9	Wed., 10/29 Fri., 10/31	**4.4.1-.2** Spin ½ & Magnetic Fields (Q5.5,6.1-.2, 8.5) **4.4.3** Addition of Angular Momenta	**Daily 9.W** **Daily 9.F**
10	Mon. 11/3 Wed., 11/5 Fri., 11/7	**Review** (Ch 3-4) **Exam** (Ch 3-4) **5.1** Identical Particles: 2-Particle Systems (Q8.6, 11.5)	**Weekly 10** **Daily 10.F**

Equipment

- Griffith's text
- Printout of roster with what pictures I have
- Spherical Schrodinger handout
- P plot.py

Check dailies

Daily 9.W Wednesday 10/29 Griffiths **4. 4.1-.4.2** Spin ½ and Magnetic Fields (Q5.5, 6.1-.2, 8.5)

4.4 Spin

What we learned about Orbital Angular Momentum

So, we've dealt with orbital angular momentum: translated it from classical to quantum mechanical; for example,

$$\hat{L}_x = \left(y\hat{p}_z - z\hat{p}_y\right) = \left(y\frac{\hbar}{i}\frac{\partial}{\partial z} - z\frac{\hbar}{i}\frac{\partial}{\partial y}\right) = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right).$$

We came up with a number of relationships relevant to the these operators:

$\left[\hat{L}_y, \hat{L}_z\right] = i\hbar\hat{L}_x$, etc.

$\hat{L}_\pm \equiv \hat{L}_x \pm i\hat{L}_y$ or, for that matter, $\hat{L}_x = \frac{1}{2}\left(\hat{L}_- + \hat{L}_+\right)$ and $\hat{L}_y = \frac{i}{2}\left(\hat{L}_- - \hat{L}_+\right)$

$\hat{L}^2 = \hat{L}_\pm\hat{L}_\mp \mp \hbar\hat{L}_z + \hat{L}_z^2$

$\hat{L}_z f_l^m = m\hbar f_l^m$ where $m = -l, \ldots l,$ and $\hat{L}^2 f_l^m = l(l+1)\hbar^2 f_l^m$ for l integer or half integer

$\hat{L}_\pm f_l^m = \underset{\pm\, l}{\lambda^m} f_l^{m\pm 1}$

Since we found the eigenvalues for L^2 and L_z, we're in a position to find the equivilant factors for $\hat{L}_\pm$, $\underset{\pm\, l}{\lambda^m}$ (not exactly an "eigenvalue" since the operator doesn't return the function back, but akin.) The reasoning goes like this:

$\hat{L}^2 f_l^m = \hat{L}_\pm\hat{L}_\mp f_l^m \mp \hbar\hat{L}_z f_l^m + \hat{L}_z^2 f_l^m$ so

$\hat{L}_\pm\hat{L}_\mp f_l^m = \hat{L}^2 f_l^m \mp \hbar\hat{L}_z f_l^m - \hat{L}_z^2 f_l^m = \left(l(l+1)\hbar^2 \mp \hbar^2 m - m^2\hbar^2\right) f_l^m = \left(l(l+1) \pm m(m-1)\right)\hbar^2 f_l^m$

But can we separate the effects of the raising and lowering operators? Yes, and it goes like this:

Since L_x and L_y correspond to observables, they must be hermitian operators; i.e., $\left\langle f \middle| \hat{L}_x g \right\rangle = \left\langle \hat{L}_x f \middle| g \right\rangle$ and $\left\langle f \middle| \hat{L}_y g \right\rangle = \left\langle \hat{L}_y f \middle| g \right\rangle$.

Then it's easy to see that $\hat{L}_+ = \hat{L}_x + i\hat{L}_y$ and $\hat{L}_- = \hat{L}_x - i\hat{L}_y$ are hermitian conjugates of each other, that is $\left\langle f \middle| \hat{L}_+ g \right\rangle = \left\langle f \middle| \left(\hat{L}_x + i\hat{L}_y\right) g \right\rangle = \left\langle \left(\hat{L}_x + i\hat{L}_y\right)^* f \middle| g \right\rangle = \left\langle \left(\hat{L}_x - i\hat{L}_y\right) f \middle| g \right\rangle = \left\langle \hat{L}_- f \middle| g \right\rangle$.

