Proofs of Theorems about Groups

Prove the following theorems by filling in the blanks and the empty spaces following implies (→) and equals (=) signs.

1. **Theorem.** Let G be a group and let $a, b \in G$. If $(ab)^2 = a^2b^2$, then $ab = ba$.

   Proof: $(ab)^2 = a^2b^2$

   →

   → $ba = ab$.

2. **Theorem.** Let G be a group and let $a, b \in G$. Then $(a^{-1}ba)^n = a^{-1}b^na$ for every positive integer n, $n \geq 2$.

   Proof by mathematical induction:

   Base step: For $n = 2$, $(a^{-1}ba)^2 =$ ________________________________ $= a^{-1}b^2a$.

   Inductive step: Inductive hypothesis: Assume ____________________________.

   Then $(a^{-1}ba)^{n+1}$

   $=$

   $=$ (by the inductive hypothesis)

   $=$

   $= a^{-1}b^{n+1}a$.

   By the Principle of Mathematical Induction, we have $(a^{-1}ba)^n = a^{-1}b^na$ for every positive integer n, $n \geq 2$.

3. **Theorem.** Let G be an Abelian group and let $a, b \in G$. Then $(ab)^n = a^n b^n$ for every positive integer n, $n \geq 2$.

Proof by mathematical induction:

Base step: For $n = 2$,

Inductive step: Inductive hypothesis: Assume ____________________________.

Then $(ab)^{n+1}$

=

= (by the inductive hypothesis)

=

$= a^{n+1} b^{n+1}$. (Be sure to point out where you use that the group G is Abelian.)

By the Principle of Mathematical Induction, we have $(ab)^n = a^n b^n$ for every positive integer n, $n \geq 2$.

4. Let G be a group and let $a, b, c \in G$. Then $(ab^{-1}c)^{-1} =$ _______________________

because $(ab^{-1}c)$___________________ $= e$ and ___________________$(ab^{-1}c) = e$.

5. **Theorem.** Let G be a group and let $a_1, a_2, \ldots, a_n \in G$, where n is a positive integer, $n \geq 2$. Then $\left(a_1 a_2 \cdots a_n\right)^{-1} =$ ________________________.

Proof by mathematical induction: