

9	Fri., 10/31	4.4.3 Addition of Angular Momenta	Daily 9.F
10	Mon. 11/3	Review (Ch 3-4)	Weekly 10
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	Fri., 11/7	5.1 Identical Particles: 2-Particle Systems (Q8.6, 11.5)	Daily 10.F

Equipment

- Griffith's text
- Moore's Text
- Spatial physics talk for particle – physics slides

Check dailies

4.4 Spin

4.4.3 Addition of Angular Momentum

Let's start by looking at how things work classically, then do the simple translation to quantum mechanically.

Classically

Classically, angular momentum adds. That is, the Earth's angular momentum about the sun is the sum of the angular momenta of each rock and dirt clod of the Earth about the sun.

$$\vec{L}_{\text{Earth.aboutsun}} = \sum_i \vec{L}_{i.\text{aboutsun}} = \sum_i \vec{p}_i \times \vec{r}_{\text{sun} \rightarrow i}$$

But that that separation vector can be rephrased as the sum of two separation vectors: that from the sun to the center of the earth and that from the center of the earth to each rock and dirt clod.

$$\vec{r}_{\text{sun} \rightarrow i} = \vec{r}_{\text{sun} \rightarrow \text{earth}} + \vec{r}_{\text{earth} \rightarrow i}$$

So,

$$\vec{L}_{\text{Earth.aboutsun}} = \sum_i \vec{p}_i \times (\vec{r}_{\text{sun} \rightarrow \text{earth}} + \vec{r}_{\text{earth} \rightarrow i}) = \sum_i \vec{p}_i \times \vec{r}_{\text{sun} \rightarrow \text{earth}} + \sum_i \vec{p}_i \times \vec{r}_{\text{earth} \rightarrow i}$$

$$\vec{L}_{\text{Earth.aboutsun}} = \vec{L}_{\text{orbitalaboutsun}} + \vec{S}_{\text{rotationalaboutcenter}}$$

Two take-aways for us:

1. The total angular momentum of a composite system is the sum of angular momenta of its parts.
2. The total angular momentum of an object about a point is the sum of its orbital angular momentum about that point and its rotational / spin about its own center.

Quantum Mechanically

Okay, if we have a composite system like Hydrogen, with the electron orbiting the proton, then the total angular momentum of that system would be

$$\vec{L}_H = \vec{L}_e + \vec{S}_e + \vec{S}_p$$

So, the operator that would act upon a wavefunction for the Hydrogen atom to return its total angular momentum would be

$$\hat{L}_H = \hat{L}_e + \hat{S}_e + \hat{S}_p.$$

Let's consider the simplest case, in which the electron's in the $l=0$ state. Then we simply have

$$\hat{S}_H \equiv \hat{S}_e + \hat{S}_p$$

to focus on.

$$\text{Similarly, } \hat{S}_{H,z} \equiv \hat{S}_{e,z} + \hat{S}_{p,z},$$

ditto for x and y components, and therefore

$$\hat{S}_{H,\pm} \equiv \hat{S}_{e,\pm} + \hat{S}_{p,\pm}$$

We know enough about separable equations to recognize that the sum of operators means a product of states. So, if we describe the spins of the electron, proton, and Hydrogen as a whole with the notation $|s, m_s\rangle$, the spin wavefunction for the hydrogen would be

$$|s, m_s\rangle_H = |s, m_s\rangle_e |s, m_s\rangle_p.$$

Of course, for the electron and proton, $s = 1/2$, so we can get more specific in this situation:

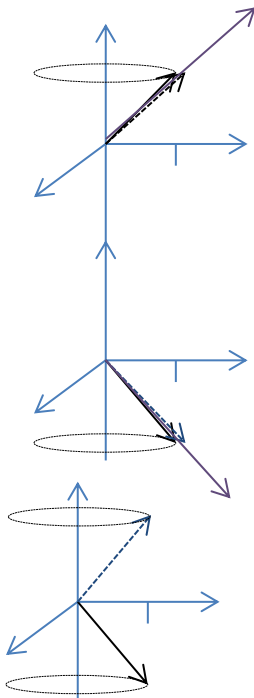
$$|s, m_s\rangle_H = |\frac{1}{2}, m_s\rangle_e |\frac{1}{2}, m_s\rangle_p$$

1. *Conceptual:* With 2 spin-1/2 particles, what are the possible results of a measurement of spin for the entire system?

For the Hydrogen as a whole, what are the possible total angular momentum due to spin, and what are the corresponding projections on the z-axis?