Therefore,

$$\left\langle f_l^m \middle| \hat{L}_\pm \hat{L}_\mp f_l^m \right\rangle = \left\langle f_l^m \middle| \left(l(l+1) \pm m(m-1)\right)\hbar^2 f_l^m \right\rangle$$

$$\left\langle \hat{L}_\mp f_l^m \middle| \hat{L}_\mp f_l^m \right\rangle = \left(l(l+1) \pm m(m-1)\right)\hbar^2 \left\langle f_l^m \middle| f_l^m \right\rangle$$

$$\left\langle \underset{\mp l}{\lambda^m} f_l^{m\mp1} \middle| \underset{\mp l}{\lambda^m} f_l^{m\mp1} \right\rangle = \left(l(l+1) \pm m(m-1)\right)\hbar^2$$

$$\left(\underset{\mp l}{\lambda^m}\right)^* \underset{\mp l}{\lambda^m} \left\langle f_l^{m\mp1} \middle| f_l^{m\mp1} \right\rangle = \left(l(l+1) \pm m(m-1)\right)\hbar^2$$

$$\left(\underset{\mp l}{\lambda^m}\right)^* \underset{\mp l}{\lambda^m} = \left(l(l+1) \pm m(m-1)\right)\hbar^2$$

$$\left|\underset{\mp l}{\lambda^m}\right| = \sqrt{\left(l(l+1) \pm m(m-1)\right)}\hbar$$

So, to within a phase, $\underset{\mp l}{\lambda^m} = \sqrt{\left(l(l+1) \pm m(m-1)\right)}\hbar$.

Translating to Intrinsic Spin Angular Momentum

Now, for orbital angular momentum, our starting point was the traditional one: start with classical expressions and then translate them to operators that, when acting upon a wavefunction, would return measurement values for those properties.

If the electron were a classical object, the kind with physical extent, then we'd expect it to also have angular momentum associated with its spinning about its own axis, like the Earth does about its axis. Quantum mechanically, we'd expect the same kinds of rules to apply; calling this angular momentum "Spin", and using an *s* to symbolize it rather than an *l*, all the above results would carry over:

$\left[\hat{S}_y, \hat{S}_z\right] = i\hbar\hat{S}_x$, etc.

$\hat{S}_\pm \equiv \hat{S}_x \pm i\hat{S}_y$ or, for that matter, $\hat{S}_x = \frac{1}{2}\left(\hat{S}_- + \hat{S}_+\right)$ and $\hat{S}_y = \frac{i}{2}\left(\hat{S}_- - \hat{S}_+\right)$

$\hat{S}^2 = \hat{S}_\pm \hat{S}_\mp \mp \hbar\hat{S}_z + \hat{S}_z^2$

Using the traditional bra – ket notation

$\hat{S}_z\left|s, m_s\right\rangle = m_s\hbar\left|s, m_s\right\rangle$ where $m_s = -s, \ldots s$, *and* $\hat{S}^2\left|s, m_z\right\rangle = s(s+1)\hbar^2\left|s, m_z\right\rangle$ for *s* integer or half integer

$S_\pm\left|s, m_z\right\rangle = \sqrt{\left(s(s+1) \pm m_s(m_s - 1)\right)}\hbar\left|s, m_z \pm 1\right\rangle$

Now, Griffiths presents, but does not prove (and none of my texts do actually prove, though he hints at a derivation) that fundamental particles have "intrinsic" spin in spite of not having physical extent, and these spins obey these same relations, in spite of their not being related to classical properties like orbital angular momentum is. Experimentally, this is certainly the case.

4.4.1 Spin ½

Electrons, neutrinos, and quarks have $s = ½$ (remember, when we derived the possible then-called l values, we found they could be integer or half-integer.)
All known matter is built of such pieces (the "delta" is a composite of 3 quarks who's individual spins are aligned, thus a composite angular momentum of $^3/_2$.) Massless 'particles' like photons have spin 1 and the graviton, if it exists, would need to have spin 2.

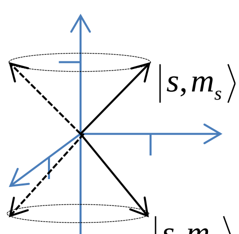
Since $m_s = -s, \ldots s$ in integer steps, a spin quantum number $s = ½$ means $m_s = -^1/_2, +^1/_2$ are the only two options.

