Mon., 11/18 Wed.,11/20 Fri., 11/22	**10.3** Point Charges *(C 14)* **4.1** Polarization *(C 14)* **4.2** Field of Polarized Object	HW9
Mon., 11/25	*(C14)* **4.3** Electric Displacement	

Last Time.

We finished convincing ourselves that

$$V_L(\vec{r},t) = \frac{1}{4\pi\varepsilon_o}\int\frac{\rho(\vec{r}',t_r)}{\mathscr{r}}d\tau' \qquad \vec{A}_L(\vec{r},t) = \frac{\mu_o}{4\pi}\int\frac{\vec{J}(\vec{r}',t_r)}{\mathscr{r}}d\tau'$$

And then proceeded to find that

$$\vec{E} = \frac{1}{4\pi\varepsilon_o}\int\left(\frac{\dot{\rho}(\vec{r}',t_r)\hat{\mathscr{r}}}{c\mathscr{r}} + \frac{\rho(\vec{r}',t_r)\hat{\mathscr{r}}}{\mathscr{r}^2} - \frac{\dot{\vec{J}}(\vec{r}',t_r)}{c^2\mathscr{r}}\right)d\tau'$$

And

$$\vec{B}(\vec{r},t) = \frac{\mu_o}{4\pi}\int\left(\frac{\dot{\vec{J}}(\vec{r}',t_r)}{c\mathscr{r}} + \frac{\vec{J}(\vec{r}',t_r)}{\mathscr{r}^2}\right)\times\hat{\mathscr{r}}d\tau'$$

Thus the general expression for a force exerted on a charged particle is

$$\vec{F}(\vec{r},t) = q\vec{E} + q\vec{v}\times\vec{B}$$

$$\vec{F}(\vec{r},t) = \frac{1}{4\pi\varepsilon_o}\int q\left(\frac{\dot{\rho}(\vec{r}',t_r)\hat{\mathscr{r}}}{c\mathscr{r}} + \frac{\rho(\vec{r}',t_r)\hat{\mathscr{r}}}{\mathscr{r}^2} - \frac{\dot{\vec{J}}(\vec{r}',t_r)}{c^2\mathscr{r}} + \vec{v}(\vec{r},t)\times\left(\left(\frac{\dot{\vec{J}}(\vec{r}',t_r)}{c^3\mathscr{r}} + \frac{\vec{J}(\vec{r}',t_r)}{c^2\mathscr{r}^2}\right)\times\hat{\mathscr{r}}\right)\right)d\tau'$$

Finally, we observed that the special case version of Coulomb's Law holds not just for static charge densities, but for constant currents since that implies charge densities that vary only linearly. Similarly, the Biot-Savart Law is approximately true if the current density varies slowly enough.

This Time: Point Charge Source.

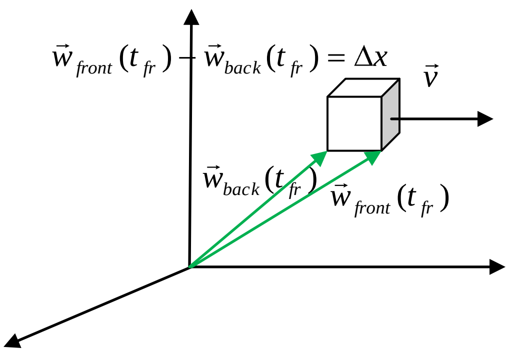
Now that we've developed our potential and field expressions for a distribution of charges and currents, it's time to narrow in and develop our expression for the fields and potentials of a point charge. Clearly, given the choice between the simple potential expressions and the rather complex field expressions, it's more appealing to convert the latter, and then use our new point-charge potentials to derive the corresponding fields.

$$V_L(\vec{r},t) = \frac{1}{4\pi\varepsilon_o}\int\frac{\rho(\vec{r}',t_r)}{\mathscr{r}}d\tau'$$

Griffith's argues that we need to be very careful about how we take the limit of this charge density down to a point. So, we'll start by considering a charged object with little physical extent, a uniformly charged cube would be convenient.

$$\rho(\vec{w},t) = \frac{q}{\Delta x \Delta y \Delta z}$$

Picture that cube instantaneously centered on location $\vec{w}$ and moving with velocity $\vec{v}$ in the x-direction.



