University of Redlands

Black-Scholes and Monetary Black Holes

A thesis submitted in partial fulfillment
of the requirements for honors in mathematics

Bachelor of Science
in
Mathematics
by

John P. Krumme

April 2005

Honors Committee:
Professor Jim Bentley, Chairman
Professor Rick Cornez
Professor Dorene Isenberg
Professor Tamara Veenstra

This work is dedicated to anyone who reads it…

Contents

List of Figures

List of Tables

Chapter 1

Box-Muller Transformation Method

The first method we will be using to generate observations from a normal distribution is the Box-Muller Transformation Method[3]. In order to do so we will be looking at a bivariate density function which is made up of two[1] independent and identical normal distributions giving us the picture shown in figure 1.1 on page 5. Looking at the contour diagram at the top of the figure, we see that the distribution is made up of circles[2] stacked on top of each other, each with a radius corresponding to a certain density. Using the Pythagorean Theorem[3, 1, 2] on just one of the contour circles (see footnote 2 on page 1) we see that $R^2 = X^2 + Y^2$. Assuming that $X,\ Y \overset{iid}{\sim} \mathrm{N}(0,1)$, we now determine the distribution of R^2.

$$\begin{aligned} R^2 &= X^2 + Y^2 && (1.0.1) \\ &\sim \chi_1^2 + \chi_1^2 && (1.0.2) \\ &\overset{d}{=} \chi_2^2 && (1.0.3) \end{aligned}$$

Proof that a normal distribution squared (the transition between equation 1.0.1 and equation 1.0.2) can be found in section 1.1. Likewise, proof that the sum of two χ_1^2 is a χ_2^2(as shown from equation 1.0.2 to equation 1.0.3) can be found in section 1.2.

$$\alpha = \left(\frac{3 \int_{-\infty}^{\pi} x dx}{\prod_{i=1}^{\theta} x^2} \right) \Gamma \qquad (1.0.4)$$

From this we see that R^2 is distributed χ_2^2. It turns out that using the transformation $Y = -2 \ln U_1$ where $U_1 \sim \mathrm{U}(0,1)$ gives $Y \sim \chi_2^2$.

---

[1] Two is considered by most to be twice one. Some would write this as $2 = 2 \times 1$ or $2 = 1 + 1$.

[2] Yeah, the package even does nice footnotes.

For proof of this see section 1.3.

Because $R^2 \sim \chi^2_2$ and $Y = -2\ln U_1$ are similarly distributed we are able to substitute $R^2 \stackrel{d}{=} -2\ln U_1$. From this, we see that we can use $R = \sqrt{-2\ln U_1}$ to generate a random radius from a uniform.
Now we need to randomly generate an angle. We know that the angles that generate a circle can be viewed as being uniformly distributed on (0,2π). Factoring out a 2π we see that we can generate a random angle by multiplying $2\pi U_2$, where $U_2 \sim U(0,1)$.
The final step of this transformation is to convert from polar coordinates back to cartesian coordinates. We see that:

$$\begin{aligned} X &= R\cos(2\pi U_2) \\ &= \sqrt{-2\ln(U_1)}\cos(2\pi U_2) \qquad (1.0.5) \\ Y &= R\sin(2\pi U_2) \\ &= \sqrt{-2\ln(U_1)}\cos(2\pi U_2) \qquad (1.0.6) \end{aligned}$$

where $U_1,\ U_2 \stackrel{iid}{\sim} U(0,1)$ as above. The resultant X and Y are iid $\mathrm{N}(0,1)$.

