Subgroups of Order 4

(a.k.a. More groups and subgroups!)

In an assignment titled, "Groups of Order 3 and 4," we discovered that there are only two groups of order 4, up to isomorphism (remember: *iso* = same, *morph* = form).

One of these two groups of order 4 is the cyclic group of order 4. Here are some familiar examples of cyclic groups of order 4. Make sure you can identify the identity element, the two generators, and the element of order 2 in each group.

$\mathbf{Z}_4$ = {0, 1, 2, 3} under addition modulo 4

U(5) = {1, 2, 3, 4} under multiplication modulo 5

U = {1, –1, *i*, –*i* } under ordinary multiplication

U(10) = {1, 3, 7, 9} under multiplication modulo 10

The other group of order 4 is Abelian, but not cyclic. Here are two familiar examples of the non-cyclic group of order 4. Since the group is not cyclic, none of its elements has order 4. Note that each non-identity element in the group has the same order and that every element is its own inverse.

U(8) = {1, 3, 5, 7} under multiplication modulo 8

Group of symmetries of the rectangle, W = {R_0, R_{180}, H, V}

Here is one more example of the non-cyclic group of order 4.

$\mathbf{Z}_2$ x $\mathbf{Z}_2$ = {00, 01, 10, 11} under component-wise addition modulo 2.

This means, for instance, 01 + 10 = 11, 10 + 11 = 01, and 10 + 10 = 00.

Exercise 1. Write the group table for $\mathbf{Z}_2$ x $\mathbf{Z}_2$.

If every group of order 4 is of one of two forms, then the same is true for every subgroup of order 4. Notice that the subgroup $< 2 > = \{0, 2, 4, 6\} = < 6 >$ of $\mathbf{Z}_8$ is a cyclic group of order 4 (under addition modulo 8). The group $\mathbf{Z}_4 \times \mathbf{Z}_2$ has both cyclic and non-cyclic subgroups of order 4. The group $\mathbf{Z}_4 \times \mathbf{Z}_2$ has 8 elements, including 01, 20, and 31. The group operation is component-wise addition, with addition modulo 4 in the first component and addition modulo 2 in the second component. Thus, for instance, $20 + 31 = 11$ and $31 + 31 = 20$.

Exercise 2. List the 8 elements of $\mathbf{Z}_4 \times \mathbf{Z}_2$. Then find two cyclic subgroups of order 4 and one non-cyclic subgroup of order 4 in $\mathbf{Z}_4 \times \mathbf{Z}_2$.

Instructions for Exercises 2-4: For a cyclic subgroup of order 4, list the elements of the subgroup, identify at least one generator for the subgroup, and show how the generator produces the elements of the subgroup. For a non-cyclic subgroup of order 4, write the group table.

Exercise 3. Find one cyclic subgroup of order 4 and two non-cyclic subgroups of order 4 in D_4, the dihedral group of order 8. Recall that D_4 consists of the symmetries of the square and that its group table appears in your textbook.

Exercise 4. Find at least one cyclic subgroup of order 4 and at least one non-cyclic subgroup of order 4 in S_4, the symmetric group of degree 4. Recall that S_4 consists of the $4! = 24$ permutations of the set $\{1, 2, 3, 4\}$.