$\vec{r}$

Now, one way to read the integral

$$V_L(\vec{r},t) = \frac{1}{4\pi\varepsilon_o}\int \frac{\rho(\vec{r}\,',t_r)}{\mathscr{r}}\, d\tau'$$

is to say we are to look at each cell of space and ask what the charge density was there *at its appropriate retarded time*. Then we multiply that cell's volume by our given density over the appropriate separation vector, and sum over all '*then* occupied' cells. Now, for any cell *that had charge in it*, the density was simply

$$\rho(\vec{w},t_r) = \frac{q}{\Delta x \Delta y \Delta z}$$

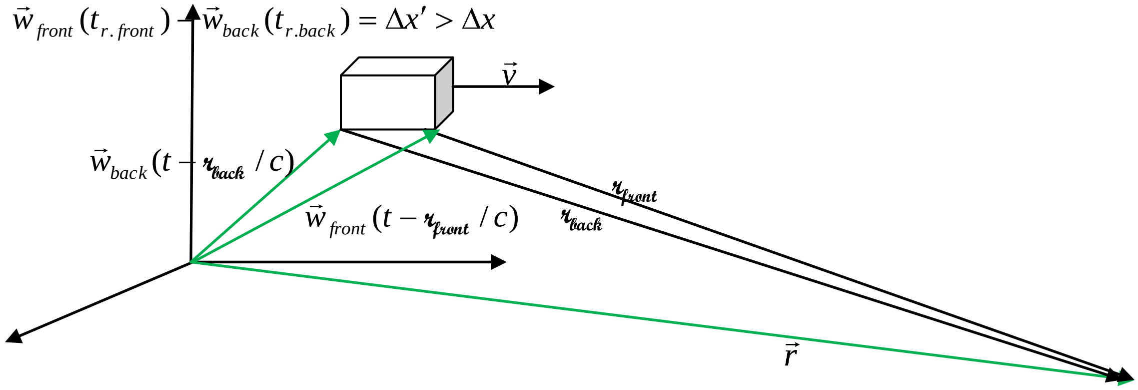
So, our task is essentially to figure out the volume of cells that, at their appropriate retardation times, had this charge density in them,

$$\Delta\tau'$$

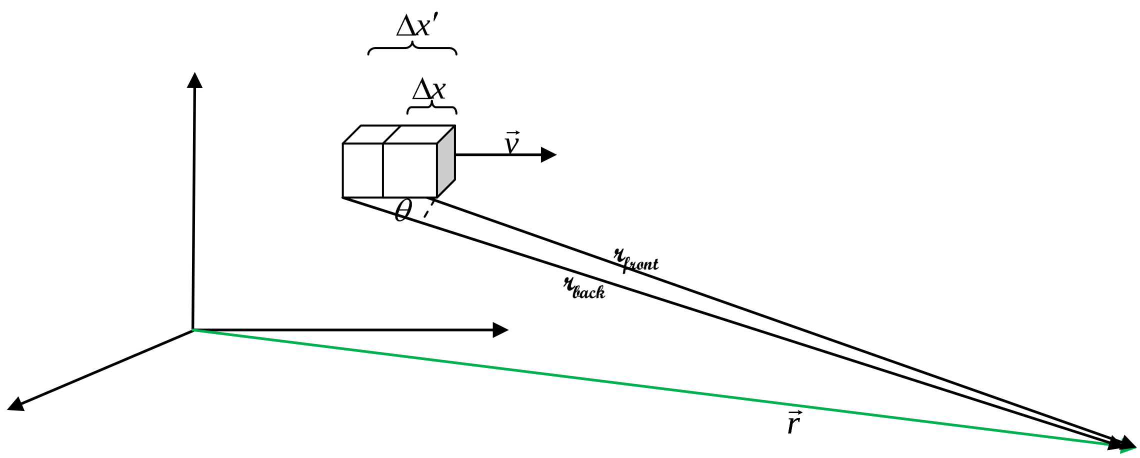
Then the integral simply reduces to

$$V_L(\vec{r},t) = \frac{1}{4\pi\varepsilon_o}\int \frac{\rho(\vec{r}\,',t_r)}{\mathscr{r}}\, d\tau' \Rightarrow \frac{1}{4\pi\varepsilon_o}\left(\frac{q}{\Delta x \Delta y \Delta z}\right)\frac{1}{\mathscr{r}}\Delta\tau'$$

Well, as the cube approaches the observation location, cells just behind the cube (relative to its motion) will have older retarded times than will the cells near the front, and back *then* some of them still had the cube in it! The cube appears stretched along the direction of its motion.



