Physics 332: E&M **2013 Induction, Inductance, Energy with B**

Wed., 11/6 Fri., 11/8	**7.2.3-7.2.5** *Inductance and Energy of B* **7.3.1-.3.3** Maxwell's Equations	
Mon., 11/11 Wed., 11/13 Fri., 11/15	**10.1 - .2.1** Potential Formulation **10.2** Continuous Distributions **10.3** Point Charges	HW8

Equipment

- Magnet and copper pipe
- Induction coils with diodes and magnets from 232
- Two inductive coils, function generator and o'scope to see induced voltage.

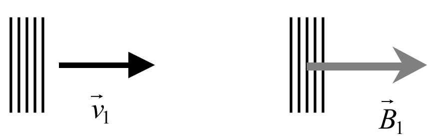
Several ways to change the Magnetic Flux:

Exercise – Come up with ways to change the magnetic flux through a coil using either a second coil or a permanent magnet

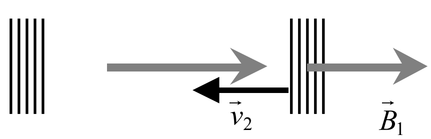
All of the following will result in an induced emf in the coil 2 on the right.

1. Change the current in coil 1

2. Move coil 1 (with current through it)



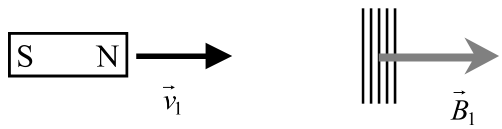
3. Move coil 2 (with current through coil 1)



4. Rotate coil 1

Rotate coil 2

5. Move the magnet relative to the coil (includes moving coil toward magnet)



6. Rotate the magnet

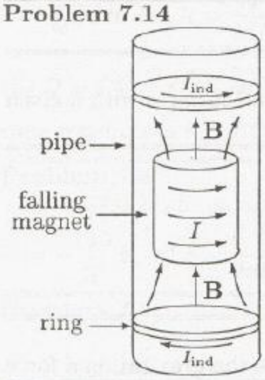
7. Rotate the coil

Problem 7.14

Explain why a cylindrical magnet takes much longer to drop through a vertical copper pipe than an unmagnetized piece of iron does.

(Ignore the part about the "current in the magnet")

Problem 7.14



Suppose the current (I) in the magnet flows counterclockwise (viewed from above), as shown, so its field, near the ends, points *upward*. A ring of pipe *below* the magnet experiences an increasing upward flux, as the magnet approaches, and hence (by Lenz's law) a current (I_{ind}) will be induced in it such as to produce a *downward* flux. Thus I_{ind} must flow *clockwise*, which is *opposite* to the current in the magnet. Since opposite currents repel, the force on the magnet is *upward*. Meanwhile, a ring *above* the magnet experiences a *decreasing* (upward) flux, so *its* induced current is *parallel* to I, and it *attracts* the magnet upward. And the flux through rings *next to* the magnet is constant, so *no* current is induced in them. *Conclusion:* the delay is due to forces exerted on the magnet by induced eddy currents in the pipe.

Demo: drop a magnet down a copper tube (not ferromagnetic) –very slow compared to free fall!

Each cross section of the pipe can be considered a loop. There will be induced currents around the pipe. These in turn produce magnetic fields, so it's like having two magnets interact. You will explain the slowing in terms of forces in Prob. 22.1 (c).

Lenz's Law

We usually just used Faraday's law to find the magnitude of the *emf* and don't worry about the minus sign. Lenz's law can be used to determine the direction of the induced current. It states that, "Nature abhors a change in magnetic flux." In other words, the induced current will produce a magnetic field that will somewhat oppose the change in flux.

Apply this to the figures above.

Examples

Pr. 7.12

Example 7.7 (an Ampere's Law type approach). A long solenoid with radius a and n tunrs per unit length, caries time varying current, I(t). What's an expression for the electric field a distance s from the axis (inside and out, quasi-static approximation)?

