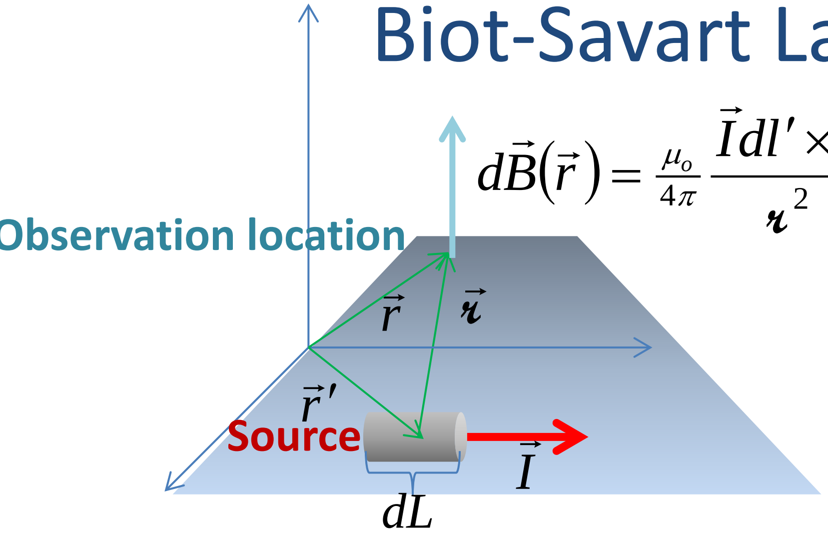
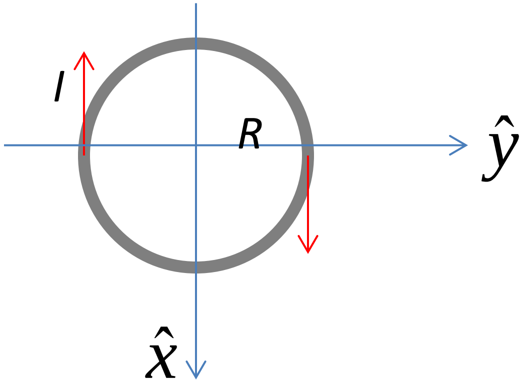
Mon. Tues. Wed. Fri.	(C 17) 5.2 Biot-Savart Law T5 Quiver Plots (C 21.6-7,.9) 1.3.4-1.3.5, 1.5.2-1.5.3, 5.3.1-.3.2 Div & Curl B (C 21.6-7,.9) 5.3.3-.3.4 Applications of Ampere's Law	 HW7
Mon. Wed. Thurs. Fri.	1.6, 5.4.1-.4.2 Magnetic Vector Potential 5.4.3 Multipole Expansion of the Vector Potential Review	 HW8

Biot-Savart Law

$$d\vec{B}(\vec{r}) = \frac{\mu_o}{4\pi}\frac{\vec{I}dl' \times \hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^2}$$



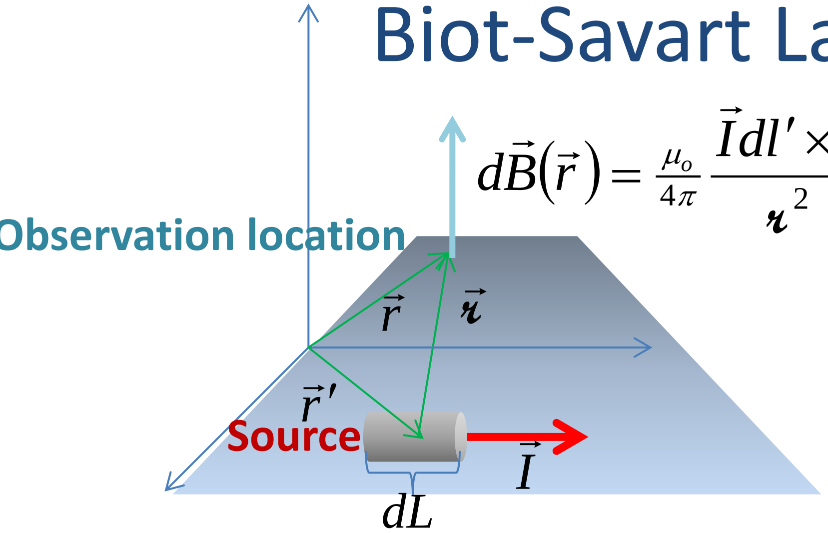
Exercise: field in center of clockwise circle of current, of radius R, lying in the x-y plane.



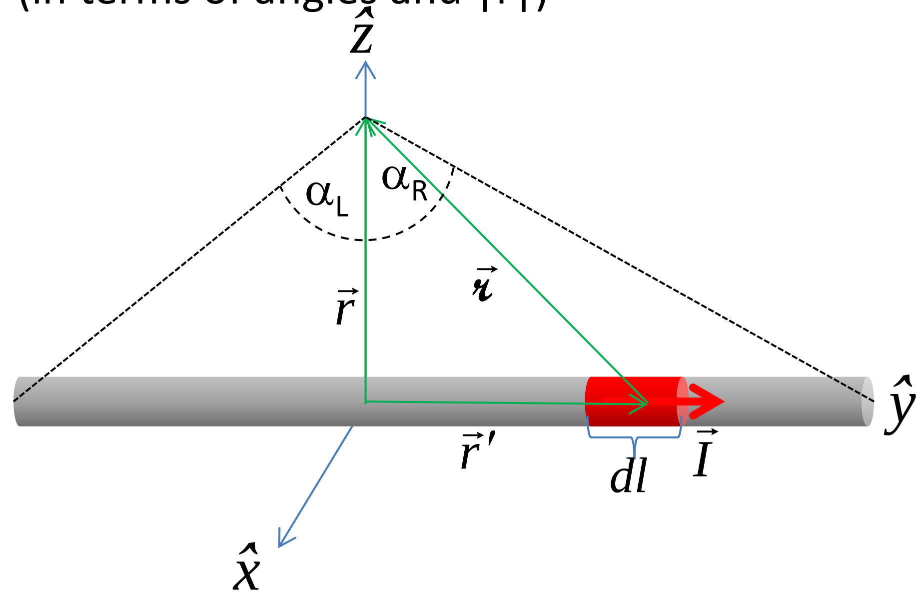
Exercise: what's the contribution of just a third of the circle?

