Wed.	10.1 - .2.1 Potential Formulation Lunch with UCR Engr – 12:20 – 1:00	
Fri.	10.2 Continuous Distributions	
Mon.	10.3 Point Charges	HW11

Maxwell's Laws

Relating Fields and Sources

Relativistically Correct since instantaneous and local

$$\vec{\nabla}\times\vec{B}-\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}=\mu_0\vec{J}$$ Maxwell – Ampere's Law $$\oint\vec{B}\cdot d\vec{\ell}-\mu_0\varepsilon_0\int\frac{\partial\vec{E}}{\partial t}\cdot d\vec{a}=\mu_0\int\vec{J}\cdot d\vec{a}$$

$$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\varepsilon_0}$$ Gauss's Law $$\oint\vec{E}\cdot d\vec{a}=\frac{Q_{enc}}{\varepsilon_0}$$

$$\vec{\nabla}\cdot\vec{B}=0$$ Gauss's Law for Magnetism $$\oint\vec{B}\cdot d\vec{a}=0$$

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$ Faraday's Law $$\oint\vec{E}\cdot d\vec{\ell}=-\left.\frac{\partial\Phi_B}{\partial t}\right|_a=-\int\frac{\partial\vec{B}}{\partial t}\cdot d\vec{a}$$

Not necessarily Relativistically Correct

Helmholtz Theorem: if you know a vector field's curl and divergence (and time derivative), you know everything

Corresponding Relations between Potentials

(on the road to general solutions for E and B)

Combine

Maxwell's Relations Between Fields & Sources with Potentials' Relations to Fields to Relate Potentials & Sources

~~$-\vec{\nabla}V \equiv \vec{E}$~~ $\quad \vec{\nabla}\times\vec{A} \equiv \vec{B}$

No effect on electro*statics.* In electro *dynamics,* work associated with V and dA/dt.

Fix by Redefining $-\vec{\nabla}V - \dfrac{\partial \vec{A}}{\partial t} \equiv \vec{E}$

(true for any scalar field *V*.)

Disagrees with $\vec{\nabla}\times\vec{E} = -\vec{\nabla}\times\vec{\nabla}V = 0$

Faraday's Law

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \longrightarrow \vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{A}\right) = \vec{\nabla}\times\left(-\frac{\partial \vec{A}}{\partial t}\right)$$

Maxwell – Ampere's Law

$$\vec{\nabla}\times\vec{B} - \mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t} = \mu_0\vec{J} \longrightarrow \vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right) + \mu_0\varepsilon_0\frac{\partial}{\partial t}\left(\vec{\nabla}V + \frac{\partial}{\partial t}\vec{A}\right) = \mu_0\vec{J}$$

Gauss's Law for Magnetism

$$\vec{\nabla}\cdot\vec{B} = 0 \longrightarrow \vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}\right) = 0$$

Gauss's Law

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \longrightarrow \vec{\nabla}\cdot\left(\vec{\nabla}V + \frac{\partial \vec{A}}{\partial t}\right) = -\frac{\rho}{\varepsilon_0}$$

Corresponding Relations between Potentials

(on the road to general solutions for E and B)

Relate Potentials & Sources

$$\vec{\nabla}\times\left(\vec{\nabla}V\right)=0 \qquad \vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}\right)=0$$

Just Mathematical Facts

$$\vec{\nabla}\cdot\left(\vec{\nabla}V+\frac{\partial\vec{A}}{\partial t}\right)=-\frac{\rho}{\varepsilon_0} \qquad \vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)+\mu_0\varepsilon_0\frac{\partial}{\partial t}\left(\vec{\nabla}V+\frac{\partial}{\partial t}\vec{A}\right)=\mu_0\vec{J}$$

Relate potentials and sources

Rearrange for future use

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)+\mu_0\varepsilon_0\left(\vec{\nabla}\frac{\partial V}{\partial t}+\frac{\partial^2\vec{A}}{\partial t^2}\right)=\mu_0\vec{J}$$

$$\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right)-\nabla^2\vec{A}+\mu_0\varepsilon_0\left(\vec{\nabla}\frac{\partial V}{\partial t}+\frac{\partial^2\vec{A}}{\partial t^2}\right)=\mu_0\vec{J}$$

$$\left(\nabla^2\vec{A}-\mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2}\right)-\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}+\mu_0\varepsilon_0\frac{\partial V}{\partial t}\right)=-\mu_0\vec{J}$$

Corresponding Relations between Potentials

(on the road to general solutions for E and B)

We want to solve for V and A given

$$\vec{\nabla}\cdot\left(\vec{\nabla}V+\frac{\partial\vec{A}}{\partial t}\right)=-\frac{\rho}{\varepsilon_0}$$

and

$$\left(\nabla^2\vec{A}-\mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2}\right)-\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}+\mu_0\varepsilon_0\frac{\partial V}{\partial t}\right)=-\mu_0\vec{J}$$

Gauge Choices

A and V can be anything that satisfy

$$\vec{\nabla}\times\vec{A}\equiv\vec{B} \qquad -\vec{\nabla}V-\frac{\partial\vec{A}}{\partial t}\equiv\vec{E}$$

Like a choice of coordinate systems – can choose a potential's gauge without changing the answers to physically meaningful questions

Can choose any functional form for A's divergence without changing its relation with B, but must compensate by modifying V

Coulomb's Gauge

$$\vec{\nabla}\cdot\vec{A}_C\equiv 0$$

$$\vec{\nabla}\cdot\left(\vec{\nabla}V_C\right)=-\frac{\rho}{\varepsilon_0}$$

Simple!

$$\left(\nabla^2\vec{A}_C-\mu_0\varepsilon_0\frac{\partial^2\vec{A}_C}{\partial t^2}\right)-\vec{\nabla}\left(\mu_0\varepsilon_0\frac{\partial V_C}{\partial t}\right)=-\mu_0\vec{J}$$

Not Simple!