Pr. 7.15 $\oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt}$ $\Phi_B = \int \vec{B}\cdot d\vec{a}$ and $\vec{B} = \mu_o I(t) n\hat{z}$ so

$$\Phi_B = \int \vec{B}\cdot d\vec{a} = \int \mu_o I(t) n da$$

Given rotational symmetry and lack of charge sources (lack of divergence)

$$\oint \vec{E}\cdot d\vec{l} = E2\pi s$$

Meanwhile

$$-\frac{d\Phi_B}{dt} = -\int \mu_o \frac{dI(t)}{dt} n da$$

For s < radius $-\frac{d\Phi_B}{dt} = -\mu_o \frac{dI(t)}{dt} n\pi s^2$ then

$$E2\pi s = -\mu_o \frac{dI(t)}{dt} n\pi s^2$$

$$E = -\tfrac{\mu_o}{2} \frac{dI(t)}{dt} ns$$

And this is such as to drive a current opposite the change in current, so create a flux opposite the loss / gain of flux.

$$E = -\tfrac{\mu_o}{2} \frac{dI(t)}{dt} ns\hat{\phi}$$

For s> radius a $-\frac{d\Phi_B}{dt} = -\mu_o \frac{dI(t)}{dt} n\pi a^2$ and so $E = -\tfrac{\mu_o}{2} \frac{dI(t)}{dt} n\left(\frac{a}{s}\right)^2 \hat{\phi}$

Problem 7.16 – Coaxial Cable with Change Current

An (slowly) alternating current $I(t) = I_0 \cos(\omega t)$ flows down a long, straight, thin wire and returns along a thin, coaxial conducting tube of radius *a*.

a. In what direction does the induced electrical field point?

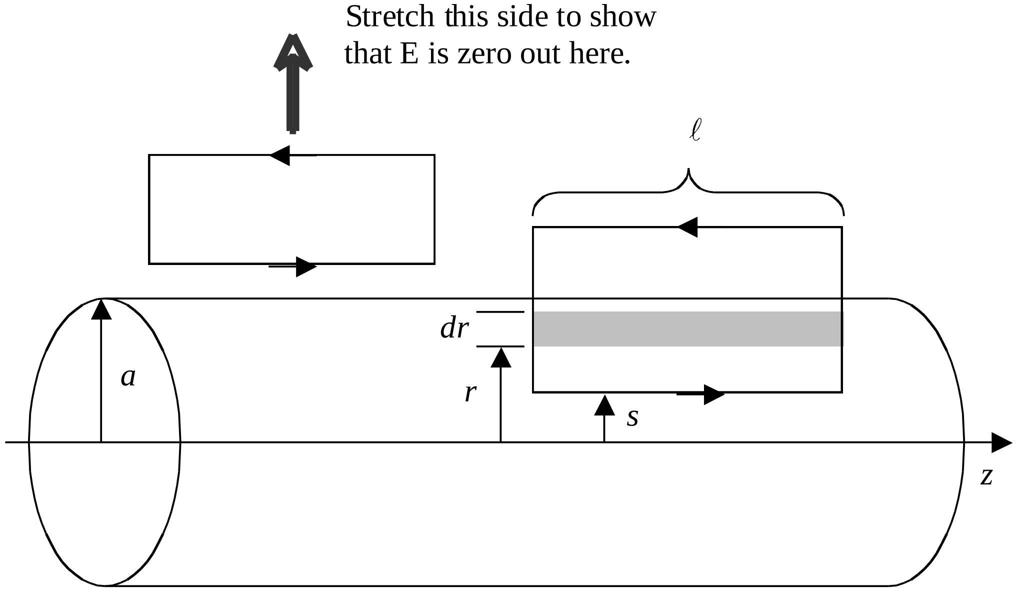
Let the current on the central wire be in the +*z* direction. In the quasistatic approximation (current changes slowly), the magnetic field is circumferential. A changing magnetic field in this direction is analogous to the current for a solenoid, which produces a longitudinal (in *z* direction) magnetic field. Therefore, the direction of the induced electric field is longitudinal. Alternatively, the field needs to be int eh direction to drive current as to oppose the changing current – so if current is growing in the z directin, it needs to point in the -z, and vice versa. Similarly out at the cylindrical shell, but clearly the field must be opposite to what it is inside.

b. Assuming that the field goes to zero as $s \to \infty$, find the induced electric field $\vec{E}(s,t)$.