Semi-classically, Qualitatively

Before getting rigorous about it, let's think of what we might expect.



the electron and proton's spins both project on the +z direction

$$S_{zH} = S_{ze} + S_{zp} = \frac{1}{2}\hbar + \frac{1}{2}\hbar = 1\hbar$$

Of course, that corresponds to a total angular momentum quantum number of 1 as well

$$|1, 1_s\rangle_H = |\frac{1}{2}, \frac{1}{2}\rangle_e |\frac{1}{2}, \frac{1}{2}\rangle_p = \uparrow\uparrow$$

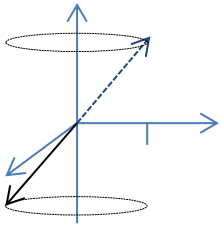
Then again, the electron and proton's spins can both be aligned with each other and along the -z axis, so

$$S_{zH} = S_{ze} + S_{zp} = -\frac{1}{2}\hbar + -\frac{1}{2}\hbar = -1\hbar$$

$$|1, -1_s\rangle_H = |\frac{1}{2}, -\frac{1}{2}\rangle_e |\frac{1}{2}, -\frac{1}{2}\rangle_p = \downarrow\downarrow$$

Now, semi-classically, you'd imagine there being the possibility of opposite alignment along z, but the same in the x-y plane, leading to a net angular momentum, it just happens to not be pointing in the z-direction at all. How would we represent that?

$$|1, 0\rangle_H = ?$$



Similarly, there's the possibility that their spins are completely anti-aligned, giving no angular momentum at all.

$$|0,0\rangle_H = ?$$

Rigorously

How do we get the m values for the 4 different states?" [Spencer](#)

Going after $|1,1_s\rangle_H$

Okay, let's see what we get. It seems like a pretty safe bet that

$$|1,1_s\rangle_H = \left| \frac{1}{2}, \frac{1}{2} \right\rangle_e \left| \frac{1}{2}, \frac{1}{2} \right\rangle_p = \uparrow_e \uparrow_p$$

We've already seen that the total spin projection is indeed 1. To confirm that the *magnitude* quantum number is also 1, we really should apply

$$\hat{S}^2 = (\hat{S}_e + \hat{S}_p) \cdot (\hat{S}_e + \hat{S}_p) = \hat{S}_e^2 + \hat{S}_p^2 + 2\hat{S}_e \cdot \hat{S}_p = \hat{S}_e^2 + \hat{S}_p^2 + 2(\hat{S}_{ex}\hat{S}_{px} + \hat{S}_{ey}\hat{S}_{py} + \hat{S}_{ez}\hat{S}_{pz})$$

Can we go through the work of getting $S^{\wedge}(1) * S^{\wedge}(2)$ (up down) below equation 4.179. Specifically how to figure out how to get from the first to the second line." [Jessica](#)
Griffiths goes back to the matrices that were developed for S_x and S_y and applies them to the state, but I'm going to derive a general expression from which we can use the raising and lowering operators.

$$\begin{aligned} \hat{S}^2 &= \hat{S}_e^2 + \hat{S}_p^2 + 2\left(\frac{1}{2}(\hat{S}_{e+} + \hat{S}_{e-})\frac{1}{2}(\hat{S}_{p+} + \hat{S}_{p-}) + \frac{1}{2i}(\hat{S}_{e+} - \hat{S}_{e-})\frac{1}{2i}(\hat{S}_{p+} - \hat{S}_{p-}) + \hat{S}_{ez}\hat{S}_{pz}\right) \\ \hat{S}^2 &= \hat{S}_e^2 + \hat{S}_p^2 + \left(\frac{1}{2}(\hat{S}_{e+}\hat{S}_{p+} + \hat{S}_{e-}\hat{S}_{p-} + \hat{S}_{e+}\hat{S}_{p-} + \hat{S}_{e-}\hat{S}_{p+}) - (\hat{S}_{e+}\hat{S}_{p+} + \hat{S}_{e-}\hat{S}_{p-} - \hat{S}_{e-}\hat{S}_{p+} - \hat{S}_{p-}\hat{S}_{e-})\frac{1}{2} + 2\hat{S}_{ez}\hat{S}_{pz}\right) \\ \hat{S}^2 &= \hat{S}_e^2 + \hat{S}_p^2 + (\hat{S}_{e+}\hat{S}_{p-} + \hat{S}_{e-}\hat{S}_{p+} + 2\hat{S}_{ez}\hat{S}_{pz}) \end{aligned}$$