That is to say, the only two possible measurements that you can make of the spin's z-component are

$$\pm\tfrac{1}{2}\hbar$$

$$\hat{S}_z\left|\tfrac{1}{2},\pm\tfrac{1}{2}\right\rangle = \pm\tfrac{1}{2}\hbar\left|\tfrac{1}{2},\pm\tfrac{1}{2}\right\rangle$$

Since there is just a discrete number of states, it's convenient to express the operator in terms of a matrix rather than a function. We've got some freedom of how we represent the eigenvectors, but the convention is to prefer the z-axis (as we've already been doing), and so define our two states as (showing all possible notations)



$$\left|s,m_s\right\rangle = \left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle = \uparrow = \begin{pmatrix}1\\0\end{pmatrix} = \chi_{+z}$$

$$\left|s,m_s\right\rangle = \left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle = \downarrow = \begin{pmatrix}0\\1\end{pmatrix} = \chi_{-z}$$

> Can we go over the Spinor/eigenspinors and how to find it for each spin component?"
>
> Jessica
>
> I would like to see this as well Jonathan

Now, since we have just two states, the operator must be representable by a 2×2 matrix so it has just enough elements to act upon each of the elements of the vector and to generate a new vector with just as many elements.

It's not too hard to generate the $\hat{S}_z$ operator since we know the eigen values it must return and we've defined the eigenvectors it acts upon

$$\hat{S}_z\left|\tfrac{1}{2},\pm\tfrac{1}{2}\right\rangle = \pm\tfrac{1}{2}\hbar\left|\tfrac{1}{2},\pm\tfrac{1}{2}\right\rangle \Rightarrow \begin{cases} \hat{S}_z\begin{pmatrix}1\\0\end{pmatrix} = \tfrac{1}{2}\hbar\begin{pmatrix}1\\0\end{pmatrix} \\ \hat{S}_z\begin{pmatrix}0\\1\end{pmatrix} = -\tfrac{1}{2}\hbar\begin{pmatrix}0\\1\end{pmatrix} \end{cases}$$

The matrix

$$\boxed{\hat{S}_z = \begin{pmatrix}\tfrac{1}{2}\hbar & 0\\ 0 & -\tfrac{1}{2}\hbar\end{pmatrix} = \tfrac{1}{2}\hbar\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}}$$

Does the trick.

Similarly,

$$S_+\left|\tfrac{1}{2},\pm\tfrac{1}{2}\right\rangle = \sqrt{\left(\tfrac{1}{2}\left(\tfrac{1}{2}+1\right)\pm\tfrac{1}{2}\left(\pm\tfrac{1}{2}+1\right)\right)}\hbar\left|\tfrac{1}{2},\pm\tfrac{1}{2}+1\right\rangle \Rightarrow \begin{cases} S_+\begin{pmatrix}1\\0\end{pmatrix} = 0 \\ S_+\begin{pmatrix}0\\1\end{pmatrix} = \hbar\begin{pmatrix}1\\0\end{pmatrix} \end{cases}$$

The matrix

$$\boxed{\hat{S}_+ = \begin{pmatrix}0 & \hbar\\ 0 & 0\end{pmatrix} = \hbar\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}}$$

Does the trick.

And

$$S_-\left|\tfrac{1}{2},\pm\tfrac{1}{2}\right\rangle = \sqrt{\left(\tfrac{1}{2}\left(\tfrac{1}{2}+1\right)\mp\tfrac{1}{2}\left(\pm\tfrac{1}{2}-1\right)\right)}\hbar\left|\tfrac{1}{2},\pm\tfrac{1}{2}+1\right\rangle \Rightarrow \begin{cases} S_-\begin{pmatrix}1\\0\end{pmatrix} = \hbar\begin{pmatrix}0\\1\end{pmatrix} \\ S_-\begin{pmatrix}0\\1\end{pmatrix} = 0 \end{cases}$$

The matrix

$$\boxed{\hat{S}_- = \begin{pmatrix}0 & 0\\ \hbar & 0\end{pmatrix} = \hbar\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}}$$

Does the trick.

Having these three, we can use our relations to construct the rest.

