Mon. 9/9	*(C 21.1-.5,.8)* **1.2 & 1.5; 2.2.1-.2.2** Gauss & Div, **T2** Numerical Quadrature	
Wed. 9/11	*(C 21.1-.5,.8)* **2.2.3** Using Gauss	
Thurs 9/12		HW1
Fri. 9/13	*(C21.1-.5,.8)* **2.2.3-.2.4** Using Gauss	

Equipment

- Bring in ppt's Gauss's Law
- Tutorial 2

Note: I should have recommended reading section 1.5 (delta function) as well.

Last Time

Last time we worked on computing the electric field at some observation location due to a continuous distribution of charge. In general, that meant summing, i.e., integrating, the contributions of all the differential morsels of charge:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_Q \frac{dq(\vec{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}$$

Depending on the geometry of the charge distribution, this sum will be parameterized in different ways so that rather than summing over charge, you're summing over locations that the charge occupies. Fairly generally,

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{dq(\vec{r}')}{\mathscr{r}^2}\hat{\mathscr{r}} = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\vec{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\, d\tau'$$

This Time

Unfortunately, that integral can be a challenge to perform. For some geometries, or under some approximations, there's a way that's often simpler – using Gauss's Law.

Back in Phys 232, we derived

$$\oint_S \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0}Q_{\text{enc}}$$

Which we call Gauss's Law.

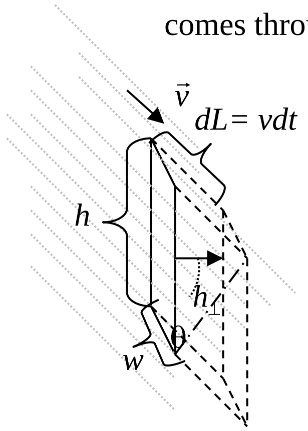
The argument went something like this:

Flux

First we got familiar with the notion of Flux, generally a measure of flow through an area / into or out of a volume.

- **Example: Rain**
    - **Flux through single, open area.** Say it's raining out and you have left your window open. Then, a reasonable question would be, at what rate is rain coming in your window. To make it concrete, let's say that we measure

amount of rain in terms of the mass of water. What's the rate at which water comes through the window?



$$\Phi_{water\to window} = \frac{dm_{water}}{dt} = \frac{\Delta m_{water}}{\Delta Vol}\frac{dVol}{dt} = \rho_{water}\frac{(dL\ wh_\perp)}{dt} = \rho_{water}\frac{dL}{dt}wh_\perp = \rho_{water}vw(h\cos\theta)$$
$$\Phi_{water\to window} = \rho_{water}vA(\cos\theta)$$
$$\Phi_{water\to window} = \rho_{water}\vec{v}\cdot\vec{A}$$

- **Flux through whole closed area.** Now, what if we were interested in not just the flux through a particular window, but into the whole room. That's simply the sum of fluxes in through all windows, out through the door, down through the floor (boy, that's gonna be a mess to clean up!).

$$\Phi_{water\to room} = \Phi_{water\to window1} + \Phi_{water\to window2} + \Phi_{water\to floor} + \Phi_{water\to door} = \sum \rho_i\vec{v}_i\cdot\vec{A}_i$$

  - **Area direction convention – "in."** In *this* case, it's convenient to have all the area vectors point *into* the room, so that you get positive contribution if the velocity points in and negative contribution if it points out (as with the door and floor.)

- **Integral Form.** More generally, this discrete sum can be written as a continuous integral.

$$\Phi_{water.into.room} = \oint \rho\vec{v}\cdot d\vec{A}_{in}$$

For the sake of future arguments, often one talks about the flux *out* through a closed area rather than *in* through it. Of course, that's just the opposite of *in* through the area.

$$\Phi_{water.out.room} = \oint \rho\vec{v}\cdot d\vec{A}_{out}$$

- **Generalizing: Any "vector field."** While our physics idea of "flux" connects best with the common usage when we're talking about something (in this case, water) being transported, the same mathematical and conceptual tool can be applied to any thing that's represented by a vector field, such as… the electric field.