So, we get 'extra' contributions to the integral – we don't just get contributions from a space Δx wide, but from a space $\Delta x'$ wide. Let's try to figure out how Δx and $\Delta x'$ are related.



Now, the time it takes for the cell to move forward distance

$$\Delta x' - \Delta x = v\Delta t$$

is the extra time that the signal from the back must have been traveling before the signal from the front that it will then travel alongside to the observer

$$(\Delta x')\cos\theta = c\Delta t$$

So, that tells us that

$$\frac{(\Delta x')\cos\theta}{c} = \Delta t = \frac{\Delta x' - \Delta x}{v}$$

$$\Delta x' = \frac{\Delta x}{1 - \frac{v\cos\theta}{c}} = \frac{\Delta x}{1 - \frac{\vec{v}\cdot\hat{\mathscr{r}}}{c}}$$

So,

The volume of points that, at their own retardation times, had the charge in them is

$$\Delta\tau' = \Delta x'\Delta y\Delta z = \frac{\Delta x\Delta y\Delta z}{1-\dfrac{\vec{v}\cdot\hat{\mathscr{r}}}{c}}$$

Thus

$$V_L(\vec{r},t) = \frac{1}{4\pi\varepsilon_o}\left(\frac{q}{\Delta x\Delta y\Delta z}\right)\frac{1}{\mathscr{r}}\Delta\tau' = \frac{1}{4\pi\varepsilon_o}\left(\frac{q}{\Delta x\Delta y\Delta z}\right)\frac{1}{\mathscr{r}}\frac{\Delta x\Delta y\Delta z}{1-\dfrac{\vec{v}\cdot\hat{\mathscr{r}}}{c}} = \frac{1}{4\pi\varepsilon_o}\frac{1}{\mathscr{r}}\frac{q}{1-\dfrac{\vec{v}\cdot\hat{\mathscr{r}}}{c}}$$

$$V_L(\vec{r},t) = \frac{1}{4\pi\varepsilon_o}\frac{qc}{\mathscr{r}c-\vec{v}\cdot\vec{\mathscr{r}}}$$

Mind you the details of this argument only work for a differentially small charge source so that we can say one r points to the whole thing and that, even if the charge is accelerating, that lag time between t_{front} and t_{back} is negligibly small so we can treat it as having a constant velocity over the time of our analysis.

One subtle point that may not be evident in this notation:

Before you did the integral, $\mathscr{r}$ was the vector that you used to, one after another, point from the observation location to every point in space: $\vec{\mathscr{r}} = \vec{r}-\vec{r}'$, where $\vec{r}'$ *was a variable of integration*.

Now we have *done* then integral, and we're only pointing at the retarded location of the charge: *now,*

Inside integral   *Outside integral*

$$\vec{\mathscr{r}} = \vec{r}-\vec{r}' \Rightarrow \vec{r}-\vec{w}(t_r)$$

(perhaps he chooses to change variables since primes are often used in relativity to differentiate between frame's perspectives on a measurement)

As for the Vector potential,

$$\vec{A}_L(\vec{r},t) = \frac{\mu_o}{4\pi}\int\frac{\vec{J}(\vec{r}',t_r)}{\mathscr{r}}d\tau'$$

Recall that the current density at a given point is

$$\vec{J}(\vec{r},t) = \rho(\vec{r},t)\vec{v}(\vec{r},t)$$

where v is the velocity of the net charge there; so the vector potential expression could be rephrased as

$$\vec{A}_L(\vec{r},t)=\frac{\mu_o}{4\pi}\int\frac{\rho(\vec{r}\,',t_r)\vec{v}(\vec{r}\,',t_r)}{\mathscr{r}}d\tau'$$

And we make the same argument about sampling space and asking what *it* registers as the density. This leads to the same place in the end.