Biot-Savart Law

$$d\vec{B}(\vec{r}) = \frac{\mu_o}{4\pi}\frac{\vec{I}\,dl' \times \hat{\boldsymbol{\imath}}}{\boldsymbol{\imath}^2}$$



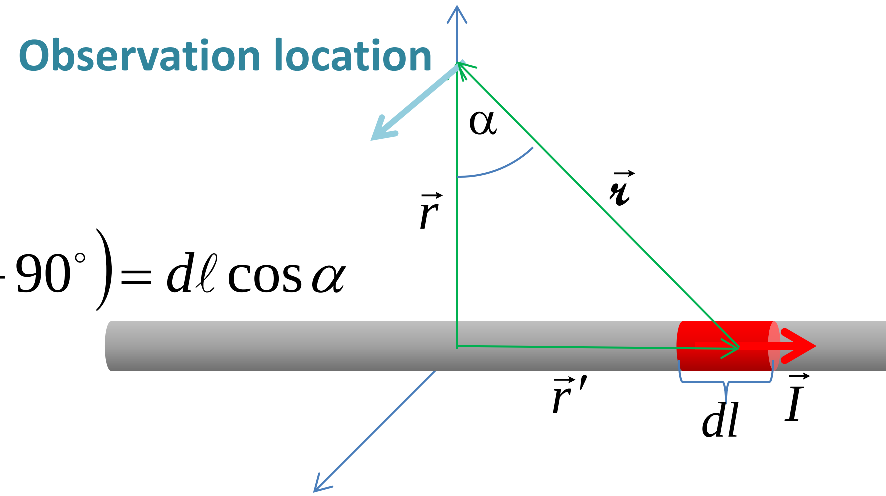
Example: field due to a line segment of current (in terms of angles and |r|)



Biot-Savart Law

Field of Finite Wire

(Book's approach)



$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\vec{\ell}' \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$d\vec{\ell}' \times \hat{\mathscr{r}} = \left| d\vec{\ell}' \times \hat{\mathscr{r}} \right| \hat{x}$$

$$\left| d\vec{\ell}' \times \hat{\mathscr{r}} \right| = |d\ell| |\hat{\mathscr{r}}| \sin\left(\alpha + 90^\circ\right) = d\ell \cos\alpha$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\ell \cos\alpha}{\mathscr{r}^2} \hat{x}$$

$$\frac{d\ell}{\mathscr{r}^2} = \frac{1}{r} \frac{r^2 d(\ell / r)}{\mathscr{r}^2} = \frac{1}{r} \cdot \left(\frac{r}{\mathscr{r}}\right)^2 \cdot d\left(\frac{\ell}{r}\right) = \frac{1}{r} \cdot \left(\frac{r}{\mathscr{r}}\right)^2 \cdot d\left(\frac{r'}{r}\right)$$

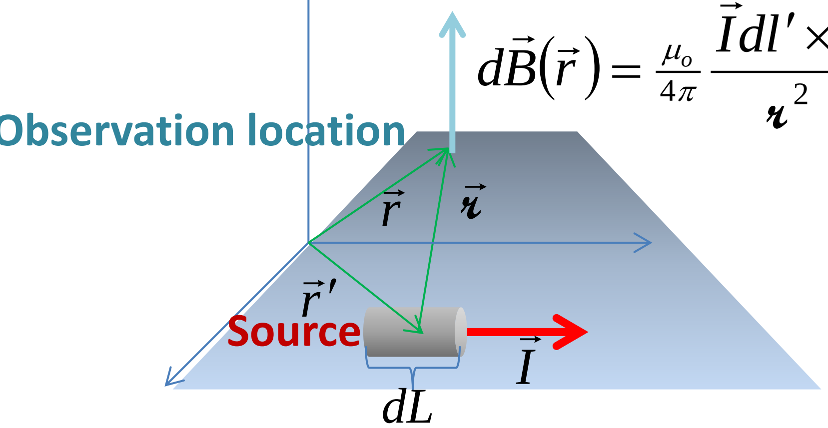
$$(\cos\alpha)^2 \qquad d(\tan\alpha)$$

$$d(\tan\alpha) = \frac{d\alpha}{\cos^2\alpha}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{1}{r} (\cos\alpha)^2 \frac{d\alpha}{(\cos\alpha)^2} \cos\alpha \hat{x} = \frac{\mu_0}{4\pi} I \int \frac{(\cos\alpha) d\alpha}{r} \hat{x} = \frac{\mu_0}{4\pi} \frac{I}{r} \int_{\alpha_R}^{\alpha_L} d(\sin\alpha) \hat{x}$$

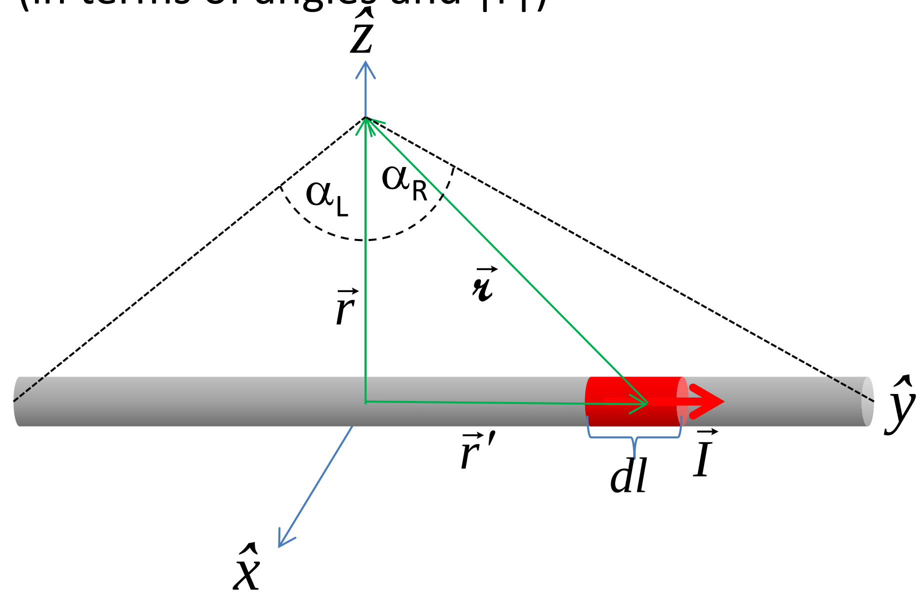
$$= \frac{\mu_0}{4\pi} \frac{2I}{r} \left(\sin(\alpha_L) - \sin(\alpha_R)\right) \hat{x}$$

Biot-Savart Law



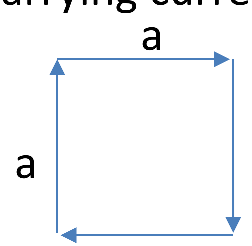
$$d\vec{B}(\vec{r}) = \frac{\mu_o}{4\pi}\frac{\vec{I}\,dl' \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

Example: field due to a line segment of current (in terms of angles and |r|)

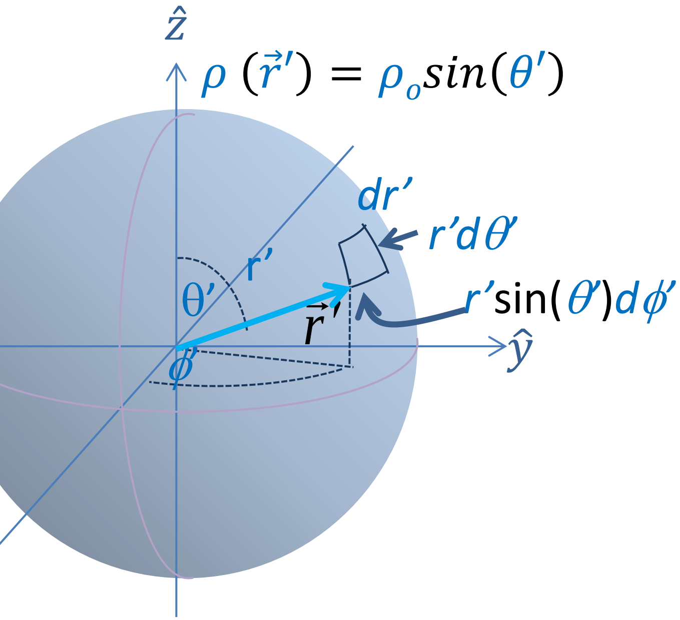


$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{2I}{r}\left(\sin(\alpha_L) - \sin(\alpha_R)\right)\hat{x}$$