For example,

$$\hat{S}_x = \tfrac{1}{2}\left(\hat{S}_- + \hat{S}_+\right) = \tfrac{1}{2}\left(\hbar\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} + \hbar\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right)$$

$$\boxed{\hat{S}_x = \tfrac{1}{2}\hbar\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}}$$

$$\hat{S}_y = \tfrac{i}{2}\left(\hat{S}_- - \hat{S}_+\right) = \tfrac{i}{2}\left(\hbar\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} - \hbar\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right)$$

$$\boxed{\hat{S}_y = \tfrac{1}{2}\hbar\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}}$$

$$\hat{S}^2 = \hat{S}_\pm \hat{S}_\mp \mp \hbar \hat{S}_z + \hat{S}_z^2 = \hbar\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\hbar\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} + \hbar\left(\tfrac{1}{2}\hbar\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right) + \tfrac{1}{2}\hbar\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\tfrac{1}{2}\hbar\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$\hat{S}^2 = \hbar^2\left(\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}\tfrac{1}{2} & 0\\ 0 & -\tfrac{1}{2}\end{pmatrix} + \tfrac{1}{4}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)$$

$$\boxed{\hat{S}^2 = \tfrac{3}{4}\hbar^2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}}$$

What about the eigenvectors for these other operators? Well, we know that S^2 and S_z share eigenvectors, so we've got that one covered, but what about S_x and S_y? Back to the old game of finding eigenvectors and eigen values:

Call it χ_x

$$\hat{S}_x \chi_x = \lambda_x \chi_x$$

$$\hat{S}_x \chi_x = \tfrac{1}{2}\hbar\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \lambda_x\begin{pmatrix}a\\ b\end{pmatrix} \Rightarrow \det\begin{vmatrix}-\lambda_x & \tfrac{1}{2}\hbar\\ \tfrac{1}{2}\hbar & -\lambda_x\end{vmatrix} = (\lambda_x)^2 - \left(\tfrac{1}{2}\hbar\right)^2 = 0 \Rightarrow \lambda_x = \pm\tfrac{1}{2}\hbar$$

Or, using the notation we've already introduced,

$\hat{S}_x \chi_x = m_x \hbar \chi_x$ where $m_x = \pm\tfrac{1}{2}$

Not too surprising, with respect to the x-axis, the spin angular momentum can have the same two projections that it can for the z-axis.

Now turning our attention to finding the eigenvector,

$$\hat{S}_x \chi_x = \begin{pmatrix}0 & \tfrac{1}{2}\hbar\\ \tfrac{1}{2}\hbar & 0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \lambda_x\begin{pmatrix}a\\ b\end{pmatrix} \Rightarrow \tfrac{1}{2}\hbar b = \lambda_x a \text{ and } \tfrac{1}{2}\hbar a = \lambda_x b$$

So, for $\lambda_x = \tfrac{1}{2}\hbar$, this simply tells us that $\tfrac{1}{2}\hbar b = \tfrac{1}{2}\hbar a$ and $\tfrac{1}{2}\hbar a = \tfrac{1}{2}\hbar b \Rightarrow a = b$

Then

$$\chi_{+x} = \begin{pmatrix}a\\ a\end{pmatrix} = a\begin{pmatrix}1\\ 1\end{pmatrix}$$

Normalizing, $1 = \langle \chi_x \| \chi_x \rangle = a^2 (1,1)\begin{pmatrix}1\\ 1\end{pmatrix} = a^2 2 \Rightarrow a = \dfrac{1}{\sqrt{2}}$.

Thus

$$a\begin{pmatrix}1\\ 1\end{pmatrix} = \boxed{\begin{pmatrix}\tfrac{1}{\sqrt{2}}\\ \tfrac{1}{\sqrt{2}}\end{pmatrix} = \chi_{+x}}$$

Similarly, if we go with the $\lambda_x = -\frac{1}{2}\hbar$ eigenvalue, we find that

$\frac{1}{2}\hbar b = -\frac{1}{2}\hbar a$ and $\frac{1}{2}\hbar a = -\frac{1}{2}\hbar b \Rightarrow a = -b$ which leads to

$$a\begin{pmatrix}1\\-1\end{pmatrix} = \boxed{\begin{pmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{pmatrix} = \chi_{-x}}$$

Taking the same approach, we find the eigen values and vectors for the y-component of the spin, S_y.