Okay, now that we've got our tool, let's apply it and see what we get

$$\begin{aligned} \hat{S}^2|1,1_s\rangle_H &= \hat{S}_e^2 \uparrow_e \uparrow_p + \hat{S}_p^2 \uparrow_e \uparrow_p + (\hat{S}_{e+}\hat{S}_{p-} \uparrow_e \uparrow_p + \hat{S}_{e-}\hat{S}_{p+} \uparrow_e \uparrow_p + 2\hat{S}_{ez}\hat{S}_{pz} \uparrow_e \uparrow_p) \\ \hat{S}^2|1,1_s\rangle_H &= \frac{1}{2}(\frac{1}{2}+1)\hbar^2 \uparrow_e \uparrow_p + \frac{1}{2}(\frac{1}{2}+1)\hbar^2 \uparrow_e \uparrow_p + (0 + 2(\frac{\hbar}{2})^2 \uparrow_e \uparrow_p) \\ \hat{S}^2|1,1_s\rangle_H &= \frac{3}{4}\hbar^2 \uparrow_e \uparrow_p + \frac{3}{4}\hbar^2 \uparrow_e \uparrow_p + \frac{1}{2}\hbar^2 \uparrow_e \uparrow_p = 2\hbar^2 \uparrow_e \uparrow_p = \\ \hat{S}^2|1,1_s\rangle_H &= 2\hbar^2|1,1_s\rangle_H \end{aligned}$$

Exactly what we'd expect for $\hat{S}^2|1,1_s\rangle_H = s(s+1)\hbar^2|1,1_s\rangle_H$ with $s = 1$.

Going After $|1,0\rangle_H$

So if we apply the lowering operator on it,

$$\hat{S}_{H-}|1,1_s\rangle_H \equiv \hat{S}_{e-}\left|\frac{1}{2}, \frac{1}{2}\right\rangle_e \left|\frac{1}{2}, \frac{1}{2}\right\rangle_p + \hat{S}_{p-}\left|\frac{1}{2}, \frac{1}{2}\right\rangle_e \left|\frac{1}{2}, \frac{1}{2}\right\rangle_p = \hat{S}_{e-} \uparrow_e \uparrow_p + \hat{S}_{p-} \uparrow_e \uparrow_p$$

Now, the electron's raising operator operates only on the electron's spin, ditto for the proton's operator. This is similar to saying that the y-momentum operator takes the derivative with respect to y and treats x as a constant and vice versa.

$$\hat{S}_{H,-}|1,1_s\rangle_H = \hbar\sqrt{\frac{1}{2}(\frac{1}{2}+1)-\frac{1}{2}(\frac{1}{2}-1)}|\frac{1}{2},-\frac{1}{2}\rangle_e|\frac{1}{2},\frac{1}{2}\rangle_p + |\frac{1}{2},\frac{1}{2}\rangle_e\hbar\sqrt{\frac{1}{2}(\frac{1}{2}+1)-\frac{1}{2}(\frac{1}{2}-1)}|\frac{1}{2},-\frac{1}{2}\rangle_p$$

$$\hat{S}_{H,-}|1,1_s\rangle_H = \hbar(\downarrow_e\uparrow_p + \uparrow_e\downarrow_p)$$

Now, hypothesizing that the lowering operator does what we'd expect to the composite system, then

$$\hat{S}_{H,-}|1,1_s\rangle_H = \hbar\sqrt{1(1+1)-1(1-1)}|1,0\rangle_H$$

$$\hbar\sqrt{2}|1,0\rangle_H = \hbar(\downarrow_e\uparrow_p + \uparrow_e\downarrow_p)$$

$$|1,0\rangle_H = \frac{1}{\sqrt{2}}(\downarrow_e\uparrow_p + \uparrow_e\downarrow_p)$$

Which gives us back a normalized wavefunction, which is good.

And we can double check that it indeed does have $m_{H_s} = 0$ as we expect:

$$\hat{S}_{H,z}|1,0\rangle_H = (\hat{S}_{e,z} + \hat{S}_{p,z})\frac{1}{\sqrt{2}}(\downarrow_e\uparrow_p + \uparrow_e\downarrow_p) = \frac{1}{\sqrt{2}}(\hat{S}_{e,z}\downarrow_e\uparrow_p + \downarrow_e\hat{S}_{p,z}\uparrow_p + \hat{S}_{e,z}\uparrow_e\downarrow_p + \uparrow_e\hat{S}_{p,z}\downarrow_p)$$

$$\frac{1}{\sqrt{2}}\left(\frac{-\hbar}{2}\downarrow_e\uparrow_p + \downarrow_e\frac{\hbar}{2}\uparrow_p + \frac{\hbar}{2}\uparrow_e\downarrow_p + \uparrow_e\frac{-\hbar}{2}\downarrow_p\right) = 0$$

Yup.