Exercise: field at center of square, of side *a*, carrying current *I*



What's the magnetic field in the center of a sphere with charge density ρ rotating about z at ω.



$$d\vec{B}(\vec{r}) = \frac{\mu_o}{4\pi}\frac{\vec{I}dl' \times \hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\mu_o}{4\pi}\frac{\vec{I}dl' \times \vec{\mathscr{r}}}{\mathscr{r}^3}$$

so, we want to figure out exactly what $d\vec{I}dl$ **is, what** $\vec{\mathscr{r}}$ **is, and what their cross product is, then what** $\mathscr{r}$ **is.**

One way of approaching $d\vec{I}dl$ **is that it's really** $\vec{v}dq$**, since that's rotating at constant rate** $\vec{v} = \omega(r'\sin\theta)\hat{\phi}$ **and the bit of charge is** $dq = \rho d\tau'$**. In our case,**

$$dq = (\rho_o \sin\theta')(r'^2 dr' \sin\theta' d\theta' d\phi')$$

So putting together $d\vec{I}dl = \vec{v}dq = \vec{v}\rho d\tau'$ **gives us**

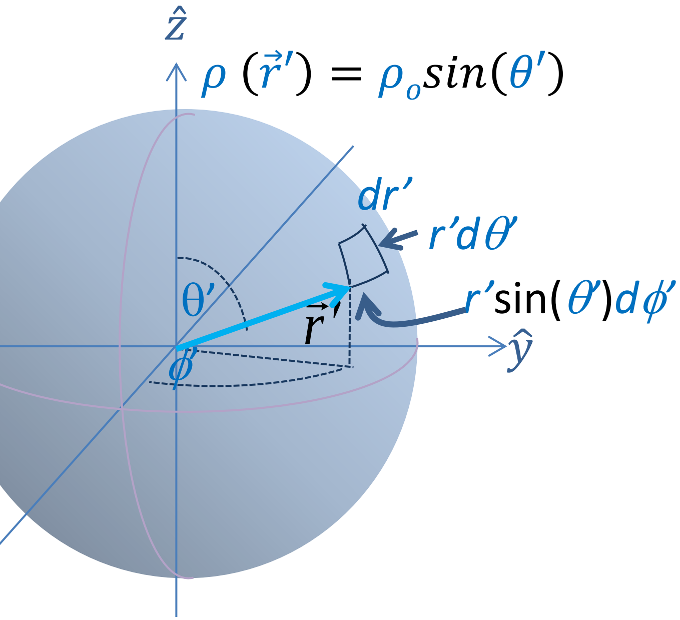
$$d\vec{I}dl = (\omega r' \sin\theta' \hat{\phi})(\rho_o \sin\theta')(r'^2 \sin\theta' d\theta' dr' d\phi')$$

$$\vec{\mathscr{r}} = \langle \cancel{x} - r'\cos\phi'\sin\theta', \cancel{y} - r'\sin\phi'\sin\theta', \cancel{z} - r'\cos\theta' \rangle$$

$$|\mathscr{r}| = r'$$

For the sake of doing the cross product, we'll want to say $\hat{\phi} = \langle -\sin\phi', \cos\phi', 0 \rangle$

What's the magnetic field in the center of a sphere with charge density ρ rotating about z at ω.



$$d\vec{B}(\vec{r}) = \frac{\mu_o}{4\pi}\frac{\vec{I}dl' \times \hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\mu_o}{4\pi}\frac{\vec{I}dl' \times \vec{\mathscr{r}}}{\mathscr{r}^3}$$

$$d\vec{I}dl = \rho\omega r'^3 \sin^3\theta' d\theta' dr' d\phi' \hat{\phi}$$

$$\vec{\mathscr{r}} = -r'\langle \cos\phi'\sin\theta', \sin\phi'\sin\theta', \cos\theta' \rangle$$

$$|\mathscr{r}| = r'$$

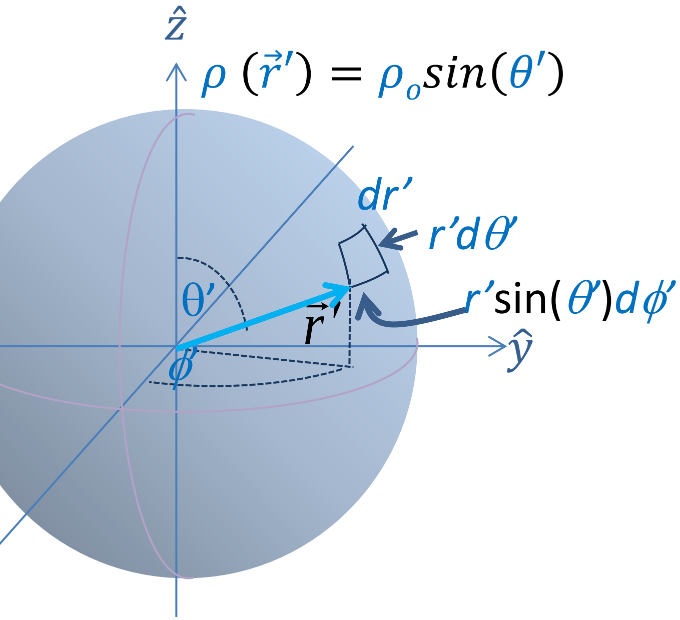
$$\hat{\phi} = \langle -\sin\phi', \cos\phi', 0 \rangle$$

$$\hat{\phi}\times\vec{\mathscr{r}} = -r'\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\sin\phi' & \cos\phi' & 0 \\ \cos\phi'\sin\theta' & \sin\phi'\sin\theta' & \cos\theta' \end{vmatrix}$$

$$\hat{\phi}\times\vec{\mathscr{r}} = -r'\left(\hat{x}\cos\phi'(\cos\theta') + \hat{y}\sin\phi'(\cos\theta') + \hat{z}\left(-\sin\phi'(\sin\phi'\sin\theta') - \cos\phi'(\cos\phi'\sin\theta')\right)\right)$$

$$\hat{\phi}\times\vec{\mathscr{r}} = -r'\left(\hat{x}\cos\phi'(\cos\theta') + \hat{y}\sin\phi'(\cos\theta') - \hat{z}(\sin\theta')\right)$$

What's the magnetic field in the center of a sphere with charge density ρ rotating about z at ω.



$$d\vec{B}(\vec{r}) = \frac{\mu_o}{4\pi}\frac{\vec{I}dl' \times \hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\mu_o}{4\pi}\frac{\vec{I}dl' \times \vec{\mathscr{r}}}{\mathscr{r}^3}$$

$$d\vec{I}dl = \rho_o \omega r'^3 \sin^3\theta' d\theta' dr' d\phi' \hat{\phi}$$

$$\vec{\mathscr{r}} = -r'\langle \cos\phi' \sin\theta', \sin\phi' \sin\theta', \cos\theta' \rangle$$

$$|\mathscr{r}| = r'$$

$$\hat{\phi} = \langle -\sin\phi', \cos\phi', 0 \rangle$$

$$\hat{\phi} \times \vec{\mathscr{r}} = -r'(\hat{x}\cos\phi'\cos\theta' + \hat{y}\sin\phi'\cos\theta' - \hat{z}\sin\theta')$$