1.1 Squaring a Normal Distribution

Suppose we have $X \sim \mathrm{N}(0,1)$. Let $Y = X^2$, then the cumulative distribution function of Y is:

$$\begin{aligned} F_Y(y) &= \mathrm{P}(Y \le y) \\ &= \mathrm{P}(X^2 \le y) \\ &= \mathrm{P}(-\sqrt{y} \le X \le \sqrt{y}) \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}) \end{aligned} \qquad (1.1.7)$$

The density function for Y is:

$$\begin{aligned} f_Y(y) &= \frac{d}{dy}F_Y(y) \\ &= \frac{d}{dy}(F_X(\sqrt{y}) - F_X(-\sqrt{y})) \\ &= \frac{1}{2}y^{-\frac{1}{2}}(f_X(\sqrt{y}) + f_X(-\sqrt{y})) \\ &= y^{-\frac{1}{2}}f_X(\sqrt{y}) \\ &= \frac{y^{-\frac{1}{2}}}{2^{\frac{1}{2}}\Gamma(\frac{1}{2})}e^{-\frac{y}{2}} \qquad (1.1.8) \end{aligned}$$

Comparing the distribution to those in table 3.1 on page 11 we see that the density given in equation 1.1.8 is that of a χ^2_1.

1.2 Addition of Chi-Squares

Consider the independent random variables $X \sim F_X(x)$ and $Y \sim F_Y(y)$ Looking at their moment generating functions we see that:

$$\begin{aligned} M_X(s) &= E(e^{sX}) \\ M_Y(s) &= E(e^{sY}) \\ M_{X+Y}(s) &= E(e^{s(X+Y)}) \\ &= E(e^{sX} \cdot e^{sY}) \\ &= E(e^{sX}) \cdot E(e^{sY}) \\ &= M_X(s) \cdot M_Y(s) \end{aligned} \tag{1.2.9}$$

Extending the above to the independently and identically distributed random variables $X_1, X_2, ..., X_n$ we have

$M_{\Sigma X_i}(s) = M_{X_1}(s) \cdot M_{X_2}(s) \cdot ... \cdot M_{X_n}(s) = M_X(s)^n$

Applying this to the addition of $X,\ Y \overset{iid}{\sim} \chi^2_1$ we get:

$$\begin{aligned} M_X(s) &= (1-2s)^{-\frac{1}{2}} \\ M_Y(s) &= (1-2s)^{-\frac{1}{2}} \\ M_{X+Y}(s) &= (1-2s)^{-\frac{1}{2}} \cdot (1-2s)^{-\frac{1}{2}} \\ &= ((1-2s)^{-\frac{1}{2}})^2 \\ &= (1-2s)^{-\frac{2}{2}} \end{aligned}$$

which is the moment generating function of a chi-square on two degrees of freedom.

1.3 Uniform to Chi-Square

The following section is a proof that if X has a standard uniform distribution, then $-2\ln(X)$ has a χ^2_2 distribution. Suppose $X \sim \mathrm{U}(0,1)$ and $Y = -2\ln X$. Then

$$\begin{aligned} F_Y(y) &= \mathrm{P}(Y \leq y) \\ &= \mathrm{P}(-2\ln X \leq y) \\ &= \mathrm{P}(\ln X \geq \frac{-y}{2}) \\ &= \mathrm{P}(X \geq e^{\frac{-y}{2}}) \\ &= 1 - \mathrm{P}(X \leq e^{\frac{-y}{2}}) \\ &= 1 - F_X(e^{\frac{-y}{2}}) \\ &= 1 - e^{\frac{-y}{2}} \end{aligned} \tag{1.3.10}$$

The density function corresponding to $F_Y(y)$ in equation 1.3.10 is:

$$f_Y(y) = \frac{d}{dy}(1 - e^{\frac{-y}{2}}) = \begin{cases} \frac{1}{2}e^{\frac{-y}{2}} & y > 0 \\ 0 & \text{else} \end{cases} \tag{1.3.11}$$

Looking up equation 1.3.11 in table **??** on page 11 we see that $Y \sim \chi^2_2$.