- **Electric Flux**

    - Now let's apply the same idea to an Electric field.

        Through a patch of area,

        $$\Phi_E \equiv \vec{E} \bullet \vec{A}$$

        In or out of a closed surface

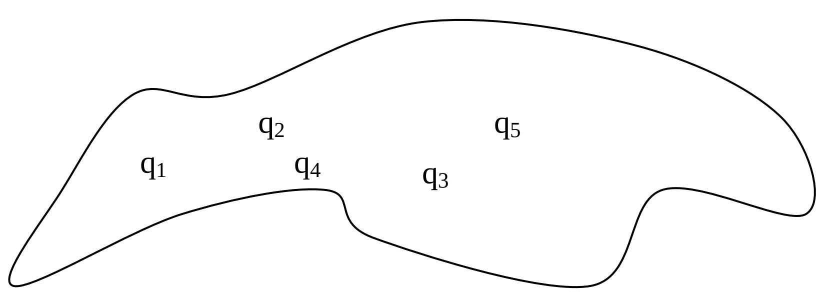
        $$\Phi_E \equiv \oint \vec{E} \bullet d\vec{A}$$

        As Griffith's points out, the analog to water stream lines is electric field lines.

    - **Motivation – What's it good for?** That's a swell definition and all, but you may be wondering why we bother – what practical use is such a definition. It turns out that there's a very simple relation between electric flux and charge distribution. That means that if you know one, it's easy to find the other, and in some cases it's easier to solve for E through a flux argument than simply summing over the sources as we've done in the past.
    - If that seems a worthy goal, then let's figure out what that but what's important is that, regardless of how messy the integral is, it has very simple solution. The book argued it out piece by piece, so now let's put all the pieces together in a coherent story.

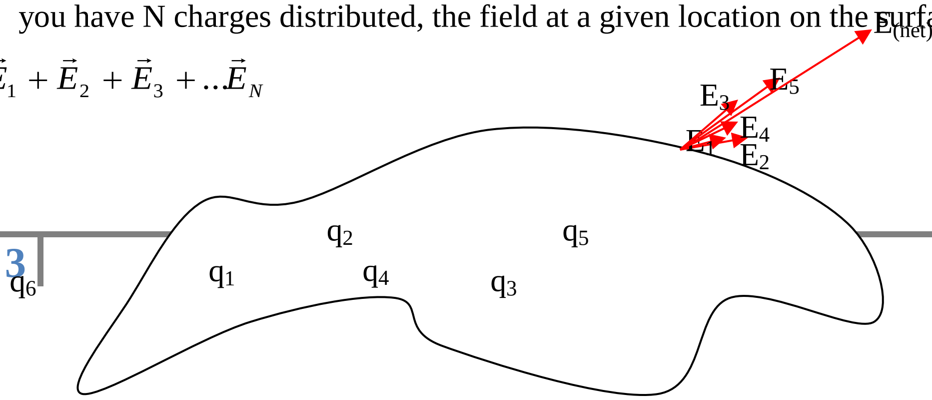
Relate Flux and Charge

- This is a very general tool, so I'll give you some vague visuals to help think about it, but don't take them too literally. Say we have some charge distribution like want to know the Flux of the electric field out a surface like this



- **First, Supperposition**: recall that the field of a charge distribution is simply the sum of the fields due to each point charge that makes up the distribution. That is, no matter how you have N charges distributed, the field at a given location on the surface is simply

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \ldots \vec{E}_N$$



That means that

$$\Phi_E \equiv \oint \vec{E} \bullet d\vec{A} = \oint \left(\vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \ldots \vec{E}_N\right) \bullet d\vec{A} = \oint \vec{E}_1 \bullet d\vec{A} + \oint \vec{E}_2 \bullet d\vec{A} + \oint \vec{E}_3 \bullet d\vec{A} + \ldots \oint \vec{E}_N \bullet d\vec{A}$$

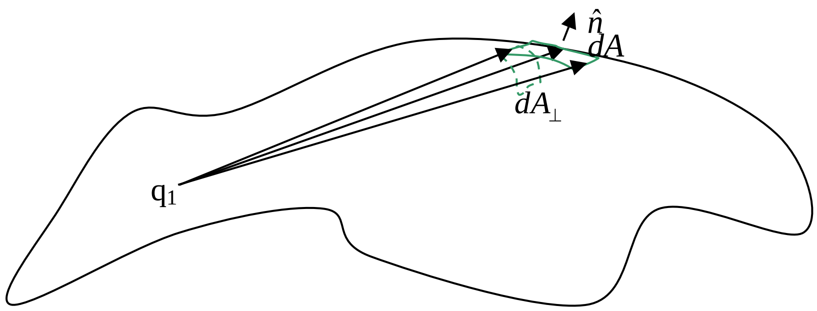
$$\Phi_E = \Phi_{E1} + \Phi_{E2} + \Phi_{E3} + \ldots \Phi_{EN}$$

Looking just at Charge 1 (which is *inside* the area). So, we can divide and conquer – find the flux due to one point charge, and then put it back together. Of course, the field at our observation location due to a point charge is simply $\vec{E}_1 = \frac{1}{4\pi\varepsilon_o}\frac{q_1}{r_{0-1}^2}\hat{r}_{0-1}$.

So,

$$\Phi_{E1} = \oint \vec{E}_1 \bullet d\vec{A} = \oint \frac{1}{4\pi\varepsilon_o}\frac{q_1}{r_{0-1}^2}\hat{r}_{0-1} \bullet d\vec{A} = \frac{q_1}{4\pi\varepsilon_o}\oint \frac{1}{r_{0-1}^2}\hat{r}_{0-1} \bullet d\vec{A} = \frac{q_1}{4\pi\varepsilon_o}\oint \frac{1}{r_{0-1}^2}\hat{r}_{0-1} dA_\perp$$

Now, let's consider that dot-product. It simply calls for the projection of the patch of area perpendicular to r-hat. So, even if we're dealing with a surface like



we still only have to worry about the perpendicular projection of the patch of area. Let's parameterize the patch in terms of variables we can hope to integrate. Given the spherical symmetry of the point charge's electric field, a good choice of coordinate systems is spherical. In that system, a differential area is

$$dA_\perp = rd\phi r\sin\theta d\theta = r^2 d\phi \sin\theta d\theta.$$

Plugging that into our flux equation, we have

- **Flux through patch.**
  - Now, across just this little patch of surface area, the flux is then just

    $$\Delta\Phi_{E1} = \frac{q_1}{4\pi\varepsilon_o}\frac{1}{r_{0-1}^2}r_{0-1}^{\ 2}\Delta\phi\left(\sin\theta\Delta\theta\right) = \frac{q_1}{4\pi\varepsilon_o}\Delta\phi\left(\sin\theta\Delta\theta\right)$$

Depends only on Angles. Remarkably, it only depends on the solid angle the area subtends.

If "the solid angel of the area subtends" is a bit foreign to you, think of it this way. Say you're a charge, radiating in all directions, and this room, it's walls, ceiling, and floor define the surface area. How much of your radiation passes through a given surface depends on how much of your view it takes up. Looking at a tile above you, it measures about 10° (or π/18 radians) by about 10° (or π/18 radians) so its solid angle is roughly the product, $\pi^2/324$ rad^2, that's what determines how much of your radiation flows through it.