$$\vec{A}_L(\vec{r},t)=\frac{\mu_o}{4\pi}\frac{qc\vec{v}}{\mathscr{r}c-\vec{v}\cdot\vec{\mathscr{r}}}=\frac{\vec{v}}{c^2}V_L(\vec{r},t)$$

Again,

Inside integral *Outside integral*

$$\vec{\mathscr{r}}=\vec{r}-\vec{r}\,'\Rightarrow\vec{r}-\vec{w}(t_r)$$

We could pause and have some fun with these potentials, but we would like to also see what the *fields* are of a point charge. Before we proceed, I'll just point out that, the potentials for an *arbitrarily moving* point charge only depend upon its velocity (not its acceleration). Now we'll take the appropriate derivatives to find the fields and see that these *do* depend upon acceleration.

$$\vec{E}=-\vec{\nabla}V-\frac{\partial}{\partial t}\vec{A}\ \text{ and }\ \vec{B}=\vec{\nabla}\times\vec{A}$$

So,

$$\vec{\nabla}V_L=V_L(\vec{r},t)=\frac{qc}{4\pi\varepsilon_o}\vec{\nabla}\frac{1}{\mathscr{r}c-\vec{v}\cdot\vec{\mathscr{r}}}=\frac{qc}{4\pi\varepsilon_o}\frac{-1}{(\mathscr{r}c-\vec{v}\cdot\vec{\mathscr{r}})^2}\vec{\nabla}(\mathscr{r}c-\vec{v}\cdot\vec{\mathscr{r}})$$

$$\vec{\nabla}(\mathscr{r}c-\vec{v}\cdot\vec{\mathscr{r}})=\vec{\nabla}(\mathscr{r}c)-\vec{\nabla}(\vec{v}\cdot\vec{\mathscr{r}})=c\vec{\nabla}\mathscr{r}-\vec{\nabla}(\vec{v}\cdot\vec{\mathscr{r}})$$

Now, product rule #4 *isn't* pretty.

$$\vec{\nabla}(\vec{v}\cdot\vec{\mathscr{r}})=\vec{v}\times(\vec{\nabla}\times\vec{\mathscr{r}})+\vec{\mathscr{r}}\times(\vec{\nabla}\times\vec{v})+(\vec{\mathscr{r}}\cdot\vec{\nabla})\vec{v}+(\vec{v}\cdot\vec{\nabla})\vec{\mathscr{r}}$$

In fact, it's around now that Griffith's says "the next two pages are rough going." The work that follows is mostly straight-forward but tedious, with some tricky stuff mixed in. Let's focus on that. One has to do with the fact that $\mathscr{r}$ isn't what it used to be: $\vec{\mathscr{r}}=\vec{r}-\vec{r}\,'\Rightarrow\vec{r}-\vec{w}(t_r)$

So it's curl *isn't* necessarily 0.

$$\vec{\nabla}\times\vec{\mathscr{r}}=\vec{\nabla}\times(\vec{r}-\vec{w}(t_r))=\vec{\nabla}\times\vec{r}-\vec{\nabla}\times\vec{w}(t_r)=0-\frac{\partial\vec{w}(t_r)}{\partial t_r}\vec{\nabla}\times\vec{w}(t_r)$$

Recall how it played out when we took the curl of the retarded current density last time; something similar happens here:

$$(\vec{\nabla}\times\vec{w}(t_r))_x=\frac{\partial w_z(t_r)}{\partial y}-\frac{\partial w_y(t_r)}{\partial z}=\frac{\partial w_z}{\partial t_r}\frac{\partial t_r}{\partial y}-\frac{\partial w_y}{\partial t_r}\frac{\partial t_r}{\partial z}=v_z\frac{\partial t_r}{\partial y}-v_y\frac{\partial t_r}{\partial z}$$

$$\vec{\nabla}\times\vec{w}(t_r)=\left(v_z\frac{\partial t_r}{\partial y}-v_y\frac{\partial t_r}{\partial z}\right)\hat{x}+\left(v_x\frac{\partial t_r}{\partial z}-v_z\frac{\partial t_x}{\partial z}\right)\hat{y}+\left(v_y\frac{\partial t_r}{\partial x}-v_x\frac{\partial t_r}{\partial y}\right)\hat{z}=\nabla t_r\times\vec{v}=-\vec{v}\times\nabla t_r$$