The magnetic field in the quasistatic approximation is (use Ampere's law)

$$\vec{B} = \begin{cases} \frac{\mu_0 I}{2\pi s}\hat{\phi} & s < a, \\ 0 & s > a. \end{cases}$$

By symmetry, we also know that the induced electric field only *s* (and *t*). Use the same shape of "amperian loop" as for a solenoid (see the diagram below).



We can argue that the induced electric field is the same at all distances outside the coaxial cable, so it must be zero (use the loop on the left).

For a loop with one side inside the cable (on the right), the line integral of the electric field around the loop is $\oint \vec{E}\cdot d\vec{\ell} = E\ell$, because only the bottom side is non-zero.

Consider a thin strip between distances *r* and *r* + *dr* from the long wire that is enclosed by the loop. The magnetic flux through this segment is

$$d\Phi = \left(\frac{\mu_0 I}{2\pi r}\right)(\ell\, dr).$$

The magnetic field comes out of the page, so the flux is positive by the RHR. The total flux through the loop is

$$\Phi = \frac{\mu_0 I\ell}{2\pi}\int_s^a \frac{dr}{r} = \frac{\mu_0 I\ell}{2\pi}[\ln r]_s^a = \frac{\mu_0 I\ell}{2\pi}\ln\left(\frac{a}{s}\right).$$

Putting in the function for the current and applying Faraday's law, $\oint \vec{E}\cdot d\vec{\ell} = -d\Phi/dt$, gives

$$\vec{E}(s,t) = \begin{cases} \dfrac{\mu_0 I_0 \omega}{2\pi}\ln\left(\dfrac{a}{s}\right)\sin(\omega t)\hat{z} & s < a, \\ 0 & s > a. \end{cases}$$

Pr. 7.17 $IR = emf = -\dfrac{d\Phi_B}{dt}$ (use Lenz's Law for direction)

Inductance.

Now, since flux is proportional to field and field is proportional to current, the flux through a loop is proportional to the source current. We can phrase this fact as

$\Phi_2 = MI_1$ where M is the proportionality constant, "Mutual Inductance."

What does M depend upon?

$$\Phi_2 = \int \vec{B}_1\cdot d\vec{a}_2 = \int \frac{\mu_o}{4\pi} I_1 \oint \frac{d\vec{l}_1 \times \vec{\mathscr{r}}}{\mathscr{r}^2}\cdot d\vec{a}_2 = I_1\left(\frac{\mu_o}{4\pi}\int\oint \frac{d\vec{l}_1 \times \vec{\mathscr{r}}}{\mathscr{r}^2}\cdot d\vec{a}_2\right)$$

Alternatively,

$$\Phi_2 = \int \vec{B}_1\cdot d\vec{a}_2 = \int \left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{a}_2 = \oint \vec{A}\cdot d\vec{l}_2 = \oint\left(\frac{\mu_o}{4\pi}\oint\frac{I_1 d\vec{l}_1}{\mathscr{r}}\right)\cdot d\vec{l}_2 = I_1\left(\frac{\mu_o}{4\pi}\oint\oint\frac{d\vec{l}_1\cdot d\vec{l}_2}{\mathscr{r}}\right)$$

Either way, it's clear that the proportionality constant just depends upon the geometry of the sensor and source loops.

Ex 7.10

A short solenoid (length ℓ, radius a, n_1 turns per length) lies on the axis of a very long solenoid (radius b, n_2 turns per length). What is the mutual inductance?

QuickTime™ and a
TIFF (Uncompressed) decompressor
are needed to see this picture.

Of course, as we've just found in Faraday's Law,

$$emf = -\frac{d\Phi_2}{dt} = -M\frac{dI_1}{dt} \text{ (assuming the geometry remains constant)}$$

Suppose there is current I_2 running through the longer, outer solenoid. The magnetic field inside it is $B_2 = \mu_0 n_2 I_2$ in the longitudinal direction. The flux through one loop of the shorter solenoid is $B_2(\pi a^2) = \mu_0 n_2 I_2 \pi a^2$. There are $n_1 \ell$ turns, so the total flux is $\Phi_1 = (\mu_0 n_2 I_2 \pi a^2)(n_1 \ell)$. The mutual inductance is defined by $\Phi_1 = MI_2$, so

$$M = \mu_0 n_1 n_2 \pi a^2 \ell .$$

A current *I* around a single loop will also produce a magnetic flux through itself which is proportional to the current. The flux can be written as $\Phi = LI$, where *L* is the *self inductance* (or just *inductance*). The induced emf that results from a change of the current is

$$\varepsilon = -L \frac{dI}{dt} .$$

The minus sign means that it is in the direction opposing the change in current.