Some *O*ther Gauge

$\vec{A}_O\equiv\vec{A}_C+\vec{\nabla}\lambda_O$ Must compensate by $V_O\equiv V_C+\frac{\partial\lambda_O}{\partial t}$

Can get away with this since curl of a gradient *must* be 0

Demo: Say $\vec{\nabla}\times\vec{A}_C=\vec{B}$

$$\vec{\nabla}\times\vec{A}_O=\vec{\nabla}\times\left(\vec{A}_C+\vec{\nabla}\lambda_C\right)$$

$$=\vec{\nabla}\times\vec{A}_C+\vec{\nabla}\times\left(\vec{\nabla}\lambda_C\right)=\vec{\nabla}\times\vec{A}_C+0=\vec{B}$$

But if we do this, then it effects E:

Demo: Say $-\vec{\nabla}V_C-\frac{\partial\vec{A}_C}{\partial t}\equiv\vec{E}$

$$-\vec{\nabla}V_C-\frac{\partial\vec{A}_O}{\partial t}=-\vec{\nabla}V_C-\frac{\partial}{\partial t}\left(\vec{A}_C-\vec{\nabla}\lambda_O\right)=\vec{E}-\vec{\nabla}\frac{\partial}{\partial t}\lambda_O$$

Corresponding Relations between Potentials

(on the road to general solutions for E and B)

We want to solve for V and A given

$$\vec{\nabla}\cdot\left(\vec{\nabla}V+\frac{\partial\vec{A}}{\partial t}\right)=-\frac{\rho}{\varepsilon_0}$$

and

$$\left(\nabla^2\vec{A}-\mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2}\right)-\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}+\mu_0\varepsilon_0\frac{\partial V}{\partial t}\right)=-\mu_0\vec{J}$$

Add and subtract $\mu_0\varepsilon_0\frac{\partial}{\partial t}\frac{\partial V}{\partial t}$ to rephrase

$$\left(\nabla^2 V-\mu_0\varepsilon_0\frac{\partial^2 V}{\partial t^2}\right)+\frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{A}+\mu_0\varepsilon_0\frac{\partial V}{\partial t}\right)=-\frac{\rho}{\varepsilon_0}$$

Second, mixed term vanishes if

$$\vec{\nabla}\cdot\vec{A}=-\mu_0\varepsilon_0\frac{\partial V}{\partial t}$$

Lorentz Gauge

$$\vec{\nabla}\cdot\vec{A}_L\equiv-\mu_0\varepsilon_0\frac{\partial V_L}{\partial t}$$

Sort of Simple

$$\left(\nabla^2 V_L-\mu_0\varepsilon_0\frac{\partial^2 V_L}{\partial t^2}\right)=-\frac{\rho}{\varepsilon_0}$$

Sort of Simple

$$\left(\nabla^2\vec{A}-\mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2}\right)=-\mu_0\vec{J}$$

To relate back to Coulomb's Gauge

$$\vec{A}_L=\vec{A}_C+\vec{\nabla}\lambda_L \qquad V_L=V_C-\frac{\partial}{\partial t}\lambda_L$$

$$\vec{\nabla}\cdot\left(\vec{A}_C+\vec{\nabla}\lambda_L\right)=-\mu_0\varepsilon_0\frac{\partial V_L}{\partial t}=-\mu_o\varepsilon_o\frac{\partial}{\partial t}\left(V_C-\frac{\partial}{\partial t}\lambda_L\right)$$

$$\vec{\nabla}\cdot\vec{A}_C+\vec{\nabla}^2\lambda_L=-\mu_0\varepsilon_0\frac{\partial V_C}{\partial t}+\mu_0\varepsilon_0\frac{\partial^2\lambda_L}{\partial t^2}$$

$$\left(\vec{\nabla}^2\lambda_L-\mu_0\varepsilon_0\frac{\partial^2\lambda_L}{\partial t^2}\right)=-\mu_0\varepsilon_0\frac{\partial V_C}{\partial t}$$

Corresponding Relations between Potentials

(on the road to general solutions for E and B)

We want to solve for V and A given

*L*orentz Gauge

$$\vec{\nabla}\cdot\vec{A}_L \equiv -\mu_0\varepsilon_0\frac{\partial V_L}{\partial t}$$

$$\left(\nabla^2 - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\right)V_L = -\frac{\rho}{\varepsilon_0}$$

$$\left(\nabla^2 - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu_0\vec{J}$$