I suppose we could run this through $\hat{S}^2 = \hat{S}_e^2 + \hat{S}_p^2 + (\hat{S}_{e+}\hat{S}_{p-} + \hat{S}_{e-}\hat{S}_{p+} + 2\hat{S}_{e,z}\hat{S}_{p,z})$ and verify that the total spin is still $s = 1$ for the hydrogen, but you get the idea.

Okay, apparently that's how one builds the mixed state that's aligned but just not along z.

1. *Starting Weekly HW*: Griffiths problem 4.34

Going After $|1,-1\rangle_H$

Notice that if we apply the lowering operator once more, we'll get

$$\hat{S}_{H,-}|1,0\rangle_H = (\hat{S}_{e-} + \hat{S}_{p-})\frac{1}{\sqrt{2}}(\downarrow_e\uparrow_p + \uparrow_e\downarrow_p) = \frac{1}{\sqrt{2}}((\hat{S}_{e-} + \hat{S}_{p-})\downarrow_e\uparrow_p + (\hat{S}_{e-} + \hat{S}_{p-})\uparrow_e\downarrow_p)$$

$$\sqrt{2}\hbar|1,-1\rangle_H = \frac{1}{\sqrt{2}}\hbar(0\uparrow_p + \downarrow_e\downarrow_p + \downarrow_e\downarrow_p + 0\downarrow_p)$$

$$|1,-1\rangle_H = \frac{1}{2}(2\downarrow_e\downarrow_p) = \downarrow_e\downarrow_p$$

Now for the completely anti-aligned state.

Going After $|0,0\rangle_H = ?$

What Griffiths proposes certainly look reasonable, but it would be good to see that it's actually derivable.

At most, it must be some linear combination of the possible solutions:

$$|0,0\rangle_H = a\uparrow_e\uparrow_p + b\downarrow_e\downarrow_p + c\downarrow_e\uparrow_p + d\uparrow_e\downarrow_p$$

If we apply

$$\hat{S}_{H.z} \equiv \hat{S}_{e.z} + \hat{S}_{p.z} \text{ we better get } 0$$

Applying it to this general expression, we get

$$\hat{S}_{H.z}|0,0\rangle_H = a\hbar \uparrow_e \uparrow_p - b\hbar \downarrow_e \downarrow_p$$

Which requires that $a = b = 0$

If we apply $\hat{S}_+ = \hat{S}_{e+} + \hat{S}_{p+} =$ we'd better get 0 too

$$\hat{S}_{H+}|0,0\rangle_H = (\hat{S}_{e+} + \hat{S}_{p+})(c \downarrow_e \uparrow_p + d \uparrow_e \downarrow_p) = c(\hat{S}_{e+} \downarrow_e \uparrow_p + \downarrow_e \hat{S}_{p+} \uparrow_p) + d(\hat{S}_{e+} \uparrow_e \downarrow_p + \uparrow_e \hat{S}_{p+} \downarrow_p)$$

$$0 = \hbar(c(\uparrow_e \hbar \uparrow_p + \downarrow_e 0) + d(0 \downarrow_p + \uparrow_e \hbar \uparrow_p)) = \hbar(c + d) \uparrow_e \uparrow_p$$

Which works only if $c = -d$. So, we've gotten as far as

$$|0,0\rangle_H = c(\downarrow_e \uparrow_p - \uparrow_e \downarrow_p)$$

To nail down the constant, we can normalize the wavefunction:

$$\langle 0,0|0,0\rangle_H = c(\downarrow_e^* \uparrow_p^* - \uparrow_e^* \downarrow_p^*)c(\downarrow_e \uparrow_p - \uparrow_e \downarrow_p) = c^2(\downarrow_e^* \uparrow_p^* \downarrow_e \uparrow_p - \downarrow_e^* \uparrow_p^* \uparrow_e \downarrow_p - \uparrow_e^* \downarrow_p^* \downarrow_e \uparrow_p + \uparrow_e^* \downarrow_p^* \uparrow_e \downarrow_p)$$

$$1 = c^2(1 - 0 - 0 + 1) = c^2 2$$

$$|c| = \frac{1}{\sqrt{2}}$$

Finally,

$$|0,0\rangle_H = \frac{1}{\sqrt{2}}(\downarrow_e \uparrow_p - \uparrow_e \downarrow_p)$$