Energy of Magnetic Fields

You wouldn't be surprised if I suggested that it took work to get massive particles, who happen to bear charge, accelerated from rest to any particular speed. Of course, once they're coasting at that speed, they're generating a magnetic field. *In addition* to what it takes to move the masses, there's also work necessary to fight the electric fields that get generated while the charges are being accelerated. How can we get at this?

Well, the rate at which energy gets transferred from an electric field to charged particles is

$$P_{E\to q} = IV = Iemf$$

For you to be pushing the charges along *in spite* of the opposing electric field, you must at least balance this rate of energy transfer

$$P_{you\to q} = -Iemf$$

But the emf that we're talking about is that induced by the changing current itself (very much like the voltage between capacitor plates is produced by the charges we're trying to put on the plates)

$$P_{you\to q} = -I\left(-L\frac{dI}{dt}\right) = \left(LI\frac{dI}{dt}\right)$$

$$\frac{dW}{dt} = \frac{d\left(\frac{1}{2}LI^2\right)}{dt}$$

$$W = \tfrac{1}{2}LI^2$$

This echoes what we showed for charging up a capacitor. As in that case, we can do a little work to rephrase this in terms of the field so that we find

$$W = \tfrac{1}{2} L I^2 = \tfrac{1}{2\mu_o} \int B^2 d\tau$$

Can we don another example on how a changing magnetic field produces current?" Davies

"We go over Griffiths argument to get from 7.34 to 7.35 the choice to integrate over all space is an interesting one and I'd like to go in to more detail." Casey P,

Can we just go over the whole derivation of 7.35 because I didn't understand how he got to it. Jessica

Yeah, I also had much more difficulty following the derivation in this section than those in the previous sections. Perhaps we can go over each step. Casey McGrath

I'm with everyone else on this one. These derivations are the hardest and least straightforward thing in the chapter. At some point I lose track of what we are doing and the physical meaning is lost, making it just a mess of variables. I can't figure out how Griffiths gets through certain steps. Anton

"Why is the energy to crank up a current not dependent on time? Would back emf push back harder to resist this change in current?"Davies

"I didn't quite understand what he meant by "the energy 'is stored in the magnetic field,' in the amount ((B^2)/2*mu naught) per unit volume" can you explain what he meant by this?"Connor W,

"I'd be interested to go over exactly how Griffiths gets those two properties from the Neumann formula and go a little more in depth as to their significance."Ben Kid

"Why is it that the Neumann formula isn't very useful for actual calculations? It doesn't seem like those line integrals would be that bad, unless I'm really missing something."Freeman,

"Can we talk more about the concept of inductance and examples of how it is useful?"Sam

"I'm trying to understand the significance of the time constant: so what is it that dictates (2/3)rds of the current's final value is substantial? Is it that the current is strong enough to overcome back emf and R to freely flow through the wire?" Rachael Hach

"Can we talk about how the "back emf" is produced? Can we also go over example 7.13 or a similar problem?" Spencer

Summary

Maxwell's Equations So Far – in both differential and integral forms

Gauss's law

$$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\varepsilon_0} \qquad \oint\vec{E}\cdot d\vec{a}=\frac{Q_{enc}}{\varepsilon_0}$$

No name

$$\vec{\nabla}\cdot\vec{B}=0 \qquad \oint\vec{B}\cdot d\vec{a}=0$$

Faraday's law

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \qquad \oint\vec{E}\cdot d\vec{\ell}=-\frac{d\Phi}{dt}=-\frac{d}{dt}\int_{surface}\vec{B}\cdot d\vec{a}$$

Ampere's law – this is incomplete!

$$\vec{\nabla}\times\vec{B}=\mu_0\vec{J} \qquad \oint\vec{B}\cdot d\vec{\ell}=\mu_0 I_{enc}$$

What's Wrong With Ampere's Law

We have both a Mathematical and a Physical reason to conclude that something's wrong with ampere's law.

