net neutral – so equal q’s on level and oblique faces)

$$PA_{end} \cos\theta = q$$

$$P\cos\theta = \frac{q}{A_{end}}$$

$$\vec{P} \cdot \hat{a}_{cross} = \frac{q}{A_{end}} = \sigma_b$$ As our more mathematical derivation concluded

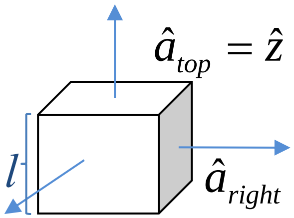
Polarization & "Bound Charge"

Example

$$V_{dips}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\int_{surface}\frac{\sigma_b}{\mathscr{r}}\,da' + \int_{volume}\frac{\rho_b}{\mathscr{r}}\,d\tau'\right]$$

where

$$\sigma_b = \vec{P}\cdot\hat{a} \quad \text{and} \quad \rho_b = -\vec{\nabla}\cdot\vec{P}$$



$\hat{a}_{top} = \hat{z}$

$\hat{a}_{right} = \hat{y}$

$\hat{a}_{front} = \hat{x}$

$\hat{a}_{back} = -\hat{x}$

$\hat{a}_{bottom} = -\hat{z}$

$\hat{a}_{left} = -\hat{y}$

A dielectric *cube* of side l, centered at the origin, carries a "frozen-in" polarization , $\vec{P} = k\vec{r}$ where k is a constant. Find all of the bound charges and check that they add up to zero.

In Cartesian (since we're talking about a cube)

$$\vec{P} = k\vec{r} = k\left(x\,\hat{x} + y\,\hat{y} + z\,\hat{z}\right)$$

Volume charge density:

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\left(\frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}\right)$$

$$= -\left[\frac{\partial(kx)}{\partial x} + \frac{\partial(ky)}{\partial y} + \frac{\partial(kx)}{\partial z}\right] = -3k$$

Surface charge density:

$\sigma_b = \vec{P}\cdot\hat{a}$ For example, $\sigma_{b.top} = \vec{P}\Big|_{z=\frac{l}{2}}\cdot\hat{z}$

$$\sigma_{b.top} = k\left[x\,\hat{x} + y\,\hat{y} + \left(l/2\right)\hat{z}\right]\cdot\hat{z} = kl/2$$

Given the symmetry, ditto for all six sides

$$\sigma_b = kl/2$$

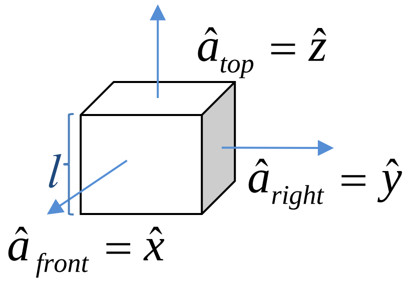
Polarization & "Bound Charge"

Example

$$V_{dips}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\int_{surface} \frac{\sigma_b}{\mathscr{r}} da' + \int_{volume} \frac{\rho_b}{\mathscr{r}} d\tau'\right]$$

where

$$\sigma_b = \vec{P}\cdot\hat{a} \quad \text{and} \quad \rho_b = -\vec{\nabla}\cdot\vec{P}$$



$\hat{a}_{back} = -\hat{x}$

$\hat{a}_{bottom} = -\hat{z}$

$\hat{a}_{left} = -\hat{y}$

A dielectric *cube* of side l, centered at the origin, carries a "frozen-in" polarization , $\vec{P} = k\vec{r}$ where k is a constant. Find all of the bound charges and check that they add up to zero.

Volume charge density: $\rho_b = -\vec{\nabla}\cdot\vec{P} = -3k$

Surface charge density: $\sigma_b = kl/2$

Total bound charge

$$Q_b = \int_{volume} \rho_b d\tau' + \oint_{surface} \sigma_b da$$

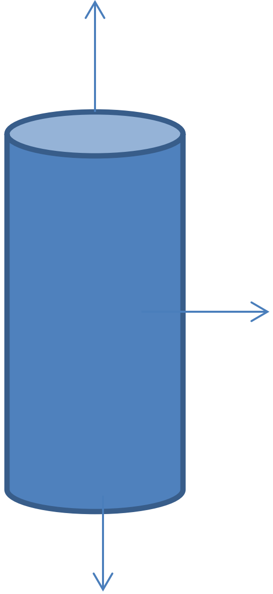
Densities are conveniently constant

$$Q_b = \rho_b Vol + \sigma_b A$$

$$Q_b = (-3k)l^3 + 6\left[(kl/2)l^2\right] = 0$$

Polarization & "Bound Charge"

Exercise



$$V_{dips}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\int_{surface} \frac{\sigma_b}{\mathscr{r}}\, da' + \int_{volume} \frac{\rho_b}{\mathscr{r}}\, d\tau'\right]$$

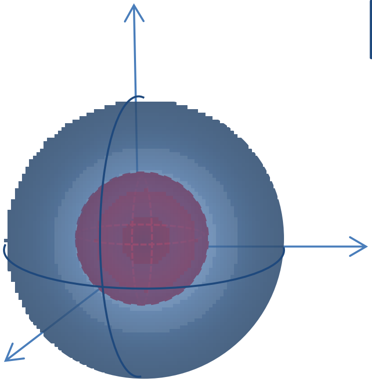
where

$$\sigma_b = \vec{P}\cdot\hat{a} \quad \text{and} \quad \rho_b = -\vec{\nabla}\cdot\vec{P}$$

A dielectric *cylinder* of radius *R* and length *L* is centered on the *z* axis. One end of the cylinder is at *z* = 0. It carries a "frozen-in" polarization , $\vec{P} = -k\left[1 + z/L\right]\hat{z}$ where *k* is a constant. Find all of the bound charges and check that they add up to zero.

Polarization & “Bound Charge”

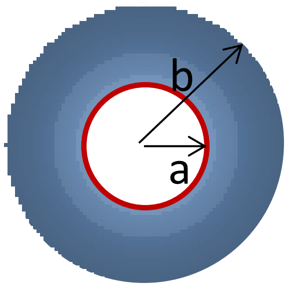
Exercise



$$\sigma_b = \vec{P} \cdot \hat{a} \quad \text{and} \quad \rho_b = -\vec{\nabla} \cdot \vec{P}$$

A thick spherical shell (inner radius *a* and outer radius *b*) is made of dielectric material with a “frozen-in” polarization $\vec{P}(\vec{r}) = \frac{k}{r}\hat{r}$

Locate all of the bound charge and use Gauss’s law to calculate the electric field in the three regions.



Cross-sectional view

[redacted]

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