Flux through whole surface. Going ahead and integrating over the whole surface area then gives

$$\Phi_{E1} = \frac{q_1}{4\pi\varepsilon_o}\oint d\phi \sin\theta d\theta = \frac{q_1}{4\pi\varepsilon_o}\int_{\phi=-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{\theta=0}^{2\pi} d\phi \sin\theta d\theta = \frac{q_1}{4\pi\varepsilon_o}4\pi = \frac{q_1}{\varepsilon_o}$$

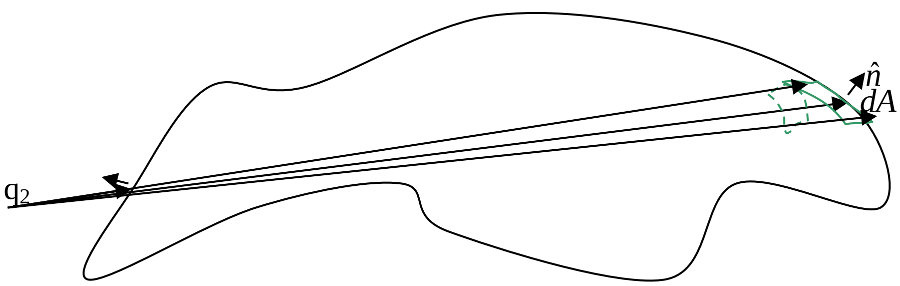
And there's a *very* simple result!

Q: What about charge on the outside?

Charge *outside* the area.

Now, before we go back to the total flux for a big charge distribution, we need to consider the other case – here we've considered a point *inside* the surface, what about one *outside* the surface?

Well, notice that the dot-product of r and n was positive, and that, as long as the charge is inside the surface, it r and n will always both point outward, so they'll always be positive.



If the charge is *outside,* on the near surface n and r will point in opposite directions, so the dot product will be negative while on the far side they'll point in the same direction, so they'll be positive. All that matters in the math is the angle subtended; perhaps you'll buy that, as you sweep through angles, for every patch through which the field flows *out* there's a canceling patch through which the field flows *in.* So summing over the whole surface, the flux comes to zero. (This is a very sketchy argument, the book is a little more thorough.)

The analogous situation would be if you had a shower head spraying water out in all directions (instead of a charge) and a loose-mesh bag (for the surface), then whatever water flows in from the left, flows out through the right, making for no net flux "into" or "out of" the bag.

So any charge *enclosed* by the surface contributes to the flux to the tune of $\frac{q_{enclosed}}{\varepsilon_o}$ while all excluded charges contribute nothing.

Thus we have

$$\Phi_E \equiv \oint \vec{E} \bullet d\vec{A} = \frac{q_1}{\varepsilon_o} + \frac{q_3}{\varepsilon_o} + \ldots \frac{q_N}{\varepsilon_o}$$

$$\boxed{\Phi_E = \oint \vec{E} \bullet d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_o}}$$

regardless of the shape of the surface.

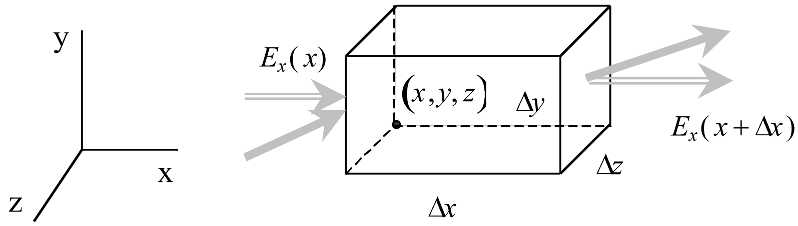
This is the integral form of **Gauss's Law.**

We also derived the differential form of this.

See PowerPoint

(1) Gauss's law

Consider a small box with edges along the coordinate axes.