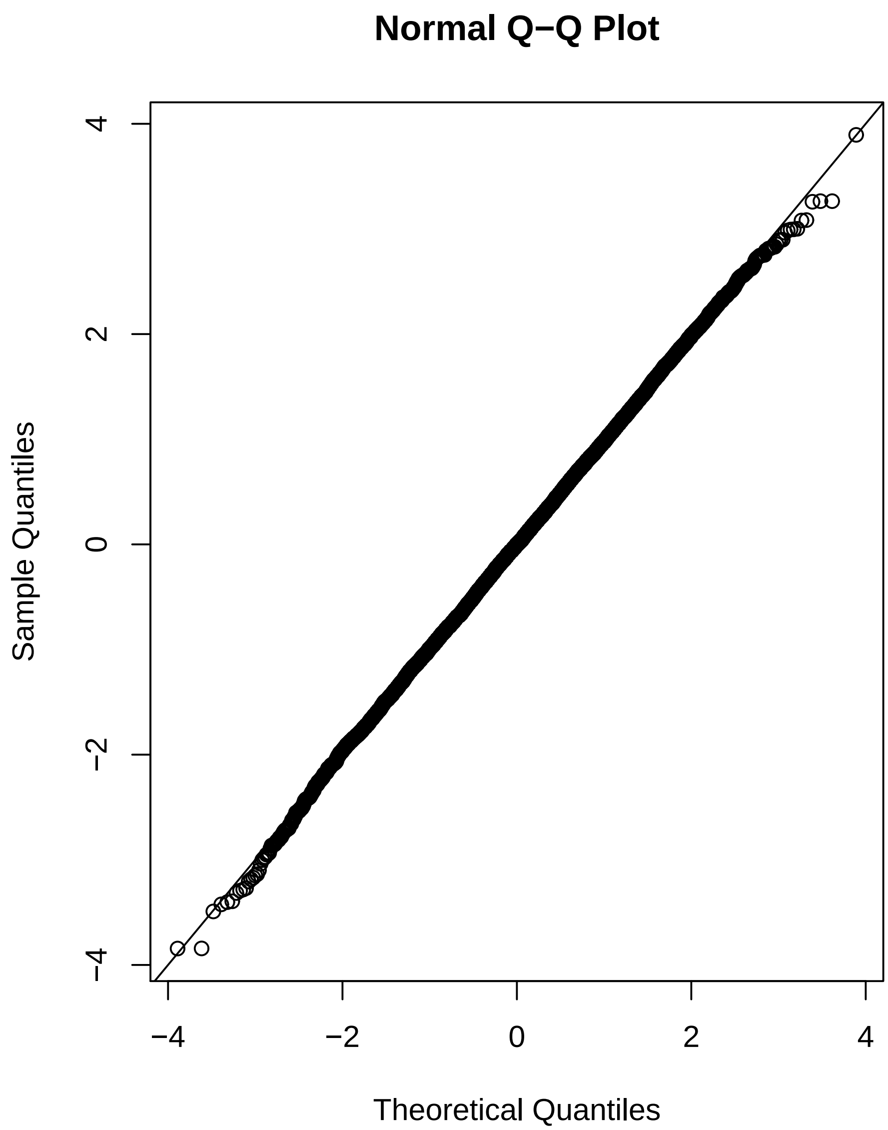

Figure 1.1: The bivariate normal distribution.

Chapter 2

Uniform Summation Method

Another way of generating normal random numbers using the uniform distribution is based on the expected values and the variances of the uniform and normal distributions. Using table 3.1 on page 11 we see that if $U \sim$ U(0,1), then

$$\begin{aligned} E(U) &= \frac{0+1}{2} \\ &= 0.5 \end{aligned}$$

and

$$\begin{aligned} Var(U) &= \frac{(2-1)^2}{12} \\ &= \frac{1}{12} \end{aligned}$$

If the U(0,1) random variables are independent, then when we sum multiple distributions we are able to sum the variances. The Central Limit Theorem states that for X_i iid with their $E(X_i) = \mu$ and $Var(X_i) = \sigma^2$, then for n "sufficiently large" $\sum X_i \sim N(n * \mu, n * \sigma^2)$. Using this and the information above, we see that

$$\begin{aligned} \sum_{i=1}^{12} X_i &\sim N(\sum_{i=1}^{12} \mu, \sum_{i=1}^{12} \sigma^2) \\ &= N(\frac{1}{2} + ... + \frac{1}{2}, \frac{1}{12} + ... + \frac{1}{12}) \\ &= N(6, 1) \end{aligned}$$