Another subtle point is that we employ some fancy footwork to get at just what ∇t_r is

$$\nabla t_r = \nabla\left(t - \frac{|\mathscr{r}|}{c}\right) = -\frac{1}{c}\nabla|\mathscr{r}| = -\frac{1}{c}\nabla\sqrt{\mathscr{r}^2} = -\frac{1}{c}\nabla\sqrt{\vec{\mathscr{r}}\cdot\vec{\mathscr{r}}} = -\frac{1}{c}\frac{1}{2}\frac{1}{\sqrt{\vec{\mathscr{r}}\cdot\vec{\mathscr{r}}}}\nabla(\vec{\mathscr{r}}\cdot\vec{\mathscr{r}}) =$$

By product rule number 4, this gradient can be expanded out; fortunately, since the two vectors are the same, there's some redundancy in the terms, and it can be collapsed down a little:

$$\nabla(\vec{\mathscr{r}}\cdot\vec{\mathscr{r}}) = 2\left[\vec{\mathscr{r}}\times(\nabla\times\vec{\mathscr{r}}) + (\vec{\mathscr{r}}\cdot\nabla)\vec{\mathscr{r}}\right]$$

Where

$$\vec{\mathscr{r}}\times(\nabla\times\vec{\mathscr{r}}) = \vec{\mathscr{r}}\times\left(\frac{\partial\vec{w}}{\partial t_r}\times\nabla t_r\right) = \vec{\mathscr{r}}\times(\vec{v}\times\nabla t_r) = \vec{v}(\vec{\mathscr{r}}\cdot\nabla t_r) - \nabla t_r(\vec{\mathscr{r}}\cdot\vec{v})$$

Using the triple cross-product rule, #2,

and

$$(\vec{\mathscr{r}}\cdot\nabla)\vec{\mathscr{r}} = (\vec{\mathscr{r}}\cdot\nabla)(\vec{r} - \vec{w}(t_r)) = (\vec{\mathscr{r}}\cdot\nabla)\vec{r} - (\vec{\mathscr{r}}\cdot\nabla)\vec{w}(t_r)$$

$$\left(\mathscr{r}_x\frac{\partial}{\partial x} + \mathscr{r}_y\frac{\partial}{\partial y} + \mathscr{r}_z\frac{\partial}{\partial z}\right)\vec{r} - \left(\mathscr{r}_x\frac{\partial}{\partial x} + \mathscr{r}_y\frac{\partial}{\partial y} + \mathscr{r}_z\frac{\partial}{\partial z}\right)\vec{w}(t_r)$$

$$(\mathscr{r}_x\hat{x} + \mathscr{r}_y\hat{y} + \mathscr{r}_z\hat{z}) - \left(\mathscr{r}_x\frac{\partial t_r}{\partial x}\frac{\partial\vec{w}}{\partial t_r} + \mathscr{r}_y\frac{\partial t_r}{\partial y}\frac{\partial\vec{w}}{\partial t_r} + \mathscr{r}_z\frac{\partial t_r}{\partial z}\frac{\partial\vec{w}}{\partial t_r}\right)$$

$$\vec{\mathscr{r}} - \vec{v}\left(\mathscr{r}_x\frac{\partial t_r}{\partial x} + \mathscr{r}_y\frac{\partial t_r}{\partial y} + \mathscr{r}_z\frac{\partial t_r}{\partial z}\right)$$

$$\vec{\mathscr{r}} - \vec{v}(\vec{\mathscr{r}}\cdot\nabla t_r)$$

So,

$$\nabla t_r = -\frac{1}{c}\frac{1}{|\mathscr{r}|}\left[\vec{\mathscr{r}} - \vec{v}(\vec{\mathscr{r}}\cdot\nabla t_r) + \vec{v}(\vec{\mathscr{r}}\cdot\nabla t_r) - \nabla t_r(\vec{\mathscr{r}}\cdot\vec{v})\right] = -\frac{1}{c}\frac{1}{|\mathscr{r}|}\left[\vec{\mathscr{r}} - \nabla t_r(\vec{\mathscr{r}}\cdot\vec{v})\right]$$

$$\nabla t_r = \frac{-\vec{\mathscr{r}}}{|\mathscr{r}|c - \vec{\mathscr{r}}\cdot\vec{v}}$$

Whew!