Note: I really have choice of phase, and the convention is to have a flipped sign, or

$$|0,0\rangle_H = \frac{1}{\sqrt{2}}(\uparrow_e \downarrow_p - \downarrow_e \uparrow_p)$$

Summing up,

$$\left\{ \begin{array}{l} |1,1\rangle_H = \uparrow_e \uparrow_p \\ |1,0\rangle_H = \frac{1}{\sqrt{2}}(\downarrow_e \uparrow_p + \uparrow_e \downarrow_p) \\ |1,-1\rangle_H = \downarrow_e \downarrow_p \end{array} \right\} \text{triplet state, } l = 1$$

$$|0,0\rangle_H = \frac{1}{\sqrt{2}}(\uparrow_e \downarrow_p - \downarrow_e \uparrow_p) \text{ singlet state, } l = 0$$

1. *Conceptual:* If we take 2 spin-1/2 particles in combined state $|1\ 1\rangle$ and measure the spin of ONE of the particles, only, what are the possible results of a measurement and what are the probabilities? What if the combined system was $|1\ 0\rangle$?

Griffiths notes that, generally, if you have two contributions of angular momentum (be that two separate particles with their spins or one particle with its orbital and spin), we can generally build the possible total spin states from linear combinations like

$$|s, m\rangle = \sum_{m_1+m_2=m} c_{m_1 m_2 m}^{s_1 s_2 s} |s_1, m_1\rangle |s_2, m_2\rangle$$

Where the ‘‘Clebsch-Gordon’’ coefficients are in the crazy table 4.8 on p. 188. These can be read two ways:

‘‘If we will be using the Clebsch-Gordon coefficients, can we run through a couple examples in class with them?’’ [Mark T.](#)

I agree, I wasn't able to follow the procedure in the textbook. Going over them would be great [Jeremy.](#)

Given Composite, Determine Composition.

For example, if you have a spin 3/2 and a spin 1, and your composite state has total angular momentum number $s = 5/2$ with z-component $m_s = 1/2$, then that's constructed of

$$\frac{1}{\sqrt{10}}|\frac{3}{2}, \frac{3}{2}\rangle|1, -1\rangle + \sqrt{\frac{3}{5}}|\frac{3}{2}, \frac{1}{2}\rangle|1, 0\rangle + \sqrt{\frac{3}{10}}|\frac{3}{2}, -\frac{1}{2}\rangle|1, 1\rangle = |\frac{5}{2}, \frac{1}{2}\rangle$$

$S_1 = 3/2, S_2 = 1$		$S_{\text{combo}} = 5/2$	$3/2$	$1/2$
		$m_{\text{combo}} = 1/2$	$1/2$	$1/2$
m_1	m_2			
$3/2$	-1	1/10	2/5	1/2
$1/2$	0	3/5	1/15	-1/3
$-1/2$	+1	3/10	-8/15	1/6

(from first column of third table of the 3/2x1 series)

So, the chance of finding the first with $S_{z,1} = \frac{3}{2}\hbar$ and the second with $S_{z,2} = -\hbar$ is 1/10.

1. *Conceptual:* Say you have 2 particles, one with spin 1 and the other with spin 1/2. You measure the total spin of the system to be 3/2 and the z-component to be -1/2. What are the possible results of measuring each individual particle's z-component of spin and what are the probabilities?

Given Individual's, Determine Possible Composites

Alternatively, if you have a pair of particles that are in $|\frac{3}{2}, \frac{3}{2}\rangle|1, -1\rangle$, then there are a few possible states they could form:

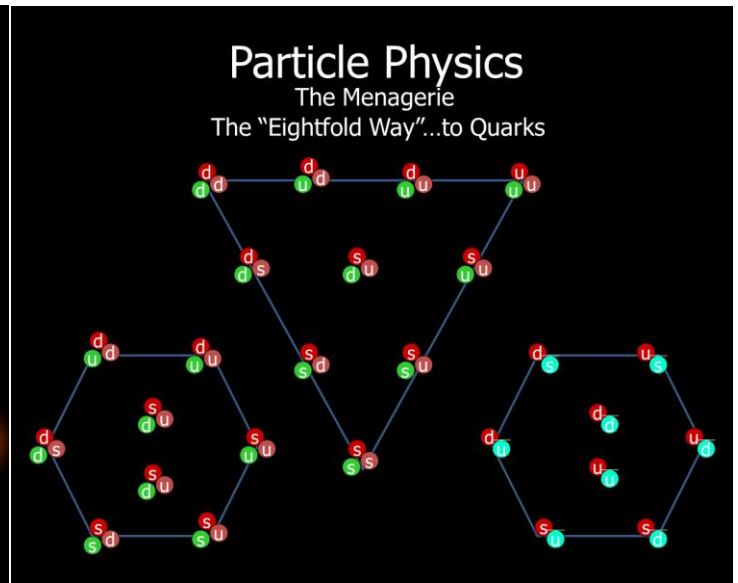
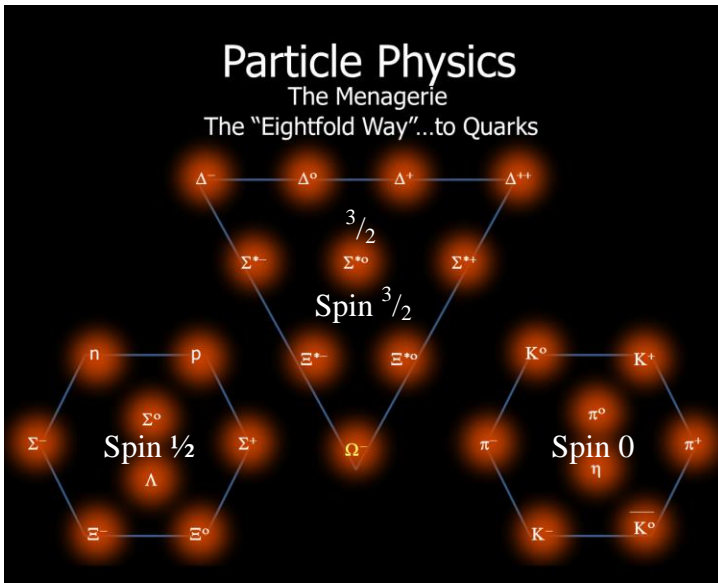
$|\frac{5}{2}, \frac{1}{2}\rangle$ with probability 1/10,
 $|\frac{3}{2}, \frac{1}{2}\rangle$ with probability 2/5,
 $|\frac{1}{2}, \frac{1}{2}\rangle$ with probability 1/2.

$S_1 = 3/2, S_2 = 1$		$S_{\text{combo}} = 5/2$	$3/2$	$1/2$
		$m_{\text{combo}} = 1/2$	$1/2$	$1/2$
m_1	m_2			
$3/2$	-1	1/10	2/5	1/2
$1/2$	0	3/5	1/15	-1/3
$-1/2$	+1	3/10	-8/15	1/6

1. *Conceptual:* Say we have 2 particles, one is in state $|3/2 -3/2\rangle$ and the other in state $|\frac{1}{2} \frac{1}{2}\rangle$. What are the possible states of the combined system of these 2 particles and what is the probability of measuring each?
2. *Starting Weekly HW:* Griffiths problem 4.36

"Problem 4.35 (I think) brings up spins of particles like mesons and such. Any chance we can talk about some of the basics of particle physics." Casey P.

See slide from Beautiful Data / Spatial Physics presentations & tables in Ch 14 of Thorton & Rex



$+\frac{2}{3}e$	up u	charm c	top (truth)
$-\frac{1}{3}e$	down d	strange s	bottom (beauty)

Notes:

Each quark has a spin of $\frac{1}{2}$, since energy of the composite varies with spin alignment, and mass of a composite system varies with its internal, a.k.a rest energy, the two possible spin combinations have different masses and initially, the pairs, like a p and a Δ^+ , got labeled as different particles rather than two states of the same system.

Spin 1 (quarks with aligned spins) versions of the Spin 0 particles (quarks with anti-aligned spins) exist. These excited versions of π^+ are called ρ^+ 's, of η is called ω , and of K is just K^* .

More accurately, $\pi^0 = \frac{1}{\sqrt{2}}(d\bar{d} - u\bar{u})$, $\eta = \frac{1}{\sqrt{2}}(d\bar{d} + u\bar{u})$, $\Lambda = \frac{1}{\sqrt{2}}(ud - du)s$, $\Sigma^0 = \frac{1}{\sqrt{2}}(ud + du)s$