Physical

Let's try applying Ampere's Law to a circuit containing a charging capacitor to rather oxymoronic effect. Consider a circuit with a battery charging a capacitor (if it makes us feel better, let's say it's a constant-current source doing the charging, so we're squarely still in magnetostatics.) Choose an Amperian loop around the wire. As the current flows, it produces a magnetic field that curls around the wire that we can relate to the current flowing through the wire if we choose a flat surface across the loop:

$$\oint\vec{B}\cdot d\vec{\ell}=\mu_o I_{pierce}=\mu_o I_{wire}\neq 0.$$

Then again, if we choose a "bubble" shaped surface that passes inside the capacitor (see the diagram below), then

$$I_{pierce}=\int_{surface}\vec{J}\cdot d\vec{a}=0,$$

Yet we've got the *same loop* around which we're evaluating the magnetic field; it can't be *both* zero and non-zero.

Mathematical

Mathematically, the divergence of a curl is 0. That's just a mathematical *fact.*

$$\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}\right)=0 \text{ (Vector Identity 9)}$$

Yet, when we take the divergence of Ampere's law, we have

$$\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{B}\right)=\mu_0\vec{\nabla}\cdot\vec{J}$$

And we'd derived the continuity equation as

$$\vec{\nabla}\cdot\vec{J}=-\frac{\partial\rho}{\partial t}\quad(5.29)$$

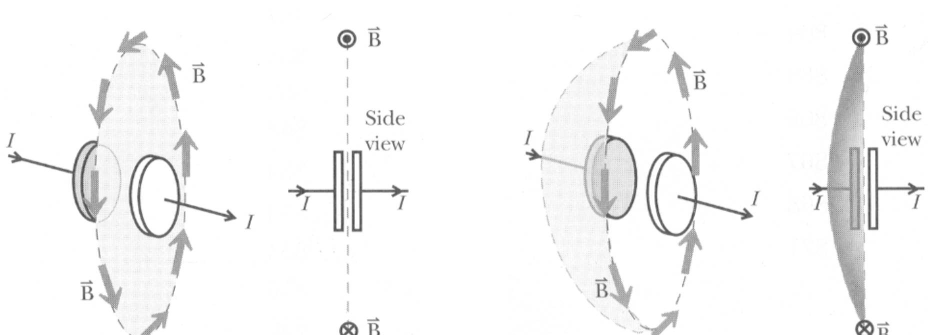
which says that a net out flow of current implies a depletion of charge concentration.

So,

$$\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{B}\right)=-\mu_0\frac{\partial\rho}{\partial t}$$

Of course, this isn't a problem in electro*statics* since charge densities aren't changing in that case; however, in general, charge densities certainly *can* change, so in general, this is a big problem. Clearly something's wrong with our math when charges are accumulating/depleting. The divergence of the curl of B *must* be 0, so there must be something *else* on the left hand side of the equation that equals this rate of change of current density.

Correcting Ampere's Law



Physical & Mathematical

Hm… a situation in which charge density is changing. Capacitors are all about that; we're always 'charging' them up or down. Now, we've got a pretty good hunch how Ampere's Law should be fixed. Let's look at the capacitor situation. So, there's not current piercing the surface that runs between the plates; *is* there anything in the gap that we *can* relate to the current in the wire? Sure, the electric field that's growing because of the onrush of charge to the plates. In fact, we can relate the offending charge density to field through

$$\vec{\nabla}\cdot\vec{E}=\rho/\varepsilon_0$$

So, we could kill off the right hand side if we amended the equation to read

$$\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{B}-\mu_0\varepsilon_o\frac{\partial\vec{E}}{\partial t}\right)=-\mu_0\frac{\partial\rho}{\partial t}$$

That way, we can merrily say that the divergence of the curl of B is zero *and* have something equaling our changing current density.

$$\left(0-\mu_0\varepsilon_o\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{E}\right)=-\mu_0\frac{\partial\rho}{\partial t}$$

Which is nothing more than the time derivative of Gauss's Law.