Minor Digression

$$\mu_0\varepsilon_0 = \left(4\pi\times10^{-7}\ {}^{N}\!/_{A^2}\right)\left(8.85\times10^{-12}\ {}^{C^2}\!/_{Nm^2}\right)$$

$$\mu_0\varepsilon_0 = \left(1.112\times10^{-17}\ {}^{s^2}\!/_{m^2}\right)$$

$$\mu_0\varepsilon_0 = \left(3.33\times10^{-9}\ {}^{s}\!/_{m}\right)^2$$

$$\mu_0\varepsilon_0 = \frac{1}{\left(2.9986\times10^{8}\ {}^{m}\!/_{s}\right)^2}$$

$$\mu_0\varepsilon_0 = \frac{1}{c^2}$$

$$\left(\nabla^2 - \tfrac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)V_L = -\frac{\rho}{\varepsilon_0}$$

$$\left(\nabla^2 - \tfrac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu_0\vec{J}$$

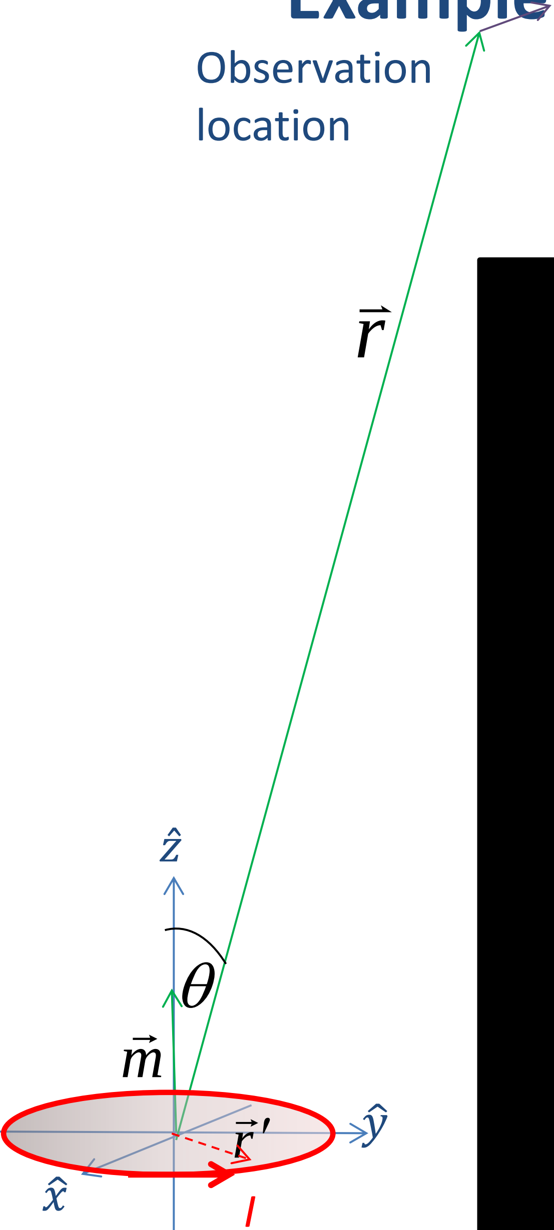
D'Alembertian

$$\Box^2 \equiv \left(\nabla^2 - \tfrac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)$$

$$\Box^2 V_L = -\frac{\rho}{\varepsilon_0}$$

$$\Box^2\vec{A} = -\mu_0\vec{J}$$

Example like Ex. 10.1 ? Time varying Dipole



$$\vec{A}(r) = \frac{\mu_o}{4\pi} \frac{m_o \sin(\omega t_r)\hat{z} \times \hat{r}}{r^2}$$

[redacted]

Side Note: Lorentz Force Law in Potential Form

(revisited now that we buy $\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$)

Consider your "system" a particle interacting with electric and magnetic fields

(*really* interacting with other charges via their electric and magnetic fields)

$$\frac{d}{dt}\vec{p} = \vec{F}_{net} = q\vec{v}\times\vec{B} + q\vec{E} = q\vec{v}\times\left(\vec{\nabla}\times\vec{A}\right) + q\left(-\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}\right) = q\vec{v}\times\left(\vec{\nabla}\times\vec{A}\right) + q\left(-\vec{\nabla}V - \frac{d}{dt}\vec{A} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{A}\right)$$

$$\frac{\partial \vec{A}}{\partial t} = \frac{d}{dt}\vec{A} - \left(\vec{v}\cdot\vec{\nabla}\right)\vec{A}$$

for any vector

$$\frac{d}{dt}\vec{p} = q\left(\vec{\nabla}\left(\vec{v}\cdot\vec{A}\right) - \left(\vec{v}\cdot\vec{\nabla}\right)\vec{A}\right) + q\left(-\vec{\nabla}V - \frac{d}{dt}\vec{A} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{A}\right)$$

$$\frac{d}{dt}\left(\vec{p} + q\vec{A}\right) = -\vec{\nabla}q\left(V - \vec{v}\cdot\vec{A}\right)$$

"p" "U"

$$\frac{d}{dt}\vec{A} = \frac{\partial \vec{A}}{\partial t} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{A}$$

Charge's experience of field varies with time because — field varies with time — and charge moves to where field may be different

$$\frac{d}{dt}\vec{A} = \frac{\partial \vec{A}}{\partial t} + \frac{dx}{dt}\frac{\partial}{\partial x}\vec{A} + \frac{dy}{dt}\frac{\partial}{\partial y}\vec{A} + \frac{dz}{dt}\frac{\partial}{\partial z}\vec{A}$$

By Product rule (4)

$$\vec{v}\times\left(\vec{\nabla}\times\vec{A}\right) = \vec{\nabla}\left(\vec{v}\cdot\vec{A}\right) - \left(\vec{A}\times\left(\vec{\nabla}\times\vec{v}\right)\right) + \left(\vec{A}\cdot\vec{\nabla}\right)\vec{v} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{A}$$

Derivative with respect to potential not source velocity

Consider your "system" a particle *and* the fields.