The end result is to get the gradient of V; similarly, time derivative of A is obtained, and

$$\vec{E}(r,t) = \frac{q}{4\pi\varepsilon_o}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\left[(c^2 - v^2)\vec{u} + \vec{\mathscr{r}}\times(\vec{u}\times\vec{a})\right]$$

Where

$\vec{u} \equiv c\hat{\mathscr{r}} - \vec{v}$

Similarly,

Applying $\nabla \times \vec{A} = \vec{B}$ where $\vec{A}_L(\vec{r},t) = \frac{\vec{v}}{c^2} V_L(\vec{r},t)$ gives

$$\vec{B} = \nabla \times \left(\frac{\vec{v}}{c^2} V_L(\vec{r},t) \right) = \frac{1}{c^2} \left[V_L(\vec{r},t)(\nabla \times \vec{v}) - \vec{v} \times \nabla V_L(\vec{r},t) \right]$$

$$\vec{B} = \frac{1}{c} \frac{q}{4\pi\varepsilon_o} \frac{\mathscr{r}}{(\vec{\mathscr{r}} \cdot \vec{u})^3} \vec{\mathscr{r}} \times \left[(c^2 - v^2)\vec{u} + \vec{\mathscr{r}} \times (\vec{u} \times \vec{a}) \right]$$

$$\vec{B} = \frac{1}{c} \hat{\mathscr{r}} \times \vec{E}$$

These are the *most* general expressions for the fields generated by a point charge.

Putting them together in the force law, we get

$$\vec{E}(r,t) = \frac{q}{4\pi\varepsilon_o} \frac{\mathscr{r}}{(\vec{\mathscr{r}} \cdot \vec{u})^3} \left[(c^2 - v^2)\vec{u} + \vec{\mathscr{r}} \times (\vec{u} \times \vec{a}) \right]$$

We just went to great pains to elucidate some of the steps in taking the gradient of V, along the way we did happen tot take the curl of v. Quoting and assembling that, we get

"I had a few questions about specific steps in these long derivations: 1. I believe I'm missing the point entirely, but I don't see how "it follows that" we get 10.46 from 10.44, 10.45, and the argument in the previous paragraph."

2. How is he solving the quadratic formula for t_r to get equation 10.48?

3. How he got equation 10.67 (I understood the equation immediately before it, just not the jump he made).

4. How he got the equation immediately after 10.67 at the top of pg 459.

Casey McGrath

For 2: He takes the equation at the top of pg 455 and solves it into this form ( )*tr^2 + ( )tr + ( )

In each of those parentheses go the functions of r, t, and v, and they become the a, b, and c for the quadratic formula. For example, a = v^2 - c^2, if you multiply out the right side of the equation and subtract over the tr^2 term. Freeman

"Can we maybe just touch on how example 10.4 would change with an accelerating charge?" Ben Kid

"Is there really any conceptual difference behind the motivations of V(r,t) and A(r,t) vs. E(r,t) and B(r,t)? I understnad mathematically we just plug in for the retarded time but I'm just asking are they all accounting for the same thing." Rachael Hach

"Can we go over some examples using the new material? I'm confused as to how to start these problems, what information to look for, etc." Sam

"The derivations use little corrections to make the formulas more accurate to the real world, but these corrections make deriving the formulas that follow rather complex. How much do these corrections change our results? Why is it worth the struggle?" Anton

"I think the math/derivations make this chapter hard to understand, especially conceptually. Can we go over the process of doing these problems, specifically where to start and how to use the equations derived in the reading." Sam

"In EX 10.3. why does griffiths square eqn 10.44 to compute the retarded time in eqn 10.45?" Jessica

I think this is to get rid of the absolute value, so then he can use the quadratic formula to find tr. Freeman

"Can we go over the proof of equation 10.42? I think the fact that is becomes only geometric is worth showing." Casey P

"Can we go over the proof of equation 10.42? I think the fact that is becomes only geometric is worth showing." Casey P

"Griffith says that only one retarded point contributes to the potentials at any given moment . I don't understand how it would be different if the particles traveled at the speed of light." Antwain

"Can we go over one of the examples in more detail? It would be helpful to do one of these problems step by step." Spencer