Then, backing out of the divergence and the continuity equation, we have

$$\vec{\nabla}\times\vec{B}-\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}=\mu_0\vec{J}$$

Or integrating and applying Stoke's Theorem, we have

$$\oint\vec{B}\cdot d\vec{l}-\mu_0\varepsilon_0\left.\frac{\partial\Phi_E}{\partial t}\right|_a=\mu_0 I_{pierce}$$

Now let's return to our charging capacitor

$$I_{wire}=\frac{dq_{plate}}{dt}=\frac{dq_{plate}}{dt}$$

But recall how the charge on a capacitor plate is related to the field between the plates (which is that piercing our surface)

$$E=\frac{q}{\varepsilon_o A}\Rightarrow q=\varepsilon_o EA=\varepsilon_o\Phi_E$$

Thus,

$$I_{wire}=\varepsilon_o\frac{d\Phi_E}{dt}$$

So, if we use the surface that slices through the wire, then we have

$$\oint\vec{B}\cdot d\vec{l}=\mu_0 I_{wire}$$

If we use the surface that slices between the capacitor plates, then we have

$$\oint\vec{B}\cdot d\vec{l}-\mu_0\varepsilon_0\left.\frac{\partial\Phi_E}{\partial t}\right|_a=0$$

Where the electric field flux term has the same value as $\mu_o I_{wire}=\mu_o\left(\varepsilon_o\frac{d\Phi_E}{dt}\right)$

Again making the integral of B evaluate to the same thing

$$\oint\vec{B}\cdot d\vec{l}=\mu_0 I_{wire}\,.$$

Our fix seems to work! Experiments have proven it correct.

Looking back at

$$\vec{\nabla}\times\vec{B}-\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}=\mu_0\vec{J}$$

The electric field term has been called the "displacement current (density)"

$$\vec{J}_d\equiv\varepsilon_0\frac{\partial\vec{E}}{\partial t}.$$

Similarly, flux of this through an area is called the "displacement current",

$$I_d\equiv\varepsilon_0\frac{\partial\Phi_E}{\partial t}.$$

As Griffith's points out, this is not a real current; however, as he mentions in a paper, this could be viewed as a "proxy" for currents elsewhere. That is, the magnetic field at location *r* may curl because of the current at location *r or* because of the currents elsewhere. We'll see this in more detail in Ch. 10.

<u>Examples/Exercises</u>:

~Problem 7.32 – Magnetic field in a charging capacitor

Thin wires connect to the centers of thin, round capacitor plates. Suppose that the current *I* is constant, the radius of the capacitor is *a*, and the separation of the plates is *w* (<< *a*). Assume that the current flows out over the plates in such a way that the surface charge is uniform at any given time and is zero at $t = 0$.

c. Find the electric field between the plates as a function of *t*.

$$\vec{E}=\frac{q(t)}{\varepsilon_o A}\hat{z}=\frac{I_{wire}t}{\varepsilon_o\pi a^2}\hat{z}\text{ T}$$

d. Find the "displacement current" through a circle of radius *s* in the plane midway between the plates. Using this circle as your "amperian loop" and the flat surface that spans it, find the magnetic field at a distance *s* from the axis.

The displacement current density is

$$I_d\equiv\varepsilon_0\frac{\partial\Phi_E}{\partial t}=\varepsilon_0\left(\int_0^s\frac{\partial\vec{E}}{\partial t}\cdot d\vec{a}\right)=\varepsilon_0\int_0^s\frac{\partial}{\partial t}\left(\frac{I_{wire}t}{\varepsilon_0\pi a^2}\right)da=\frac{I_{wire}}{\pi a^2}\pi s^2=\frac{I_{wire}}{a^2}s^2,$$

Note, this is exactly what you'd get for Ienclosed if you had a wire of radius *a* and you wanted to know how much flowed within a smaller radius *s*.

By symmetry we know that the magnetic field is of the form $\vec{B}=B(s)\hat{\phi}$. Applying the Ampere-Maxwell equation gives

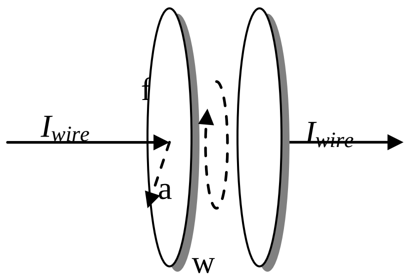
$$\oint \vec{B}\cdot d\vec{\ell} = B\cdot(2\pi s) = \mu_0\left(I_{enc} + I_{d,enc}\right) = \mu_0\left(I_{wire}\frac{s^2}{a^2}\right)$$

$$\vec{B} = \frac{\mu_0 I_{wire}\, s}{2\pi a^2}\hat{\phi}$$

Which is exactly that the same fraction of the radius out from the center of the wire.

e. We'll find the current running *along the surface of the capacitor plate* by considering two different Amperian surfaces for the same Amperian loop.