$$\hat{S}_y\chi_y = \tfrac{1}{2}\hbar\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\chi_y = \lambda_y\chi_y \Rightarrow \det\left|\begin{pmatrix}-\lambda_y & -i\frac{1}{2}\hbar\\ i\frac{1}{2}\hbar & -\lambda_y\end{pmatrix}\right| = (\lambda_y)^2 - (\tfrac{1}{2}\hbar)^2 = 0 \Rightarrow \lambda_y = \pm\tfrac{1}{2}\hbar$$

$$\hat{S}_y\chi_y = \tfrac{1}{2}\hbar\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \lambda_y\begin{pmatrix}a\\b\end{pmatrix} \Rightarrow -i\tfrac{1}{2}\hbar b = \lambda_y a, \text{ and } i\tfrac{1}{2}\hbar a = \lambda_y b$$

Going with $\lambda_x = \frac{1}{2}\hbar$, we get that $-i\frac{1}{2}\hbar b = \frac{1}{2}\hbar a \Rightarrow b = ia$

$$a\begin{pmatrix}1\\i\end{pmatrix} = \boxed{\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{i}{\sqrt{2}}\end{pmatrix} = \chi_{+y}}$$

And going with $\lambda_x = -\frac{1}{2}\hbar$, we get that $-i\frac{1}{2}\hbar b = -\frac{1}{2}\hbar a \Rightarrow b = -ia$

$$a\begin{pmatrix}1\\-i\end{pmatrix} = \boxed{\begin{pmatrix}\frac{1}{\sqrt{2}}\\-\frac{i}{\sqrt{2}}\end{pmatrix} = \chi_{-y}}$$

Now, each of these pairs represents a complete basis set, that is, you can construct any other possible spin state from any one of these pairs, that includes each other.

For example, $\chi_{-y} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\-\frac{i}{\sqrt{2}}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} - \frac{i}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}(\uparrow - i\downarrow)$

Now, this particular projection was trivial enough to do by inspection, but more generally, how do we resolve a vector into components of a particular ortonomal basis set?

$$\chi = \sum_n c_{z.n}\chi_{z.n} \quad \text{where the coeffcients } c_{z.n} = \langle\chi_{z.n}|\chi\rangle$$

And the probability of measuring $\chi_{z.n}$'s eigenvalue is $|c_{z.n}|^2 = |\langle\chi_{z.n}|\chi\rangle|^2$.

Now, in the case of these ½-spins, the basis set has only 2 eigenvectors, so it's easy enough to do this explicitly for all of two eigenvectors. Let's work a case.

"Could we go over how Griffiths got 4.152?" Spencer

Here's a concrete example; it may be less abstract, and once you've considered it, it'll be easier to go back and consider the general kind of case.

Say you've initially got some weird mixed state,

$$\chi = \begin{pmatrix} i\frac{3}{5} \\ \frac{4}{5} \end{pmatrix}$$

You set up an experiment to measure its y-axis alignment (that's right, you pass a beam of such particles through an "SG(y)" machine.)

What is the probability that you'll measure $S_y = -\frac{1}{2}\hbar$?

That's the eigenvalue for χ_{-y}, so the probability of measuring it when you start with our given initial state is

$$\left|c_{-y}\right|^2 = \left|\langle \chi_{-y} | \chi \rangle\right|^2 = \left| \begin{pmatrix} \frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} i\frac{3}{5} \\ \frac{4}{5} \end{pmatrix} \right|^2 = \left| \left(i\frac{1}{\sqrt{2}}\frac{3}{5} + \frac{i}{\sqrt{2}}\frac{4}{5} \right) \right|^2 = \frac{1}{2}\left|\left(i\frac{3}{5} + i\frac{4}{5}\right)\right|^2 = \frac{1}{2}\left|\left(\frac{7}{5}\right)\right|^2 = \frac{1}{2}\frac{49}{25} = \frac{49}{50}$$

For that matter, if you measured the x-projection over and over again, would be the average of your measurements (note: here's a case where "expectation" value is quite misleading, since none of your measurements would produce it – you'd never 'expect' to measure it, just to average it.)