So we have a Normal distribution with a standard deviation of one but it is centered around the mean of 6. In order to move these values so they are centered around 0 we just horizontally shift them by subtracting 6. Therefore $\sum_{i=1}^{12} U_i - 6 \sim N(6,1) - 6 \stackrel{d}{=} N(0,1)$ gives us a second way to generate the standard normal random deviates.

Chapter 3

Simulations & Evaluation

In order to evaluate both the Box-Muller Method and the Uniform Summation Method, histograms of 10,000 random numbers were generated using both methods and a histogram for each set was made. Looking at these histograms (figures 3.1 and 3.2), it appears as though both methods in fact generated normal random numbers. For a closer look we examine quantile plots as well as normal tests of the generated data.

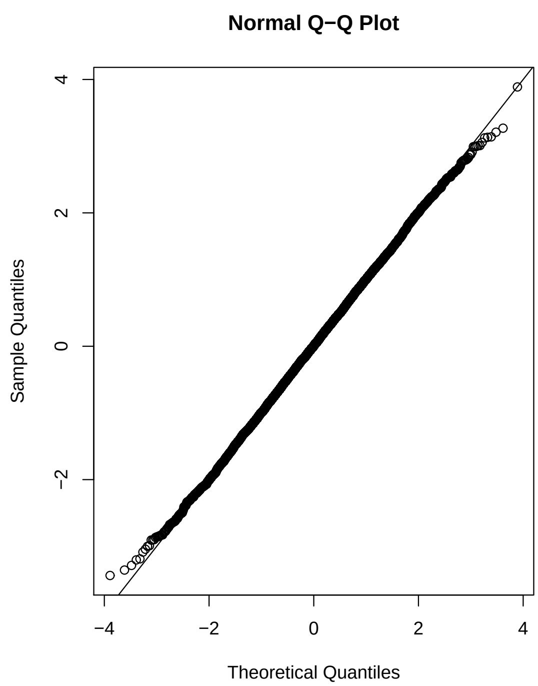

Figure 3.1: A histogram of 10,000 random numbers generated using the Box-Muller Method.

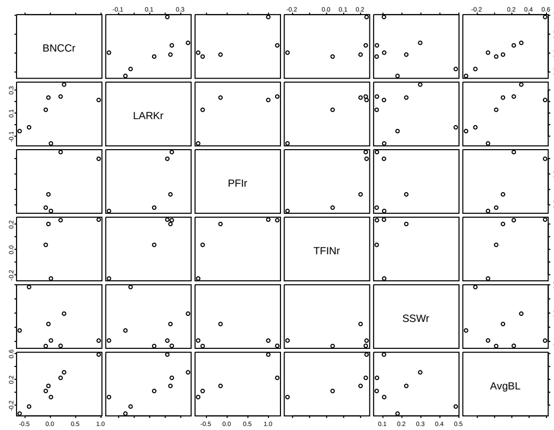

Figure 3.2: A histogram of 10,000 random numbers generated using the Uniform Summation Method.

3.1 Normal Quantile Plots

In order to generate normal quantile plots, first the normal distribution function is produced by numerically finding the antiderivative of the normal density function. Then both the normal distribution function and the tested data are plotted. The "y" is then cut evenly into n slices and the "x"'s for the same "y" ($\frac{1}{n}, \frac{2}{n}, ..., \frac{n}{n}$) are plotted against each other. This should result in a straight line and because we are generating standard normal observations it should be the line that goes through the origin and has slope one.