Putting all this together,

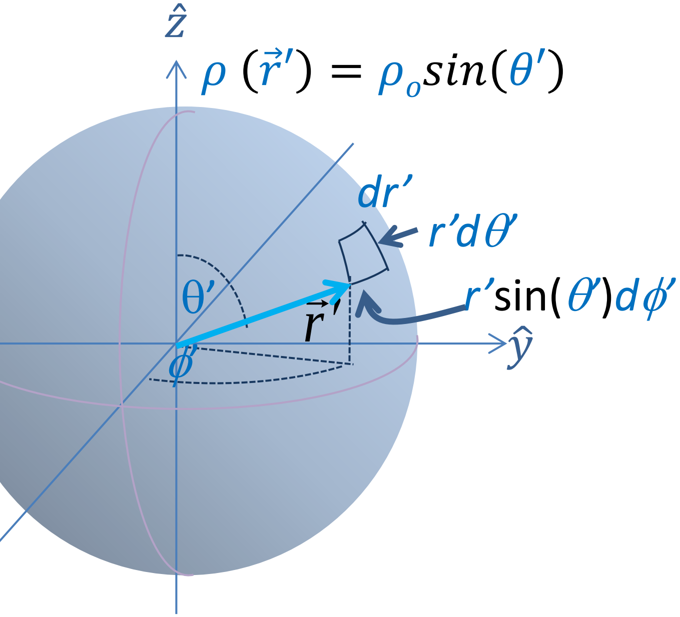
$$\vec{B}(0) = \frac{\mu_o}{4\pi}\int \frac{(\rho_o \omega r'^3 \sin^3\theta' d\theta' dr' d\phi')(-r')(\hat{x}\cos\phi'\cos\theta' + \hat{y}\sin\phi'\cos\theta' - \hat{z}\sin\theta')}{r'^3}$$

$$\vec{B}(0) = -\frac{\mu_o}{4\pi}\rho_o \omega \int (r'\sin^3\theta' d\theta' dr' d\phi')(\cos\phi'\cos\theta'\hat{x} + \sin\phi'\cos\theta'\hat{y} - \sin\theta'\hat{z})$$

Symmetry tells us there will only be a z-component in the end, so skipping the other two:

$$\vec{B}(0) = -\frac{\mu_o}{4\pi}\rho_o \omega \int (r'\sin^3\theta' d\theta' dr' d\phi')(-\sin\theta')\hat{z}$$

What's the magnetic field in the center of a sphere with charge density ρ rotating about z at ω.



$$\vec{B}(0) = \tfrac{\mu_o}{4\pi}\,\rho_o\omega\int\left(r'\sin^4\theta' d\theta' dr' d\phi'\right)\hat{z}$$

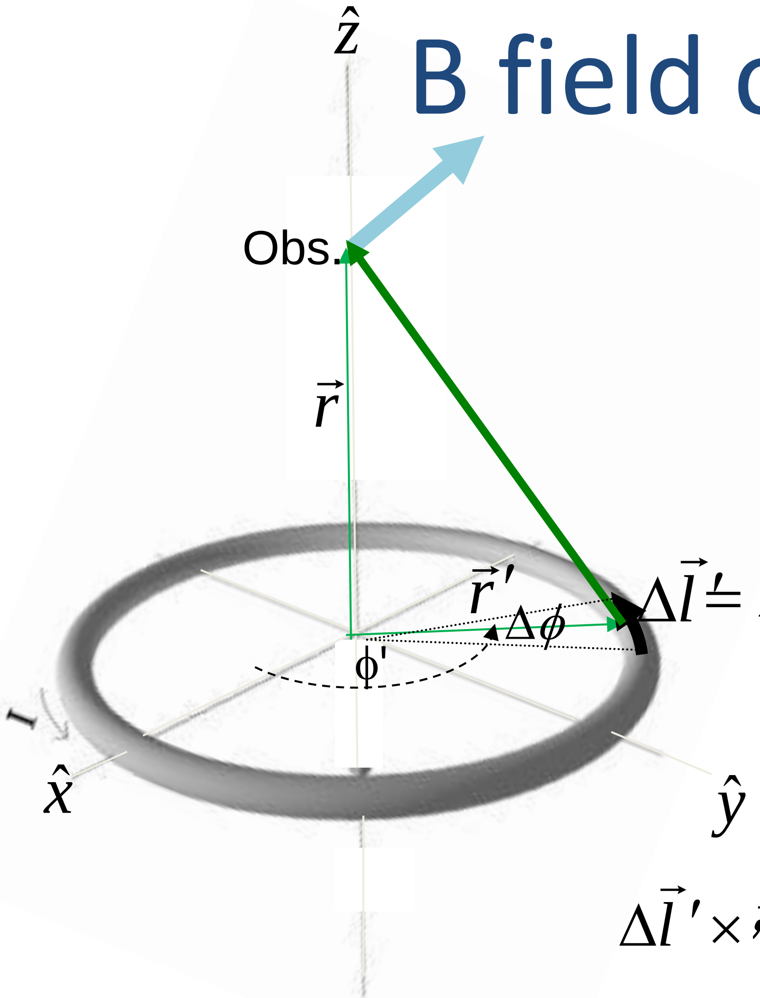
$$\vec{B}(0) = \tfrac{\mu_o}{4\pi}\,\rho_o\omega\int_0^{r'=R}\int_0^{\theta=\pi}\int_0^{\phi=2\pi} r'\sin^4\theta' d\theta' dr' d\phi'\hat{z}$$

$$\vec{B}(0) = \tfrac{\mu_o}{4\pi}\,\rho_o\omega(2\pi)\int_0^{r'=R}\int_0^{\theta=\pi} r'\sin^4\theta' d\theta' dr'\hat{z}$$

$$\vec{B}(0) = \tfrac{\mu_o}{2}\,\rho_o\omega\left(\tfrac{1}{2}R^2\right)\int_0^{\theta=\pi}\sin^4\theta' d\theta'\hat{z}$$

$$\vec{B}(0) = \tfrac{\mu_o}{4}\,\rho_o\omega R^2\left(\frac{3\pi}{8}\right)\hat{z} = \frac{\mu_o 3\pi R^2\rho_o\omega}{64}\,\hat{z}$$

B field of loop – on axis



$$\vec{r} = \langle 0,0,z \rangle$$

$$\Delta\vec{B} = \frac{\mu_o}{4\pi}\frac{I\Delta\vec{l}\,'\times\hat{\mathfrak{r}}}{\mathfrak{r}^2} = \frac{\mu_o}{4\pi}\frac{I\Delta\vec{l}\,'\times\vec{\mathfrak{r}}}{\mathfrak{r}^3}$$