$$\left\langle \chi \middle| \hat{S}_y \chi \right\rangle = \left(-i\tfrac{3}{5}, \tfrac{4}{5}\right)\tfrac{1}{2}\hbar \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} i\frac{3}{5} \\ \frac{4}{5} \end{pmatrix} = \tfrac{1}{2}\hbar\left(-i\tfrac{3}{5}, \tfrac{4}{5}\right) \begin{pmatrix} -i\frac{4}{5} \\ -\frac{3}{5} \end{pmatrix} = \tfrac{1}{2}\hbar\left(\left(-i\tfrac{3}{5}\right)\left(-i\tfrac{4}{5}\right) + \tfrac{4}{5}\left(-\tfrac{3}{5}\right)\right) = \tfrac{1}{2}\hbar\left(-\tfrac{12}{25} - \tfrac{12}{25}\right) = -\hbar\tfrac{24}{50}$$

1. *Conceptual*: Find the eigenvectors in Table Q6.1 in Griffiths. Give equation numbers.
2. *Conceptual*: Q5S.3 AND Q5R.1 (note: SG(-θ) means SG(ϕ) where ϕ=- θ, rather than simply switching which output is + and which is -)
3. *Math*: Using the eigenvectors in Table Q6.1,
    a. Calculate the probability of measuring up and down in an SGy device if the particles entering the device are in |+x>.
    b. Calculate the probability of measuring up and down in an SGθ device if the particles entering the device are in |+y>.
4. *Starting Weekly HW*: Griffiths 4.27
5. *Starting Weekly HW*: Griffiths 4.31

"Could we go over the Levi-Civita symbol, is it another approach for cycling indices prior to Monday's lecture? (i.e. [L_x, p_y])" Jeremy, (problem 4.26) – It relates to that; notice that it's +1 when you keep the right-handed order (x y z, z x y, y z x) and it's -1 if you cycle in a left-handed order – sort of the price for going out of order. Without looking too closely at the derivation, I expect where it's coming from in this problem is that underlying all this spin is cross products, and x × y points z; y × z points x; z × x points y; of course anything crossed with itself is 0.

4.4.2 Electron in a Magnetic Field

Rotation (Torque) and Translation (Force)

Classically,

Borrowing a bit from E&M, just as angular momentum keeps track of the circulation of a mass around some point, magnetic moment, μ, keeps track of the circulation of a charge about some point; for a simple circular current loop, it is the current times the area it encircles. The important connection for us now is simply that the two are proportional:

$$\vec{\mu} = \gamma\vec{s}$$

Of course, a magnetic field exerts a force upon a moving charge / upon a current, and if that current forms a loop, then, as each part is moving in a different direction, it experiences a force in a different direction such that the loop is forced to flip so its dipole moment aligns with the field, (a common example is a compass needle). In terms of energy,

$$E = -\vec{\mu}\cdot\vec{B}$$

Quantum Mechanically

So the Hamiltonian would be

$$\hat{H} = -\hat{\mu}\cdot\vec{B}$$

$$\hat{H} = -\gamma\hat{S}\cdot\vec{B}$$

Defining the z-direction to be that of the magnetic field, we'd have

$$\hat{H} = -\gamma\hat{S}_z B = -\gamma\frac{\hbar}{2}B\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

Clearly, the spin-dependent factors of the energy eigenvectors would simply be the z-eigenvectors:

$$\chi_{+z} = \begin{pmatrix}1\\0\end{pmatrix} = \uparrow \text{ with energy eigenvalue } E_+ = -\gamma\frac{\hbar}{2}B$$

And $\chi_{-z} = \begin{pmatrix}0\\1\end{pmatrix} = \downarrow$ with energy eigenvalue $E_- = \gamma\frac{\hbar}{2}B$

Since we've got energy states, we know how they time evolve,

$$\Psi_E(t,s,m_s) = \Psi_E\left(t,\tfrac{1}{2},\pm\tfrac{1}{2}\right) = \chi_{\pm z}e^{-iE_{\pm}t/\hbar}$$

This time evolution in a magnetic field can be used to demonstrate two interesting behaviors:

"Can we run through the Larmor precession (ex 4.3)?" Mark T,

Lamor Precession

Initially, say you've got some mixed state,

$$\Psi(t=0) = c_+\chi_{+z} + c_-\chi_{-z}$$

Where $c_+{}^2 + c_-{}^2 = 1$, so we could use them to define an angle $\alpha \equiv 2\tan^{-1}\frac{c_-}{c_+}$ and thus write them as $c_+ = \cos(\alpha/2)$ and $c_- = \sin(\alpha/2)$