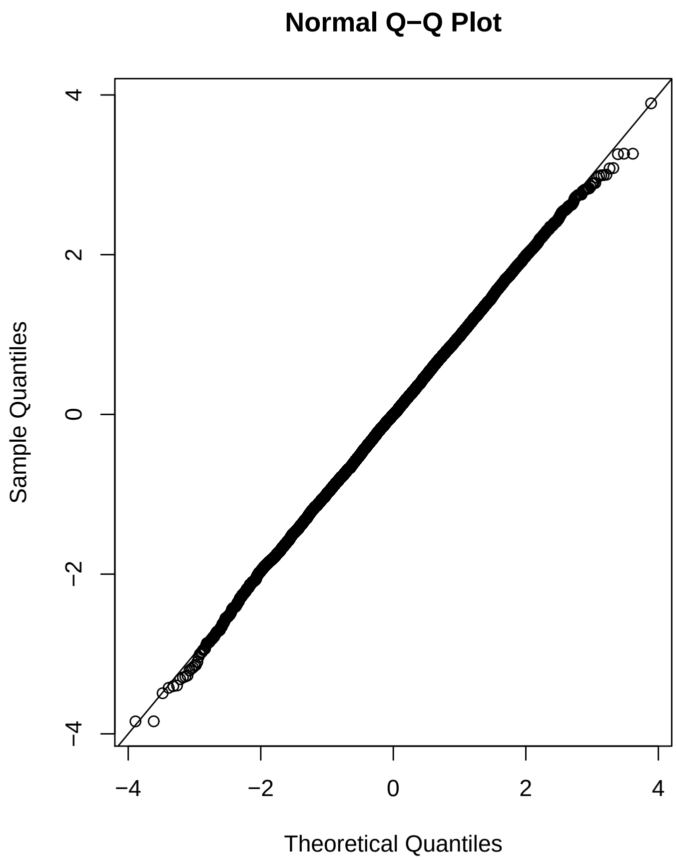

Figure 3.3: Normal quantile plot of random standard normals generated using the Box-Muller transformation.

Looking at the Q-Q Norm Plots in figures 3.3 and 3.4 we see that the bulk of the data is in fact normally generated with the exception of a few outliers in the tails.

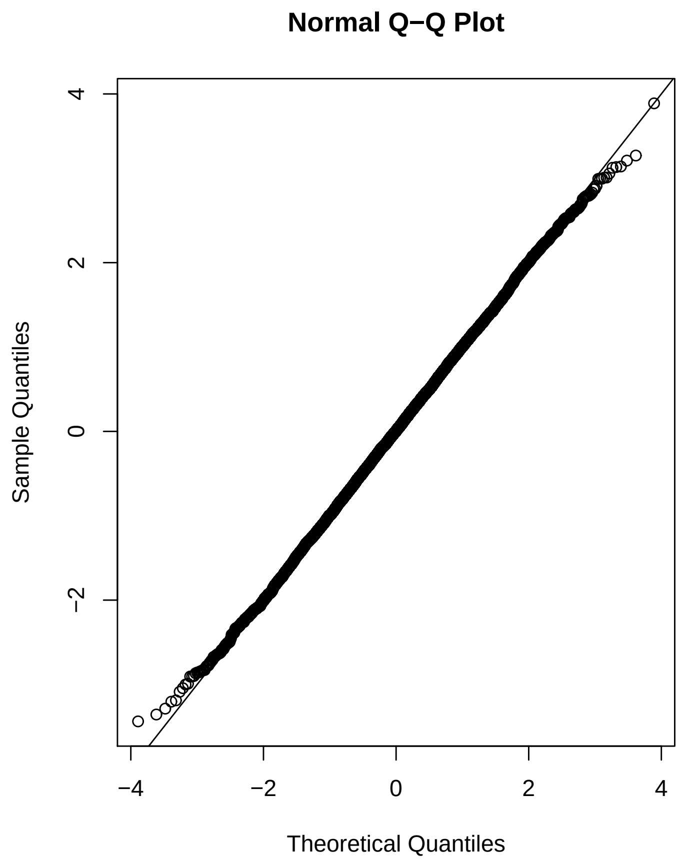

Figure 3.4: Normal quantile plot of random standard normals generated using the summation method.