The force is negative gradient the potential energy

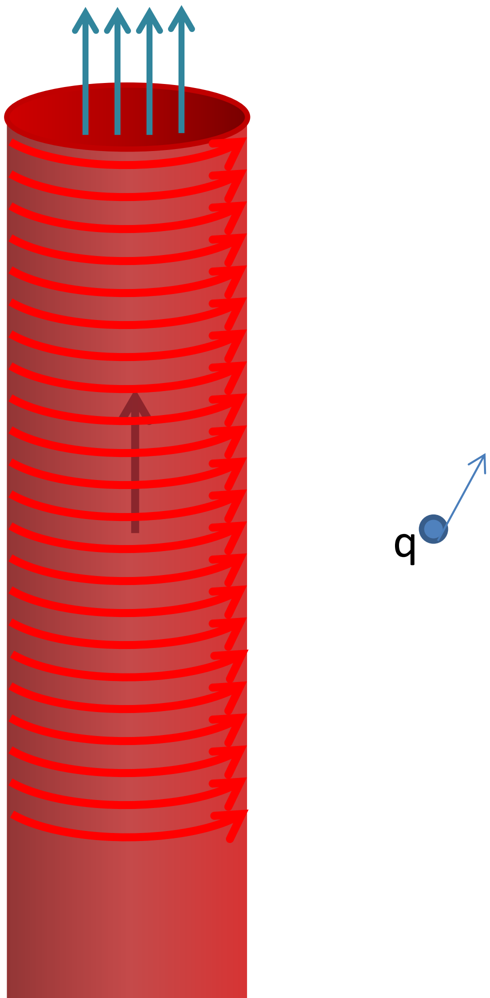
if $-\vec{\nabla}q\left(V - \vec{v}\cdot\vec{A}\right) = 0$ then $\vec{p}_i + q\vec{A}_i = \vec{p}_f + q\vec{A}_f = const$

'potential momentum'

Finding Vector Potential

$$\oint \vec{A}\cdot d\vec{\ell} = \int \vec{B}\cdot d\vec{a} = \Phi$$

Charged particle outside a disappearing solenoid



$$\vec{A}_{initially} = \begin{cases} (\mu_0 nIs/2)\hat{\phi} & s < R, \\ (\mu_0 nIR^2/2s)\hat{\phi} & s > R. \end{cases}$$

$$\vec{A}_{finally} = 0$$

initially

$$\frac{d}{dt}\left(m\vec{v} + q\vec{A}\right) = -\vec{\nabla}q\left(V - \vec{v}\cdot\vec{A}\right) = 0$$

$$m\vec{v}_i + q\vec{A}_i = m\vec{v}_f + q\vec{A}_f$$

$$q\left(\mu_0 nIR^2/2s\right)\hat{\phi} = m\vec{v}_f$$

$$\frac{q}{m}\left(\mu_0 nIR^2/2s\right)\hat{\phi} = \vec{v}_f$$

$-\hat{x}$

Continuous Source Distribution

Solve

$$\left(\nabla^2 - \tfrac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)V_L = -\frac{\rho}{\varepsilon_0} \qquad \left(\nabla^2 - \tfrac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu_0 \vec{J}$$

As with solving any differential equation, "inspired guess" is a valid solution method

a) We already know for static charge or current distributions

$$\nabla^2 V_L = -\frac{\rho}{\varepsilon_0} \quad \text{and} \quad \nabla^2 \vec{A} = -\mu_0 \vec{J}$$

Are solved by

$$V(\vec{r}) = \tfrac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,')}{\mathscr{r}}\,d\tau' \quad \text{and} \quad \vec{A}(\vec{r}) = \tfrac{\mu_o}{4\pi}\int \frac{\vec{J}(\vec{r}\,')}{\mathscr{r}}\,d\tau'$$

b) Without sources, we have the classic wave equation, so variations in *V* and *A* propagate

$$\left(\nabla^2 - \tfrac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)V_L = 0$$

$$\nabla^2 V_L = \tfrac{1}{c^2}\frac{\partial^2}{\partial t^2}V_L$$

$$V_L(t) \propto e^{i\vec{k}\cdot(\vec{\mathscr{r}} - \vec{c}t)}$$

So a variation in *V* observed by an observer at time *t* was generated at a distance *r* away at previous time

$$t_r \equiv t - \frac{\mathscr{r}}{c}$$

Combining what we know about these two special cases (constant or free space), we can guess

$$V(\vec{r},t) = \tfrac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,',t_r)}{\mathscr{r}}\,d\tau' \quad \text{and} \quad \vec{A}(\vec{r},t) = \tfrac{\mu_o}{4\pi}\int \frac{\vec{J}(\vec{r}\,',t_r)}{\mathscr{r}}\,d\tau'$$