So, at a later time, *t*,

$$\Psi(t) = \cos(\alpha/2)\chi_{+z}e^{-iE_+t/\hbar} + \sin(\alpha/2)\chi_{-z}e^{-iE\_t/\hbar} = \begin{pmatrix} \cos(\alpha/2)e^{i\gamma Bt/2} \\ \sin(\alpha/2)e^{-i\gamma Bt/2} \end{pmatrix}$$

While the projection on the z-axis will remain constant at

$$\langle S_z \rangle = \langle \Psi^*(t)|\hat{S}_z\Psi(t)\rangle = \left(\cos(\alpha/2)e^{-i\gamma Bt/2}, \sin(\alpha/2)e^{i\gamma Bt/2}\right)\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \cos(\alpha/2)e^{i\gamma Bt/2} \\ \sin(\alpha/2)e^{-i\gamma Bt/2} \end{pmatrix} = \frac{\hbar}{2}\cos(\alpha)$$

Working out the projection on the y and x give

$$\langle S_x \rangle = \langle \Psi^*(t)|\hat{S}_x\Psi(t)\rangle == \frac{\hbar}{2}\sin(\alpha)\cos(\gamma Bt)$$

and $\langle S_y \rangle = \langle \Psi^*(t)|\hat{S}_y\Psi(t)\rangle == \frac{\hbar}{2}\sin(\alpha)\sin(\gamma Bt)$

So it's like a vector of length $\frac{\hbar}{2}$ cocked at an angle a down from the z-axis, and rotating counter-clockwise with frequency $\omega_{Lamor} \equiv \gamma B$.

Stern-Gerhlach

Classically

If you have a varying magnetic field, say one whose field lines narrow as you rise up in the z-direction, then applying the right-hand-rule to a few spots on a current loop circulating ccw around the z-axis, you'll find a net force upward; similarly for one circulating cw around the z-axis, you'll find a net force down.

$$\vec{F} = \vec{\nabla}(\vec{\mu}\cdot\vec{B})$$

For example, if $\vec{B} = -\alpha x\hat{i} + (B_o + \alpha z)\hat{k}$ then

$$\vec{F} = \vec{\nabla}(-\alpha x\mu_x + (B_o + \alpha z)\mu_z)$$

$$\vec{F} = -\alpha\mu_x\hat{i} + \alpha\mu_z\hat{k}$$

If we can simultaneously apply a horizontal electric field to oppose the force in the x-direction, then we'd simply have,

And so the change in energy of a current loop traveling through this region would be

$\Delta E = W = -\alpha z\mu_z$

On top of that, there would be the regular energy associated with alignment,

$E_o = -\mu_z B_o$

So, at the end of the travel, $E = -\mu_z B_o - \alpha z\mu_z$

Quantum Mechanically

Now we're ready to translate to quantum mechanics. The energy of our spin-1/2 particle after traveling through such a region is

$$\hat{H} = (-\alpha z - B_o)\gamma\hat{S}_x$$

if it's in state $\chi_{\pm z}$ to begin with, then

$$E_{\pm} = \mp(\alpha z + B_o)\gamma \tfrac{1}{2}\hbar$$

And $\chi(t)_{\pm z} = \chi(0)_{\pm z} e^{-iE_{\pm}t/\hbar} = \chi(0)_{\pm z} e^{\pm i(\alpha z + B_o)\gamma t/2}$

Operating upon this with the z momentum operator, we see that it has momentum up or down, depending on its spin alignment.

$$\hat{p}_z \chi(t)_{\pm z} = \frac{\hbar}{i}\frac{\partial}{\partial z}\chi(0)_{\pm z} e^{\pm i(\alpha z + B_o)\gamma t/2}$$

$$p_z \chi(t)_{\pm z} = (\pm \hbar \alpha \gamma t/2)\chi(t)_{\pm z}$$

So, if it's exposed to the field for time *T*, then the momentum it's acquired is

$p_z = (\pm \hbar \alpha \gamma T/2)$

Up if spin up and down if spin down.

"I see that quantum numbers come in, at least for spin, like in Chemistry. Even though we have to go over the hard stuff, could we take a little time connect the math of these numbers to what is going on in the real world?" Anton - Coming up next (Ch 5, Section 2)