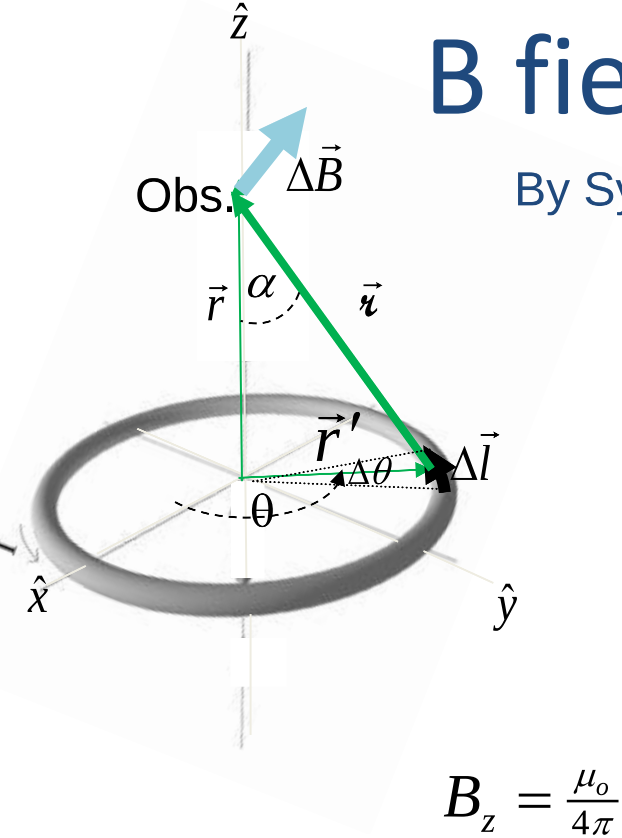
$$\vec{\mathfrak{r}} = \vec{r} - \vec{r}\,' = \langle -R\cos\theta', -R\sin\theta', z \rangle$$

$$\Delta\vec{l} \doteq R\Delta\vec{\phi}\,' = R\Delta\phi\left(\Delta\hat{\phi}\right) = R\Delta\phi'\langle -\sin\phi', \cos\phi', 0 \rangle$$

$$\Delta\vec{l}\,'\times\vec{\mathfrak{r}} = \det\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -R\Delta\phi'\sin\phi' & R\Delta\phi'\cos\phi' & 0 \\ -R\cos\phi' & -R\sin\phi' & z \end{vmatrix}$$

$$\Delta\vec{l}\,'\times\vec{\mathfrak{r}} = R\Delta\phi'\langle z\cos\phi', z\sin\phi', R \rangle$$

B field of loop – on axis



By Symmetry (could do math, but why bother)

$$B_x = \frac{\mu_o}{4\pi} IRz \int_{\theta'=0}^{\theta'=2\pi} \frac{\cos\theta' d\theta'}{\left(r^2+R^2\right)^{\frac{3}{2}}} = \ldots = 0$$

$$B_y = \frac{\mu_o}{4\pi} IRz \int_{\theta'=0}^{\theta'=2\pi} \frac{\sin\theta' d\theta'}{\left(r^2+R^2\right)^{\frac{3}{2}}} = \ldots = 0$$

$$B_z = \frac{\mu_o}{4\pi} IR^2 \int_{\theta'=0}^{\theta'=2\pi} \frac{d\theta'}{\left(r^2+R^2\right)^{\frac{3}{2}}} = \frac{\mu_o}{4\pi} \frac{IR^2 z}{\left(z^2+R^2\right)^{\frac{3}{2}}} \int_{\theta=0}^{\theta=2\pi} d\theta$$

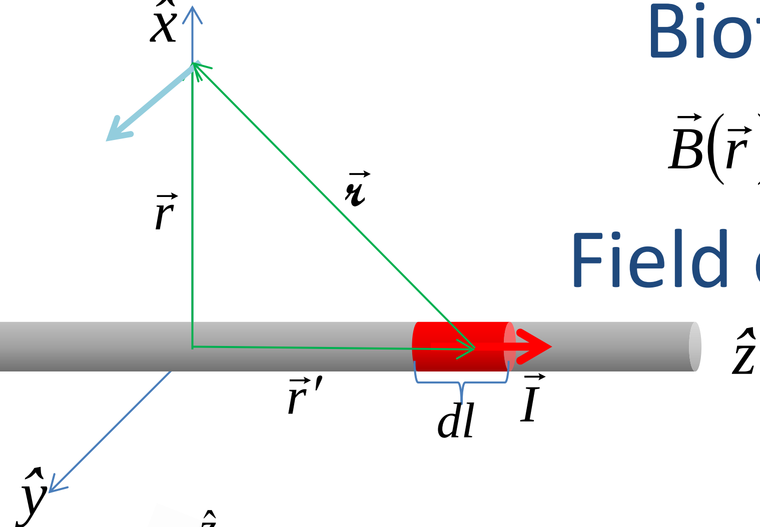
Rephrase in terms of angle α…

$$= \frac{\mu_o}{4\pi} \frac{IR^2}{\left(z^2+R^2\right)^{\frac{3}{2}}} 2\pi$$

In special case of z = 0, reduces to…

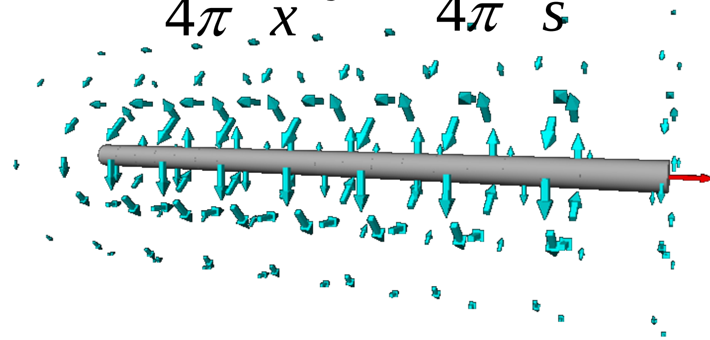
Biot-Savart Law

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\vec{\ell}' \times \hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^2}$$

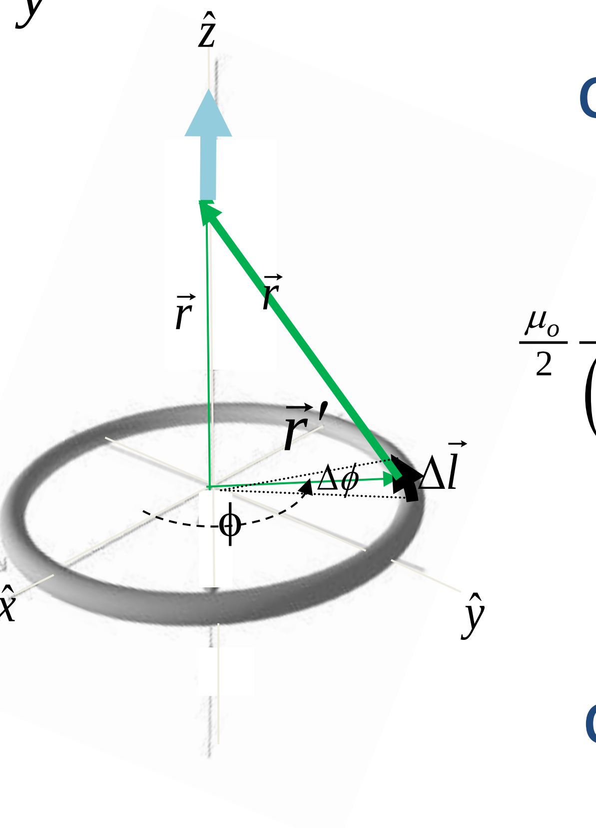


Field of Infinite Wire

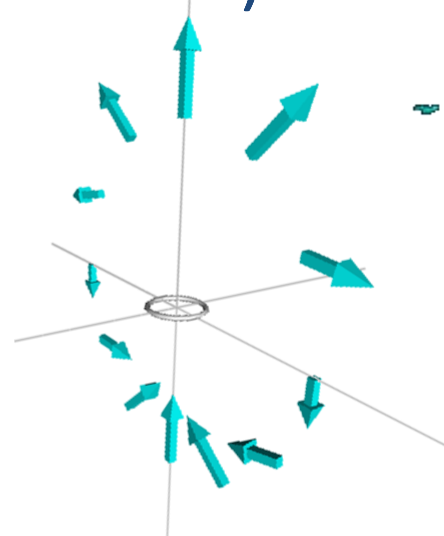
Cartesian		Polar
$\frac{\mu_0}{4\pi}\frac{2I}{x}\hat{y}$	$=$	$\frac{\mu_0}{4\pi}\frac{2I}{s}\hat{\phi}$