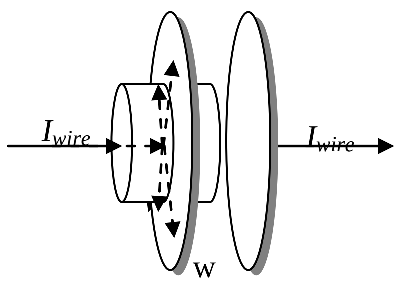
Surface 1: the obvious, flat one bound by this loop



$$\oint \vec{B}\cdot d\vec{l} = \mu_0\varepsilon_0 \left.\frac{\partial\Phi_E}{\partial t}\right|_{\substack{a\\ surf\,1}} + \mu_0 I_{\substack{pierce\\ surf\,1}}$$

$$\oint \vec{B}\cdot d\vec{l} = \mu_0\varepsilon_0 \left.\frac{\partial\Phi_E}{\partial t}\right|_{\substack{a\\ surf\,1}} + 0$$

Surface 2: the not-so obvious, can-shaped one bound by this loop



$$\oint \vec{B}\cdot d\vec{l} = \mu_0\varepsilon_0 \left.\frac{\partial\Phi_E}{\partial t}\right|_{\substack{a\\ surf\,2}} + \mu_0 I_{\substack{pierce\\ surf\,2}}$$

$$\oint \vec{B}\cdot d\vec{l} = 0 + \mu_0\left(I_{wire} - I_{plate}\right)$$

(the minus sign since, while the wire's current flows *in* through the surface, the current in the plate flows radially *out* through the surface.)

Of course, regardless of the area you choose, it's the same loop over which you're evaluating the magnetic field, so

$$\mu_0\varepsilon_0 \left.\frac{\partial\Phi_E}{\partial t}\right|_{\substack{a\\ surf\,1}} = \oint \vec{B}\cdot d\vec{l} = \mu_0\left(I_{wire} - I_{plate}\right)$$

Thus

$$I_{plate} = I_{wire} - \varepsilon_0 \left.\frac{\partial\Phi_E}{\partial t}\right|_{\substack{a\\ surf\,1}}$$

We found in part b) $\varepsilon_0 \frac{\partial \Phi_E}{\partial t} = \frac{I_{wire}}{a^2} s^2$

So, $I_{plate} = I_{wire} - \frac{I_{wire}}{a^2} s^2 = I_{wire}\left(1 - \left(\frac{s}{a}\right)^2\right)$

Problem 7.33 – Displacement current in a coaxial cable with alternating current

For Problem 7.16, a current $I(t) = I_0 \cos(\omega t)$ flows down a long, straight, thin wire and returns along a thin, coaxial conducting tube of radius *a*. The electric field for the region $s < a$ is

$$\vec{E}(s,t) = \frac{\mu_0 I_0 \ell \omega}{2\pi} \ln\left(\frac{a}{s}\right) \sin(\omega t)\hat{z}.$$

a. Find the displacement current density $\vec{J}_d$.

$$\vec{J}_d = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \frac{\varepsilon_0 \mu_0 I_0 \omega^2}{2\pi} \ln\left(\frac{a}{s}\right) \cos(\omega t)\hat{z} = \frac{\varepsilon_0 \mu_0 I \omega^2}{2\pi} \ln\left(\frac{a}{s}\right)\hat{z},$$

because $I = I_0 \cos(\omega t)$.

b. Integrate to get the total displacement current I_d. (What direction does it "flow"?)

The displacement current "flows" in the *z* direction inside the coaxial cable, so we need to integrate over a circle of radius *a*. Since $\vec{J}_d$ depends on *s*, so divide the area into thin rings between *s* and *s* + *ds*.

$$I_d = \int \vec{J}_d \cdot d\vec{a} = \int_0^a J_d(s) \cdot (2\pi s\, ds) = \varepsilon_0 \mu_0 I \omega^2 \int_0^a \ln\left(\frac{a}{s}\right) s\, ds$$

$$= \varepsilon_0 \mu_0 I \omega^2 \int_0^a (s \ln a - s \ln s) ds = \varepsilon_0 \mu_0 I \omega^2 \left[\frac{s^2}{2} \ln a - \frac{s^2}{2} \ln s + \frac{s^2}{4}\right]_0^a$$

$$= \frac{\varepsilon_0 \mu_0 I \omega^2 a^2}{4}$$

c. Compare *I* and I_d. What is their ratio? If the outer cylinder has a diameter of 2 mm, how high would the frequency have to be for I_d to be 1% of *I*?

