Rao-Blackwell

Theorem 1.1 *Let $\mathbf{X} \sim f_{\mathbf{X}}(\mathbf{x}, \theta)$ and T be sufficient for θ, $\mathbf{x} \in \mathfrak{X}$ and $t \in \mathfrak{T}$. Let U be any unbiased estimator for $g(\theta)$. Define $V_t = \mathrm{E}(U|T = t)$. Then V is an unbiased estimator for $g(\theta)$ and $\mathrm{Var}(V) \leq \mathrm{Var}(U)$ with equality iff $V = U$ with probability one.*

Proof 1.1 *Since $U = U(\mathbf{X})$ is an estimator, it is also a statistic. And, since T is sufficient for θ we have*

$$\begin{aligned} V &= \mathrm{E}(U|T = t) && (1) \\ &= \int_{\mathfrak{T}} u(x) f_{X|T}(x|T = t)\ dx && (2) \end{aligned}$$

By Fisher, and noting that $u(x)$ is a function of x and not θ, we see that V is θ-free. Thus, V is a statistic as well.

Further,

$$\begin{aligned} \mathrm{E}(U) &= g(\theta) && (3) \\ &= \int_{\mathfrak{X}} u(x) f_X(x, \theta)\ dx && (4) \\ &= \int_{\mathfrak{T}} \left[\int_{X \in T = t} u(x) f_{X|T}(x|T = t)\ dx \right] f_T(t, \theta)\ dt && (5) \\ &= \int_{\mathfrak{T}} v(t) f_T(t, \theta)\ dt && (6) \\ &= \mathrm{E}(V) && (7) \end{aligned}$$

So, V is unbiased.

Now,

$$\begin{aligned} \mathrm{Var}(U) &= \mathrm{E}\left(U - \mathrm{E}(U)\right)^2 && (8) \\ &= \mathrm{E}\left(U - \mathrm{E}(V)\right)^2 && (9) \\ &= \mathrm{E}\left((U - V)^2\right) + \mathrm{E}\left((V - \mathrm{E}(V))^2\right) + 2\mathrm{E}\left((U - V)(V - \mathrm{E}(V))\right) && (10) \end{aligned}$$

Since we know that $\mathrm{E}(U) = \mathrm{E}(V)$ by above,

$$\begin{aligned} \mathrm{E}\left((U - V)(V - \mathrm{E}(V))\right) &= \int_{\mathfrak{X}} (V - \mathrm{E}(V))(U - V) f_X(x, \theta)\ dx && (11) \\ &= \int_{\mathfrak{T}} (V - \mathrm{E}(V)) \left[\int_{X \in T = t} (U - V) f_{X|T}(x|T = t)\ dx \right] f_T(t, \theta)\ dt && (12) \\ &= \int_{\mathfrak{T}} (V - \mathrm{E}(V))[0] f_T(t, \theta)\ dt && (13) \\ &= 0 && (14) \end{aligned}$$

and thus

$$\begin{aligned} \mathrm{Var}(U) &= \mathrm{E}\left((U - V)^2\right) + \mathrm{E}\left((V - \mathrm{E}(V))^2\right) && (15) \\ &\geq \mathrm{E}\left((V - \mathrm{E}(V))^2\right) && (16) \\ &\geq \mathrm{Var}(V) && (17) \end{aligned}$$

with equality iff $\mathrm{E}\left((U - V)^2\right) = 0$ or $V = U$ with probability one.

Example 1.1 *Let* $X_i \overset{iid}{\sim} N\left(\mu, \sigma_0^2\right)$ *so that* $\theta = \mu$*. By exponential family we see that* $T = \sum_{i=1}^n X_i$ *is min suff for* $\theta = \mu$*.*

Let $U = X_1$ *with* $\mathrm{E}(U) = \mathrm{E}(X_1) = \mu$*. Thus,* U *is an unbiased estimator for* θ *with variance* $\mathrm{Var}(U) = \mathrm{Var}(X_1) = \sigma_0^2$*.*

Note that $T = \sum_{i=1}^n X_i = U + \sum_{i=2}^n$ *and that*

$$f_{U|T}(u|T=T) = \frac{f_{T|U}(t|U=u) f_U(u)}{f_T(t)}$$

where $U \sim N(\mu, \sigma_0^2)$*,* $T \sim N(n\mu, n\sigma_0^2)$*, and* $T|U \sim N\left((n-1)\mu + u, (n-1)\sigma_0^2\right)$*. Hence*

$$\begin{aligned}
f_{U|T}(u|T=t) &= \frac{\left(2\pi(n-1)\sigma_0^2\right)^{-1/2} \exp\left(-\frac{(t-(n-1)\mu-u)^2}{2(n-1)\sigma_0^2}\right) \left(2\pi\sigma_0^2\right)^{-1/2} \exp\left(-\frac{(u-\mu)^2}{2\sigma_0^2}\right)}{(2\pi n\sigma_0^2)^{-1/2} \exp\left(-\frac{(t-n\mu)^2}{2n\sigma_0^2}\right)} && (18)\\
&= \sqrt{\frac{n}{n-1}} \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left[-\frac{1}{2\sigma_0^2}\left(\frac{(t-(n-1)\mu-u)^2}{n-1} + \frac{(u-\mu)^2}{1} - \frac{(t-n\mu)^2}{n}\right)\right] && (19)\\
&= \sqrt{\frac{n}{n-1}} \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left[-\frac{1}{2\sigma_0^2}\left(\frac{t^2}{n-1} + (n-1)\mu^2 + \frac{u^2}{n-1} - 2t\mu - \frac{2tu}{n-1} + 2\mu u\right.\right. && (20)\\
&\qquad \left.\left. -u^2 + \mu^2 - 2\mu u - \frac{t^2}{n} - n\mu^2 + 2t\mu\right)\right] && (21)\\
&= \sqrt{\frac{n}{n-1}} \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left[-\frac{1}{2\sigma_0^2}\left(\frac{u^2}{n-1} + u^2 - \frac{2tu}{n-1} + \frac{t^2}{n-1} - \frac{t^2}{n}\right)\right] && (22)\\
&= \sqrt{\frac{n}{n-1}} \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left[-\frac{1}{2\sigma_0^2}\left(\frac{n}{n-1}u^2 - \frac{2t}{n-1}u + \frac{t^2}{(n-1)n}\right)\right] && (23)\\
&= \sqrt{\frac{n}{n-1}} \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left[-\frac{1}{2\sigma_0^2\left(\frac{n-1}{n}\right)}\left(\left(u - \frac{t}{n}\right)^2\right)\right] && (24)
\end{aligned}$$

So, $U|T \sim N\left(\frac{t}{n}, \frac{n-1}{n}\sigma_0^2\right)$ *and thus*

$$\begin{aligned}
V &= \mathrm{E}(U|T=t) && (25)\\
&= \frac{T}{n} && (26)\\
&= \frac{\sum_{i=1}^n X_i}{n} && (27)\\
&= \overline{X} && (28)
\end{aligned}$$

with $\mathrm{E}(V) = \mathrm{E}(\overline{X}) = \mu$ *and* $\mathrm{Var}(V) = \mathrm{Var}(\overline{X}) = \frac{\sigma_0^2}{n}$*. Note that* $\frac{\sigma_0^2}{n} \to 0$ *as* $n \to \infty$ *which is better than* $\mathrm{Var}(U) = \sigma_0^2$ *unless* $n = 1$*, in which case* $V = U$*.*