# of a Loop (on axis)

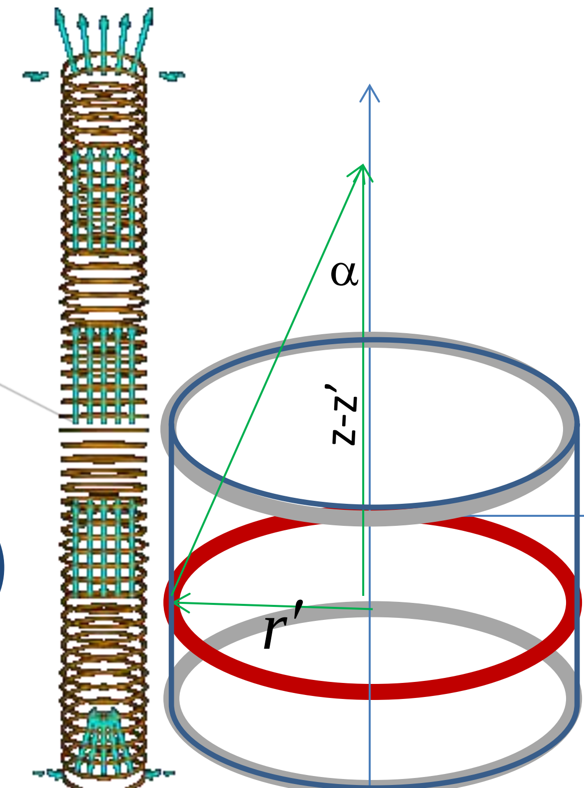


$$\frac{\mu_o}{2}\frac{IR^2}{\left(z^2+R^2\right)^{\frac{3}{2}}}\hat{z}$$



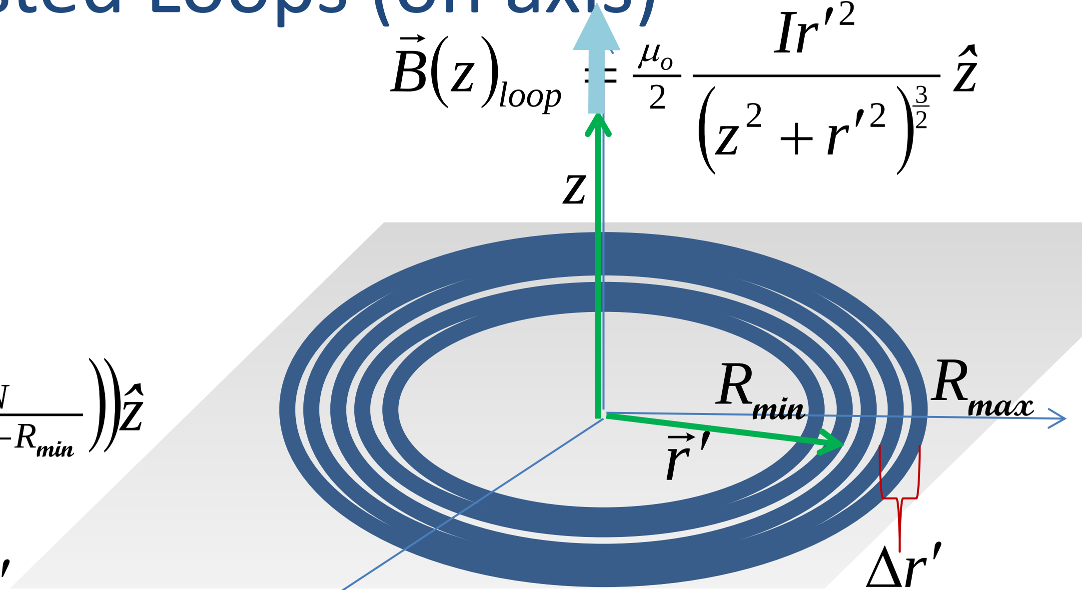
# of a Solenoid (on axis)

$$\frac{\mu_o}{2}\frac{IN}{L}\left(\cos\alpha_{bottom} - \cos\alpha_{top}\right)\hat{z}$$



Biot-Savart Law
of a disc of nested Loops (on axis)

$$\vec{B}(z)_{loop} = \frac{\mu_o}{2}\frac{Ir'^2}{\left(z^2+r'^2\right)^{\frac{3}{2}}}\hat{z}$$



$$\vec{B}(z)_{nest} = \sum_{loops}\vec{B}(z)_{loop.i}$$

$$\vec{B}(z)_{nest} = \sum_{loops}\frac{\mu_o}{2}\frac{Ir'^2}{\left(z^2+r'^2\right)^{\frac{3}{2}}}\hat{z}$$

$$\vec{B}(z)_{nest} = \sum_{r'}\frac{\mu_o}{2}\frac{Ir'^2}{\left(z^2+r'^2\right)^{\frac{3}{2}}}\left(\Delta r'\left(\tfrac{N}{R_{max}-R_{min}}\right)\right)\hat{z}$$

$$\frac{1loop}{\Delta r'} = \frac{N}{R_{max}-R_{min}}$$

$$1loop = \frac{N}{R_{max}-R_{min}}\Delta r'$$

$$\vec{B}(z)_{nest} = \frac{\mu_o}{2}I\,\frac{N}{R_{max}-R_{min}}\int_{r'=R_{min}}^{R_{max}}\frac{r'^2dr'}{\left(z^2+r'^2\right)^{\frac{3}{2}}}\hat{z}$$

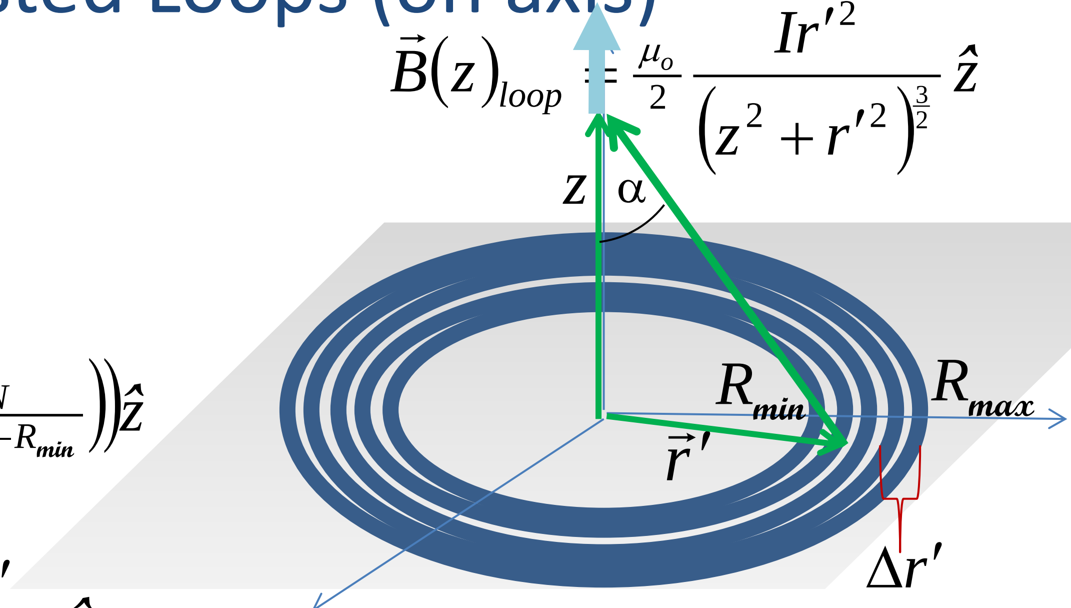
$$\vec{B}(z)_{nest} = \frac{\mu_o}{2}I\,\frac{N}{R_{max}-R_{min}}\left[\ln\left(\sqrt{z^2+r'^2}+r'\right)-\frac{r'}{\sqrt{z^2+r'^2}}\right]_{R_{min}}^{R_{max}}\hat{z}$$