Continuous Source Distribution

Solve

$$\left(\nabla^2 - \tfrac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)V_L = -\frac{\rho}{\varepsilon_0} \qquad \left(\nabla^2 - \tfrac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu_0 \vec{J}$$

Our guess

$$V(\vec{r},t) = -\tfrac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,',t_r)}{\mathscr{r}}\,d\tau' \qquad \vec{A}(\vec{r},t) = -\tfrac{\mu_o}{4\pi}\int \frac{\vec{J}(\vec{r}\,',t_r)}{\mathscr{r}}\,d\tau'$$

where

$$t_r = t - \frac{\mathscr{r}}{c} \qquad \vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$$

Plug in to test

$$\nabla^2 V = \vec{\nabla}_r \cdot \left(\vec{\nabla}_r V(\vec{r}\,', t_r)\right)$$

$$\vec{\nabla}_r \cdot \left(\vec{\nabla}_r \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,', t_r)}{\mathscr{r}}\,d\tau'\right)$$

$$\left(\vec{\nabla}_r \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,',t_r)}{\mathscr{r}}\,d\tau'\right) = \frac{1}{4\pi\varepsilon_0}\int \left(\frac{\vec{\nabla}_r\rho(\vec{r}\,',t_r)}{\mathscr{r}} + \rho(\vec{r}\,', t_r)\vec{\nabla}_r\left(\frac{1}{\mathscr{r}}\right)\right) d\tau'$$

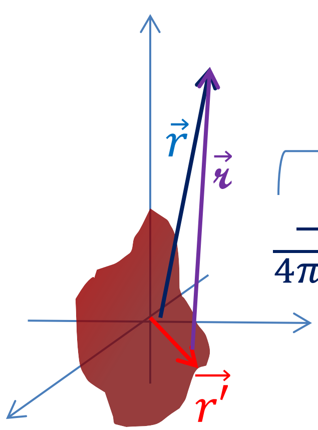
$$\vec{\nabla}_r\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$\frac{-1}{4\pi\varepsilon_o}\left(\int \frac{\dot{\rho}(\vec{r}\,', t_r)\hat{\mathscr{r}}}{c\mathscr{r}} + \frac{\rho(\vec{r}\,', t_r)\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'\right)$$

$$\frac{\partial t_r}{\partial \mathscr{r}} = -\frac{1}{c} \qquad \vec{\nabla}_r(\mathscr{r}) = \hat{\mathscr{r}}$$

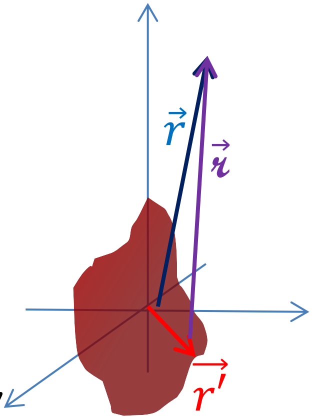
$$\vec{\nabla}_r\rho(\vec{r}\,', t_r) = \frac{\partial\rho(\vec{r}\,', t_r)}{\partial t_r}\frac{\partial t_r}{\partial \mathscr{r}}\vec{\nabla}_r(\mathscr{r})$$

$$\frac{\partial\rho(\vec{r}\,', t_r)}{\partial t_r} = \frac{\partial\rho(\vec{r}\,', t_r)}{\partial t} = \dot{\rho}(\vec{r}\,', t_r)$$



Del asks how detected voltage changes as we change observation locations *not* source locations.

Continuous Source Distribution



Solve

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)V_L = -\frac{\rho}{\varepsilon_0} \qquad \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu_0\vec{J}$$

Our guess

$$V(\vec{r},t) = -\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{r}',t_r)}{\mathscr{r}}d\tau' \qquad \vec{A}(\vec{r},t) = -\frac{\mu_o}{4\pi}\int\frac{\vec{J}(\vec{r}',t_r)}{\mathscr{r}}d\tau'$$

where

$$t_r = t - \frac{\mathscr{r}}{c} \qquad \vec{\mathscr{r}} = \vec{r} - \vec{r}'$$

Plug in to test

$$\nabla^2 V = \vec{\nabla}_r \cdot \left(\vec{\nabla}_r V(\vec{r}',t_r)\right)$$

$$\vec{\nabla}_r \cdot \left(\vec{\nabla}_r \frac{1}{4\pi\varepsilon_o}\int\frac{\rho(\vec{r}',t_r)}{\mathscr{r}}d\tau'\right)$$

$$\vec{\nabla}_r \cdot \frac{-1}{4\pi\varepsilon_o}\left(\int\frac{\dot{\rho}(\vec{r}',t_r)\hat{\mathscr{r}}}{c\mathscr{r}} + \frac{\rho(\vec{r}',t_r)\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau'\right) = \frac{-1}{4\pi\varepsilon_o}\int\left(\vec{\nabla}_r \cdot \left(\frac{\dot{\rho}(\vec{r}',t_r)\hat{\mathscr{r}}}{c\mathscr{r}}\right) + \vec{\nabla}_r \cdot \left(\frac{\rho(\vec{r}',t_r)\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\right)d\tau$$

$$\vec{\nabla}_r \cdot \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}} \qquad \vec{\nabla}_r \cdot \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\vec{\mathscr{r}})$$