Calculate the electric flux per volume in the limit that the volume goes to zero, which is the divergence of $\vec{E}$:

Divergence

- **Motivation. Return to Rain**
  - Recall that the general idea of a "flux" is a flow rate: the charge flux down a wire, dq/dt, is the current. Similarly, in the example of rain that we used to motivate the definition of flux, the rate at which water enters a room through some open windows would be a flux. $\Phi_w = \frac{dm_w}{dt} = \oint \rho_w \vec{v}_w \cdot d\vec{A}$
  - **Normalizing per Volume.** Now, if I told you that 1 kg of water rained in per minute, you'd be pretty worried – until I told you that the room was the Superdome- that volume's *huge.* 1 kg / minute leak isn't so bad as if we were talking about, say , *this* room. This example illustrates that *flux* alone doesn't tell the whole story. Sometimes you're more interested in flux *per volume.* On a per volume basis, the same flux into the Superdome is *nothing* compared to that into

this room. Flux out per volume is "Divergence." (Conversely, I suppose we'd call Flux *in* per volume "Convergence"=–Divergence)

$$div \equiv \frac{\Phi_w}{Vol} = \frac{dm_w}{dt}\Big/ Vol = \frac{\oint \rho_w \vec{v}_w \cdot d\vec{A}}{Vol}$$

Math.

Now for a little math.

$$\text{div}\left(\vec{E}\right) = \lim_{\Delta V \to 0} \frac{\oint \vec{E}\cdot\hat{n}\, dA}{\Delta V}$$

$$= \lim_{\Delta V \to 0}\left\{\frac{\left[E_x(x+\Delta x) - E_x(x)\right]\Delta y \Delta z}{\Delta x \Delta y \Delta z} + \frac{\left[E_y(y+\Delta y) - E_y(y)\right]\Delta x \Delta z}{\Delta x \Delta y \Delta z} + \frac{\left[E_z(z+\Delta z) - E_z(z)\right]\Delta x \Delta y}{\Delta x \Delta y \Delta z}\right\}$$

$$= \lim_{\Delta x \to 0}\frac{\left[E_x(x+\Delta x) - E_x(x)\right]}{\Delta x} + \lim_{\Delta y \to 0}\frac{\left[E_y(y+\Delta y) - E_y(y)\right]}{\Delta y} + \lim_{\Delta z \to 0}\frac{\left[E_z(z+\Delta z) - E_z(z)\right]}{\Delta z}$$

$$= \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

The other side of Gauss's law over the volume in the limit that the volume goes to zero is:

$$\lim_{\Delta V \to 0}\left(\frac{1}{\varepsilon_0}\frac{\sum q_{\text{inside}}}{\Delta V}\right) = \frac{\rho}{\varepsilon_0},$$

where ρ is the charge density. The differential form of Gauss's law is:

$$\text{div}\left(\vec{E}\right) = \vec{\nabla}\cdot\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\varepsilon_0}$$

Note that this is a scalar equation. In the second form, the "del" operator is $\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$.

<u>**Examples/Exercises**</u>:

Problem (from answer of 2.16)

Suppose the electric field (in cylindrical coordinates) is

$$\vec{E} = \begin{cases} Cs\hat{s} & s < a \\ \left(Ca^2/s\right)\hat{s} & s > a \end{cases}$$

What is the charge density in each region?

The charge density is $\rho = \varepsilon_0\left(\vec{\nabla}\cdot\vec{E}\right)$. Since the electric field only has an *s* (radial) component, the divergence (from the front cover) is

$$\rho = \varepsilon_0 \left(\vec{\nabla} \cdot \vec{E}\right) = \begin{cases} \varepsilon_0 \frac{1}{s} \frac{\partial}{\partial s}\left[s\left(Cs\right)\right] = \varepsilon_0 \frac{1}{s}\left(2Cs\right) = 2\varepsilon_0 C & s < a \\ \varepsilon_0 \frac{1}{s} \frac{\partial}{\partial s}\left[s\left(Ca^2/s\right)\right] = 0 & s > a \end{cases}$$

Computational Tutorial #2 – Numerical Integration (have students go through it)

<u>Preview</u>

The first attempt at HW #1 is due tomorrow at 3 pm. You must make a first attempt in order to get any credit for a problem!

For Monday, you'll read about applying Gauss's Law.