$$\vec{B}(z)_{nest} = \frac{\mu_o}{2}I\,\frac{N}{R_{max}-R_{min}}\left[\ln\left(\frac{\sqrt{\left(\frac{z}{R_{max}}\right)^2+1}+1}{\sqrt{\left(\frac{z}{R_{min}}\right)^2+1}+1}\right)+\frac{1}{\sqrt{\left(\frac{z}{R_{min}}\right)^2+1}}-\frac{1}{\sqrt{\left(\frac{z}{R_{max}}\right)^2+1}}\right]\hat{z}$$

Biot-Savart Law
of a disc of nested Loops (on axis)

$$\vec{B}(z)_{nest} = \sum_{loops} \vec{B}(z)_{loop.i}$$

$$\vec{B}(z)_{loop} = \frac{\mu_o}{2}\frac{Ir'^2}{\left(z^2+r'^2\right)^{\frac{3}{2}}}\hat{z}$$



$$\vec{B}(z)_{nest} = \sum_{loops} \frac{\mu_o}{2}\frac{Ir'^2}{\left(z^2+r'^2\right)^{\frac{3}{2}}}\hat{z}$$

$$\vec{B}(z)_{nest} = \sum_{r'} \frac{\mu_o}{2}\frac{Ir'^2}{\left(z^2+r'^2\right)^{\frac{3}{2}}}\left(\Delta r'\left(\frac{N}{R_{max}-R_{min}}\right)\right)\hat{z}$$

$$\frac{1 loop}{\Delta r'} = \frac{N}{R_{max}-R_{min}}$$

$$\vec{B}(z)_{nest} = \frac{\mu_o}{2} I \frac{N}{R_{max}-R_{min}} \int_{r'=R_{min}}^{R_{max}} \frac{r'^2 dr'}{\left(z^2+r'^2\right)^{\frac{3}{2}}}\hat{z}$$

$$1 loop = \frac{N}{R_{max}-R_{min}}\Delta r'$$

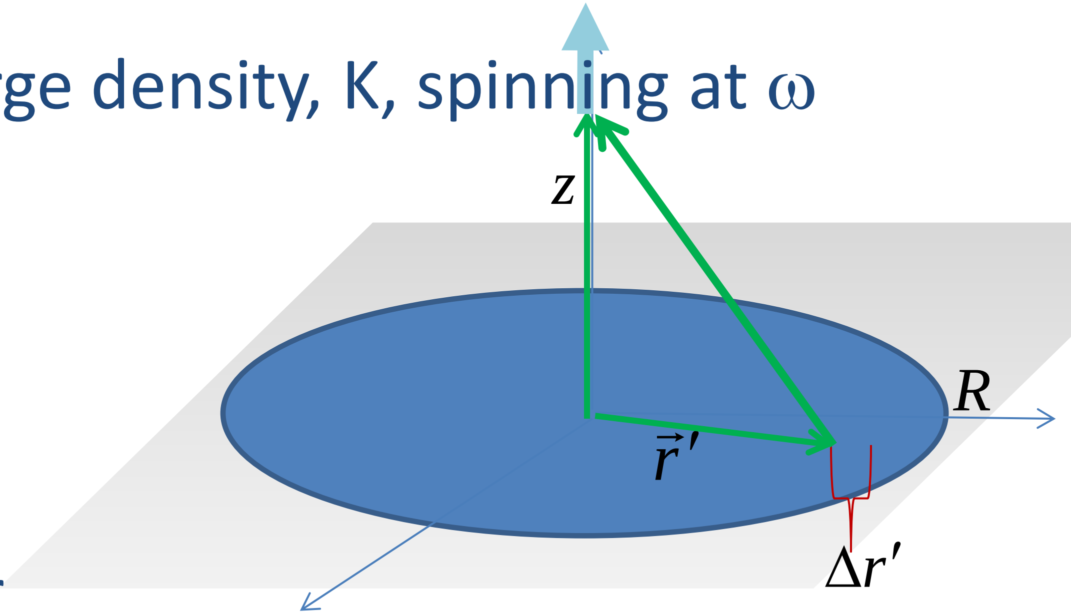
Exercise: translate into integral of trig function of α.

$$\frac{r'^2 dr'}{\left(z^2+r'^2\right)^{\frac{3}{2}}} = \left(\frac{r'}{\sqrt{z^2+r'^2}}\right)^2 \frac{dr'}{\sqrt{z^2+r'^2}} = (\sin\alpha)^2 \frac{d(z\tan\alpha)}{(z/\cos\alpha)} = (\sin\alpha)^2\frac{\cos\alpha\, d\alpha}{(\cos\alpha)^2}$$

$$= \left(1-\cos^2\alpha\right)\frac{\cos\alpha\, d\alpha}{(\cos\alpha)^2} \qquad = \left(\frac{d\alpha}{\cos\alpha} - \cos\alpha\, d\alpha\right)$$

Exercise:

disc of uniform surface charge density, K, spinning at ω



$$\vec{B}(z)_{nest} = \sum_{loops} \vec{B}(z)_{loop.i}$$

$$\vec{B}(z)_{nest} = \sum_{loops} \frac{\mu_o}{2} \frac{(dI)r'^2}{\left(z^2 + r'^2\right)^{\frac{3}{2}}} \hat{z}$$

Recall: $K = \dfrac{dI}{dl_\perp} \Rightarrow dI = K dl_\perp$

Where $dl_\perp$ **is a small step perpendicular to current flow, i.e** dr'