Product Rule 5 $\quad \vec{\nabla}\cdot\left(f\vec{A}\right) = \left(\vec{\nabla}f\right)\cdot\vec{A} + f\left(\vec{\nabla}\cdot\vec{A}\right)$

$$= \frac{-1}{4\pi\varepsilon_o}\int\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}}\cdot\left(\vec{\nabla}_r\frac{\dot{\rho}(\vec{r}',t_r)}{c}\right) + \frac{\dot{\rho}(\vec{r}',t_r)}{c}\left(\vec{\nabla}_r\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}}\right)\right) + \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot\left(\vec{\nabla}_r\rho(\vec{r}',t_r)\right) + \rho(\vec{r}',t_r)\left(\vec{\nabla}_r\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\right)\right)d\tau'$$

$$\vec{\nabla}_r\dot{\rho}(\vec{r}',t_r) = -\frac{\ddot{\rho}(\vec{r}',t_r)}{c}\hat{\mathscr{r}} \qquad \vec{\nabla}_r\rho(\vec{r}',t_r) = -\frac{\dot{\rho}(\vec{r}',t_r)}{c}\hat{\mathscr{r}}$$

Continuous Source Distribution

Solve

$$\left(\nabla^2 - \tfrac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)V_L = -\frac{\rho}{\varepsilon_0} \qquad \left(\nabla^2 - \tfrac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu_0 \vec{J}$$

Our guess

$$V(\vec{r},t) = \tfrac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}',t_r)}{\mathscr{r}}\,d\tau' \qquad \vec{A}(\vec{r},t) = \tfrac{\mu_o}{4\pi}\int \frac{\vec{J}(\vec{r}',t_r)}{\mathscr{r}}\,d\tau'$$

where

$$t_r \equiv t - \frac{\mathscr{r}}{c}$$

Plug in to test

$$\nabla^2 V = \vec{\nabla}\cdot\left(\vec{\nabla}V\right)$$

$$\nabla^2 V = \frac{-1}{4\pi\varepsilon_o}\left(\int\left(-\frac{\ddot{\rho}(\vec{r}',t_r)}{\mathscr{r}c^2} + \frac{\dot{\rho}(\vec{r}',t_r)}{c\mathscr{r}^2}\right) + \left(-\frac{\dot{\rho}(\vec{r}',t_r)}{\mathscr{r}^2 c} + \rho(\vec{r}',t_r)4\pi\delta^3(\mathscr{r})\right)d\tau'\right)$$

$$\nabla^2 V = \frac{1}{4\pi\varepsilon_o}\int \frac{\ddot{\rho}(\vec{r}',t_r)}{\mathscr{r}c^2}\,d\tau' - \frac{\rho(r,t)}{\varepsilon_o}$$

Of course $\ddot{\rho}(\vec{r}',t_r) = \dfrac{\partial^2 \rho(\vec{r}',t_r)}{\partial t^2}$

$$\nabla^2 V = \tfrac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\frac{1}{4\pi\varepsilon_o}\int \frac{\rho(\vec{r}',t_r)}{\mathscr{r}}\,d\tau'\right) - \frac{\rho(r,t)}{\varepsilon_o}$$

$$\nabla^2 V = \tfrac{1}{c^2}\frac{\partial^2}{\partial t^2}V - \frac{\rho(r,t)}{\varepsilon_o}$$

✓

Continuous Source Distribution

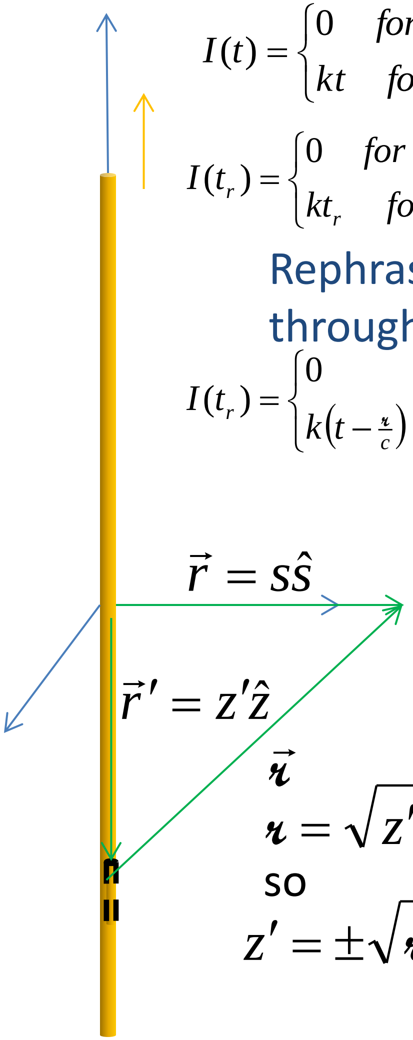
$$\vec{A}(\vec{r},t) = -\frac{\mu_o}{4\pi}\int \frac{\vec{J}(\vec{r}\,',t_r)}{\mathscr{r}}\,d\tau' \quad \text{where} \quad t_r \equiv t - \frac{\mathscr{r}}{c}$$

Example: find the Vector potential for a wire carrying a linearly growing current.