The product of the electric and magnetic constants is $\varepsilon_0 \mu_0 = 1/c^2$, so

$$\frac{I_d}{I} = \frac{\varepsilon_0 \mu_0 \omega^2 a^2}{4} = \left(\frac{\omega a}{2c}\right)^2 = \frac{1}{100},$$

$$\frac{\omega a}{2c} = \frac{1}{10},$$

$$\omega = \frac{c}{5a} = \frac{3\times 10^{8}\,\text{m/s}}{5(0.001\text{ m})} = 6\times 10^{10}\,\text{Hz}.$$

Note that a is the radius, not the diameter. The (ordinary, not angular frequency) is

$$f = \frac{\omega}{2\pi} = \frac{6\times 10^{10}\,\text{Hz}}{2\pi} \approx 10^{10}\,\text{Hz},$$

Or 10 GHz, which is way above radio, but some devices are pushing these frequencies.

7.34 (unless its homework)

Summary

Maxwell's Equations (Complete!) – in both differential and integral forms

Gauss's law

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \oint \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$$

No name

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \oint \vec{B}\cdot d\vec{a} = 0$$

Faraday's law

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \oint \vec{E}\cdot d\vec{\ell} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_{surface}\vec{B}\cdot d\vec{a}$$

Ampere's law – this is incomplete!

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} = \mu_0\left(\vec{J}+\vec{J}_d\right)$$

$$\oint \vec{B}\cdot d\vec{\ell} = \mu_0 I_{enc} + \mu_0\varepsilon_0\frac{d}{dt}\int \vec{E}\cdot d\vec{a} = \mu_0 I_{enc} + \mu_0\varepsilon_0\frac{d\Phi_{elec}}{dt} = \mu_0\left(I_{enc}+I_{d,enc}\right)$$

Derivations

There's a long history of people trying to derive Maxwell's relations from something more concise. For example, Heras (AJP 75 p 652) demonstrates that more generally, if you have a scalar a vector "source" that are time dependent and related by a continuity equation, then you can define associated fields that obey, essentially, Maxwell's Equations. Another paper shows that, if you start with coulomb's law for a stationary charge, and transform into a frame in which the charge is accelerating, you get all the right fields. So, it seems that these laws are not necessarily fundamental in the sense that 'that's just the way nature works' rather they may be considered to follow from something still more fundamental. Feynman offers the word of caution that most 'derivations' involve assumptions that themselves bear justification – thus it just rephrases the question. On the other hand, the reasonability of an assumption is in the eye of the beholder.

Boundary Conditions

Suppose there is a surface with a charge density σ (may depend on position). The electric field just above and below the boundary are related by

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{1}{\varepsilon_0}\sigma \qquad \vec{E}^{||}_{\text{above}} = \vec{E}^{||}_{\text{below}}$$

These relations can be summarized by

$$\vec{E}_{\text{above}} - \vec{E}_{\text{below}} = \frac{\sigma}{\varepsilon_0}\hat{n},$$

where $\hat{n}$ is a unit normal vector that points "above." The electric (scalar) potential is continuous across the boundary, $V_{\text{above}} = V_{\text{below}}$, but its derivatives are not all ($\vec{E} = -\vec{\nabla}V$).

Suppose that there is a surface current $\vec{K}$ on a boundary.

$$B^{\perp}_{above} = B^{\perp}_{below} \qquad B^{||}_{above} - B^{||}_{below} = \mu_0 K \text{ (not in same direction)}$$

These relations can be summarized by

$$\vec{B}_{above} - \vec{B}_{below} = \mu_0\left(\vec{K} \times \hat{n}\right)$$

The magnetic vector potential is continuous across the boundary, $\vec{A}_{above} = \vec{A}_{below}$, but not all of its derivatives are ($\vec{B} = \vec{\nabla} \times \vec{A}$).