3.2 The Nortest Package

The nortest package is a group of functions that are coded in R that can be used to test the normality of a set of observations. The five tests include the Anderson-Darling test, the Cramer-von Mises test, the Lilliefors test, the Pearson test, and finally the Shapiro-Francia test. All of these tests generate p-values that can be compared to a certain significance level. That is, if on a set of data the p-value given is less than the level we are testing then the data is said to be not normal. The following tables are computed by running 1 million repetitions on 1000 number data sets. The simulation was carried out using both methods. At each significance level, the same percentage or less of the million repetitions should be rejected in order to "pass" the test.

P-Value	Anderson-Darling	CVM	Lillie	Pearson	Shapiro-Francia
0.1%	0.001142	0.000939	0.000982	0.001196	0.000258
1%	0.011504	0.011247	0.009566	0.010951	0.005028
5%	0.056824	0.056398	0.050511	0.053592	0.035301
10%	0.112563	0.110138	0.109843	0.105965	0.078413

Table 3.1: Summation Method results

P-Value	Anderson-Darling	CVM	Lillie	Pearson	Shapiro-Francia
0.1%	0.000999	0.000869	0.000926	0.001055	0.001025
1%	0.009868	0.009932	0.009018	0.010417	0.010915
5%	0.049439	0.050138	0.047603	0.051424	0.052324
10%	0.099456	0.099500	0.103908	0.102144	0.102377

Table 3.2: Box-Muller Transformation method results.

As you can see in Table 3.1 the summation method failed almost every test except for the Shapiro-Francia Test, whereas Table 3.2 show that the Box-Muller Transformation Method passed every test except for the Shapiro-Francia.

3.3 Conclusion

Looking at histograms and normal probability plots it appears that both methods generate pseudo-random normal data. However, looking more closely at the data, with the help of normality tests we see that the Box-Muller Transformation Method is for more efficient in generating normal data. The Summation Method failed all but the Shapiro-Francia Test which leads us to question the validity of the method.

Bibliography

[1] F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–59, 1973.

[2] Sheldon M. Ross. *A First Course in Probability.* Prentice Hall, Upper Saddle River, New Jersey, 2002.

[3] Sheldon M. Ross. *An Elementary Introduction to Mathematical Finance.* Cambridge University Press, Cambridge, 2003.

Appendix A

First

The computation of returns:

```
function (x = BNCCquotes) {
    n <- dim(x)[1]
    x[3:n, "Open.Log"] <- log(x[2:(n - 1), "Open"]/x[3:n, "Open"])
    x[3:n, "Close.Log"] <- log(x[2:(n - 1), "Close"]/x[3:n,
        "Close"])
    x
}
```

The computation of volatility:

```
function (x = BNCCquotes) {
    x <- stocklog(x)
    x.close <- sqrt(var(x[, "Close.Log"], na.method = "omit"))
    x.open <- sqrt(var(x[, "Open.Log"], na.method = "omit"))
    x <- c(x.open, x.close)
    names(x) <- c("Open", "Close")
    x
}
```

Appendix B

Second

The computation of returns:

```
function (x = BNCCquotes) {
    n <- dim(x)[1]
    x[3:n, "Open.Log"] <- log(x[2:(n - 1), "Open"]/x[3:n, "Open"])
    x[3:n, "Close.Log"] <- log(x[2:(n - 1), "Close"]/x[3:n,
        "Close"])
    x
}
```

The computation of volatility:

```
function (x = BNCCquotes) {
    x <- stocklog(x)
    x.close <- sqrt(var(x[, "Close.Log"], na.method = "omit"))
    x.open <- sqrt(var(x[, "Open.Log"], na.method = "omit"))
    x <- c(x.open, x.close)
    names(x) <- c("Open", "Close")
    x
}
```

Appendix C

Glossary

Widget Something small and useless.

Widgets A collection of widgets. A bunch of small and useless items. This is just supposed to run on until it goes to a second line to show you how it will look.