Defined piecewise through time

$$I(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ kt & \text{for} \quad t > 0 \end{cases}$$

$$I(t_r) = \begin{cases} 0 & \text{for} \quad t_r < 0 \\ kt_r & \text{for} \quad t_r > 0 \end{cases}$$



$$\vec{A}(\vec{r},t) = -\frac{\mu_o}{4\pi}\int_{-\infty}^{\infty} \frac{I(\vec{r}\,',t_r)}{\mathscr{r}}\,d\vec{l}\,' = -\frac{\mu_o}{4\pi}\int_{-\infty}^{\infty} \frac{I(\vec{r}\,',t-\frac{\mathscr{r}}{c})}{\mathscr{r}}\,dz'\hat{z}$$

As time goes on, observer becomes aware of more and more of wire starting to carry current. At any time, some morsels are just too far away to contribute. Limits should reflect that.

Rephrase as piecewise through space

$$I(t_r) = \begin{cases} 0 & \text{for} \quad t - \frac{\mathscr{r}}{c} < 0 \\ k\left(t - \frac{\mathscr{r}}{c}\right) & \text{for} \quad t - \frac{\mathscr{r}}{c} > 0 \end{cases} \quad \text{or} \quad \begin{matrix} t < \frac{\mathscr{r}}{c} \\ t > \frac{\mathscr{r}}{c} \end{matrix} \quad \text{or} \quad \begin{matrix} ct < \mathscr{r} \\ ct > \mathscr{r} \end{matrix} \quad \text{or} \quad \begin{matrix} |z'| < \sqrt{(ct)^2 - s^2} \\ |z'| > \sqrt{(ct)^2 - s^2} \end{matrix}$$

$$\mathscr{r} = \sqrt{z'^2 + s^2}$$

so

$$z' = \pm\sqrt{\mathscr{r}^2 - s^2}$$

$$\vec{A}(\vec{r},t) = -\frac{\mu_o}{4\pi}\int_{z'=-\sqrt{(ct)^2-s^2}}^{z'=\sqrt{(ct)^2-s^2}} \frac{k\left(t-\frac{\mathscr{r}}{c}\right)}{\mathscr{r}}\,dz'\hat{z}$$

$$= -\frac{\mu_o}{4\pi}k\left(t\int_{z'=-\sqrt{(ct)^2-s^2}}^{z'=\sqrt{(ct)^2-s^2}} \frac{dz'}{\sqrt{z'^2+s^2}} - \frac{1}{c}\int_{z'=-\sqrt{(ct)^2-s^2}}^{z'=\sqrt{(ct)^2-s^2}} dz'\right)$$

For first integral $\displaystyle\int_{z'_{\min}}^{z'_{\max}} \frac{dz'}{\sqrt{s^2+z'^2}} = \ln\left(\sqrt{s^2+z'^2}+z'\right)\Big|_{z'_{\min}}^{z'_{\max}}$

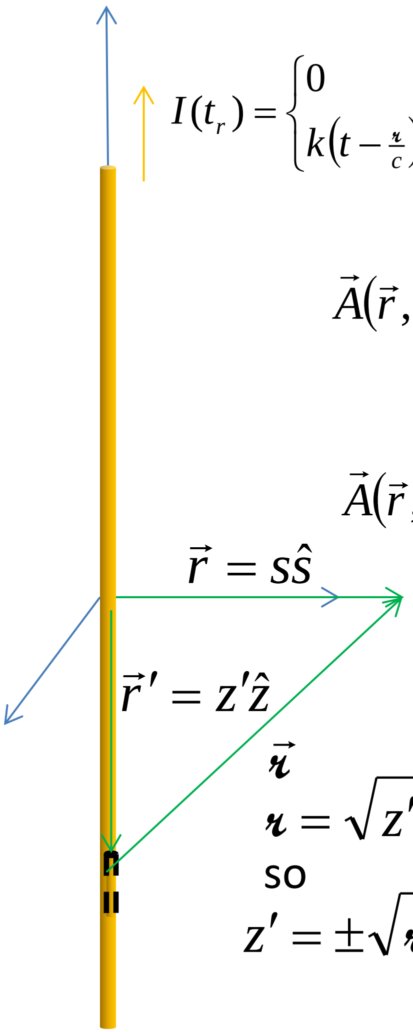
Continuous Source Distribution

$$\vec{A}(\vec{r},t) = -\frac{\mu_o}{4\pi}\int \frac{\vec{J}(\vec{r}\,',t_r)}{\mathscr{r}}\,d\tau' \quad \text{where} \quad t_r \equiv t - \frac{\mathscr{r}}{c}$$

Example: find the Vector potential for a wire carrying a linearly growing current.