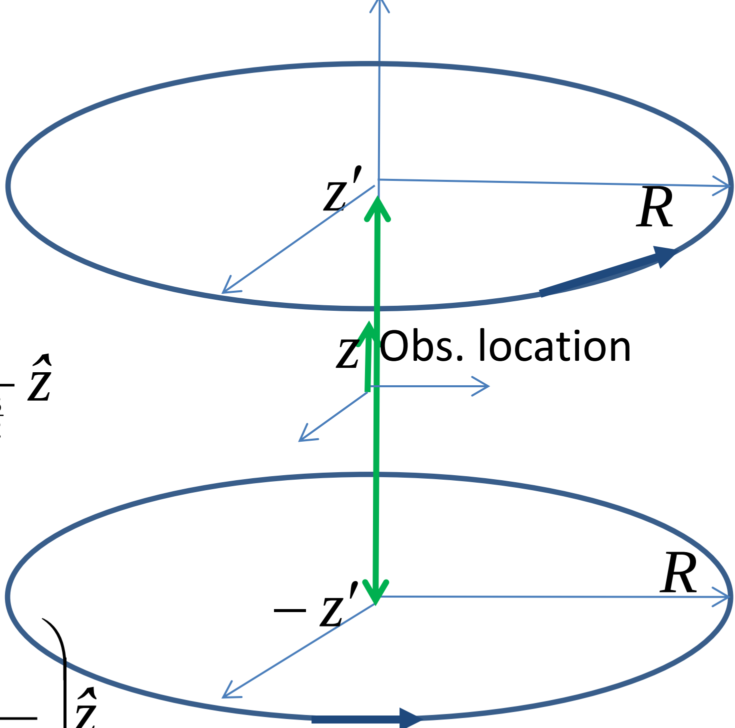
From the homework that you've just done, $K = \sigma v = \sigma\omega r'$ **so,** $dI = \sigma\omega r' dr'$

$$\vec{B}(z)_{nest} = \int_0^R \frac{\mu_o}{2} \frac{(\sigma\omega r' dr')r'^2}{\left(z^2 + r'^2\right)^{\frac{3}{2}}} \hat{z} = \frac{\mu_o}{2}\sigma\omega \int_0^R \frac{r'^3 dr'}{\left(z^2 + r'^2\right)^{\frac{3}{2}}} \hat{z} = \frac{\mu_o}{2}\sigma\omega z \int_0^{\frac{R}{z}} \frac{\left(\frac{r'}{z}\right)^3 d\left(\frac{r'}{z}\right)}{\left(1 + \left(\frac{r'}{z}\right)^2\right)^{\frac{3}{2}}} \hat{z}$$

$$= \frac{\mu_o}{4}\sigma\omega z \int_0^{\frac{R}{z}} \frac{\left(\frac{r'}{z}\right)^2 d\left(\frac{r'}{z}\right)^2}{\left(1 + \left(\frac{r'}{z}\right)^2\right)^{\frac{3}{2}}} \hat{z} = \frac{\mu_o}{4}\sigma\omega z \int_0^{\left(\frac{R}{z}\right)^2} \frac{\xi d\xi}{(1+\xi)^{\frac{3}{2}}} \hat{z} = \frac{\mu_o}{4}\sigma\omega z \left[\frac{2(\xi + 2)}{(1+\xi)^{\frac{1}{2}}}\right]_0^{\left(\frac{R}{z}\right)^2}$$

$$= \frac{\mu_o}{2}\sigma\omega z \left[\frac{\left(\left(\frac{R}{z}\right)^2 + 2\right)}{\left(1 + \left(\frac{R}{z}\right)^2\right)^{\frac{1}{2}}} - 2\right]$$

Helmholtz Coils



$$\vec{B}(z)_{Hc} = \vec{B}(z)_{loop.top} + \vec{B}(z)_{loop.bottom}$$

$$\vec{B}(z)_{Hc} = \frac{\mu_o}{2}\frac{IR^2}{\left((z-z')^2+R^2\right)^{\frac{3}{2}}}\hat{z} + \frac{\mu_o}{2}\frac{IR^2}{\left((z+z')^2+R^2\right)^{\frac{3}{2}}}\hat{z}$$

$$\vec{B}(z)_{Hc} = \frac{\mu_o}{2} IR^2\left(\frac{1}{\left((z-z')^2+R^2\right)^{\frac{3}{2}}} + \frac{1}{\left((z+z')^2+R^2\right)^{\frac{3}{2}}}\right)\hat{z}$$

Why they're useful

$$\left.\frac{dB(z)_{Hc}}{dz}\right|_{z=0} = -\frac{3\mu_o}{2} IR^2\left(\frac{(0-z')}{\left((0-z')^2+R^2\right)^{\frac{5}{2}}} + \frac{(0+z')}{\left((0+z')^2+R^2\right)^{\frac{5}{2}}}\right) = 0$$

and

$$\left.\frac{d^2B(z)_{Hc}}{dz^2}\right|_{z=0} = 0$$ **If $z' = {}^{R}/_{2}$**

Produces fairly uniform field near middle

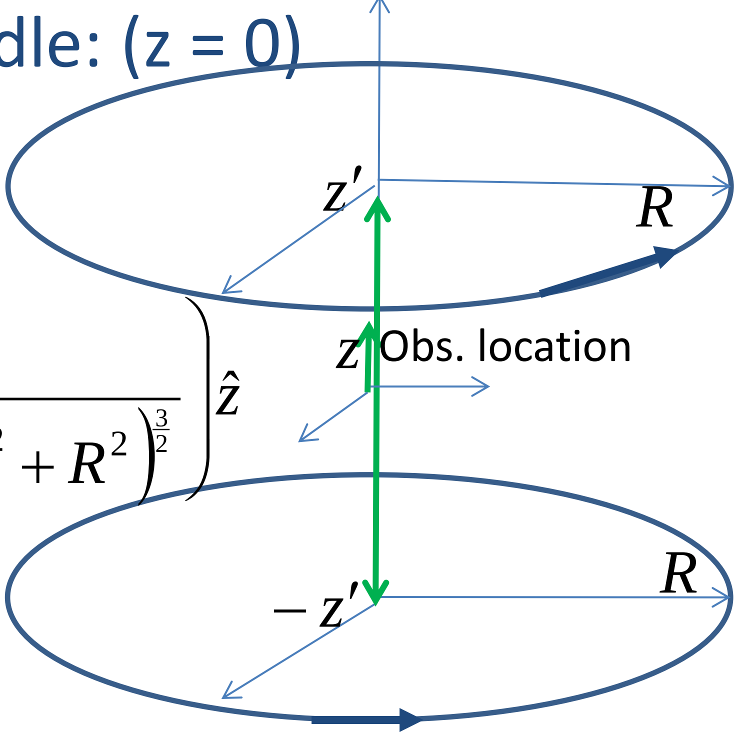
Helmholtz Coils **If $z' = {}^{R}/_{2}$** field in the middle: (z = 0)

$$\vec{B}(0)_{Hc} = \vec{B}(0)_{loop.top} + \vec{B}(0)_{loop.bottom}$$

$$\vec{B}(0)_{Hc} = \tfrac{\mu_o}{2} IR^2 \left(\frac{1}{\left((0 - R/2)^2 + R^2\right)^{\frac{3}{2}}} + \frac{1}{\left((0 - R/2)^2 + R^2\right)^{\frac{3}{2}}} \right) \hat{z}$$

$$\vec{B}(0)_{Hc} = \tfrac{\mu_o}{2} IR^2 \left(\frac{2}{\left(5R^2/4\right)^{\frac{3}{2}}} \right) \hat{z}$$

$$\vec{B}(0)_{Hc} = \left(\frac{4}{5}\right)^{\frac{3}{2}} \frac{\mu_o I}{R} \hat{z}$$



Mon. Tues. Wed. Fri.	(C 17) 5.2 Biot-Savart Law T5 Quiver Plots (C 21.6-7,.9) 1.3.4-1.3.5, 1.5.2-1.5.3, 5.3.1-.3.2 Div & Curl B (C 21.6-7,.9) 5.3.3-.3.4 Applications of Ampere’s Law	 HW7
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