Defined piecewise through time

$$\vec{A}(\vec{r},t) = -\frac{\mu_o}{4\pi}\int_{-\infty}^{\infty} \frac{I(\vec{r}\,',t_r)}{\mathscr{r}}\,d\vec{l}\,'$$



$$I(t_r) = \begin{cases} 0 & \text{for} \quad t - \frac{\mathscr{r}}{c} < 0 \\ k\left(t - \frac{\mathscr{r}}{c}\right) & \text{for} \quad t - \frac{\mathscr{r}}{c} > 0 \end{cases}$$

or

$$z' < \sqrt{(ct)^2 - s^2}$$

$$z' > -\sqrt{(ct)^2 - s^2}$$

$$\vec{A}(\vec{r},t) = -\frac{\mu_o}{4\pi}k\left(t\ln\left(\frac{\sqrt{\left((ct)^2 - s^2\right) + s^2} + \sqrt{(ct)^2 - s^2}}{\sqrt{\left((ct)^2 - s^2\right) + s^2} - \sqrt{(ct)^2 - s^2}}\right) - \frac{1}{c}\left(\sqrt{(ct)^2 - s^2} - -\sqrt{(ct)^2 - s^2}\right)\right)\hat{z}$$

$$\vec{A}(\vec{r},t) = -\frac{\mu_o}{4\pi}k\left(t\ln\left(\frac{ct + \sqrt{(ct)^2 - s^2}}{ct - \sqrt{(ct)^2 - s^2}}\right) - \frac{2\sqrt{(ct)^2 - s^2}}{c}\right)\hat{z}$$

$$\mathscr{r} = \sqrt{z'^2 + s^2}$$

so

$$z' = \pm\sqrt{\mathscr{r}^2 - s^2}$$

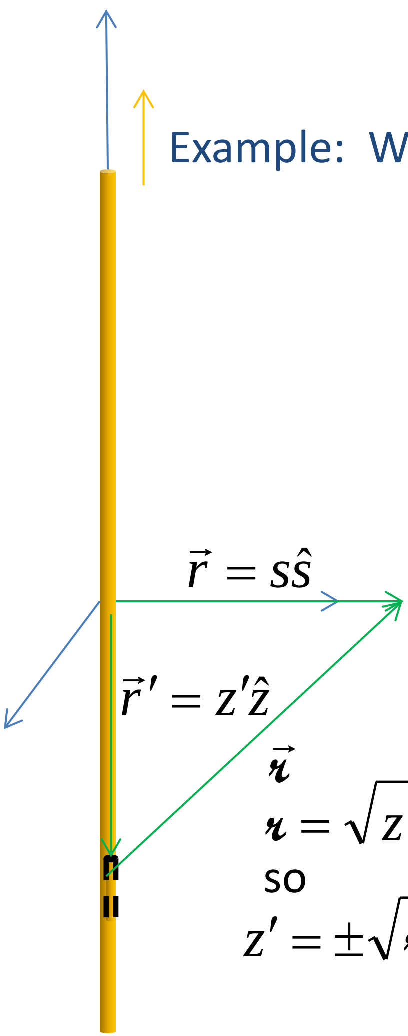
$$\vec{A}(\vec{r},t) = \begin{cases} -\frac{\mu_o}{4\pi}kt\left(\ln\left(\frac{1 + \sqrt{1 - \left(\frac{s}{ct}\right)^2}}{1 - \sqrt{1 - \left(\frac{s}{ct}\right)^2}}\right) - 2\sqrt{1 - \left(\frac{s}{ct}\right)^2}\right)\hat{z} & \text{for} \quad s < ct \\ 0 & \text{for} \quad s > ct \end{cases}$$

Continuous Source Distribution

$$\vec{A}(\vec{r},t) = -\frac{\mu_o}{4\pi}\int \frac{\vec{J}(\vec{r}',t_r)}{\mathscr{r}}\,d\tau' \quad \text{where} \quad t_r \equiv t - \frac{\mathscr{r}}{c}$$

$$\vec{A}(\vec{r},t) = \begin{cases} -\frac{\mu_o}{4\pi}kt\left(\ln\left(\dfrac{1+\sqrt{1-\left(\frac{s}{ct}\right)^2}}{1-\sqrt{1-\left(\frac{s}{ct}\right)^2}}\right) - 2\sqrt{1-\left(\frac{s}{ct}\right)^2}\right)\hat{z} & \text{for } s < ct \\ 0 & \text{for } s > ct \end{cases}$$

Example: What are B and E?



$$\vec{\nabla}\times\vec{A} \equiv \vec{B}$$

$$\vec{B}(\vec{r},t) = \begin{cases} -\dfrac{\partial}{\partial s}\left(-\frac{\mu_o}{4\pi}kt\left(\ln\left(\dfrac{1+\sqrt{1-\left(\frac{s}{ct}\right)^2}}{1-\sqrt{1-\left(\frac{s}{ct}\right)^2}}\right) - 2\sqrt{1-\left(\frac{s}{ct}\right)^2}\right)\right)\hat{\phi} & \text{for } s < ct \\ 0 & \text{for } s > ct \end{cases}$$

A bit of math later:

$$\vec{B}(\vec{r},t) = \begin{cases} -\frac{\mu_o}{4\pi c}2k\sqrt{\left(\frac{ct}{s}\right)^2 - 1}\,\hat{\phi} & \text{for } s < ct \\ 0 & \text{for } s > ct \end{cases}$$

$$\mathscr{r} = \sqrt{z'^2 + s^2}$$

so

$$z' = \pm\sqrt{\mathscr{r}^2 - s^2}$$

Continuous Source Distribution

$$\vec{A}(\vec{r},t) = -\frac{\mu_o}{4\pi}\int \frac{\vec{J}(\vec{r}\,',t_r)}{\mathscr{r}}\,d\tau' \quad \text{where} \quad t_r \equiv t - \frac{\mathscr{r}}{c}$$

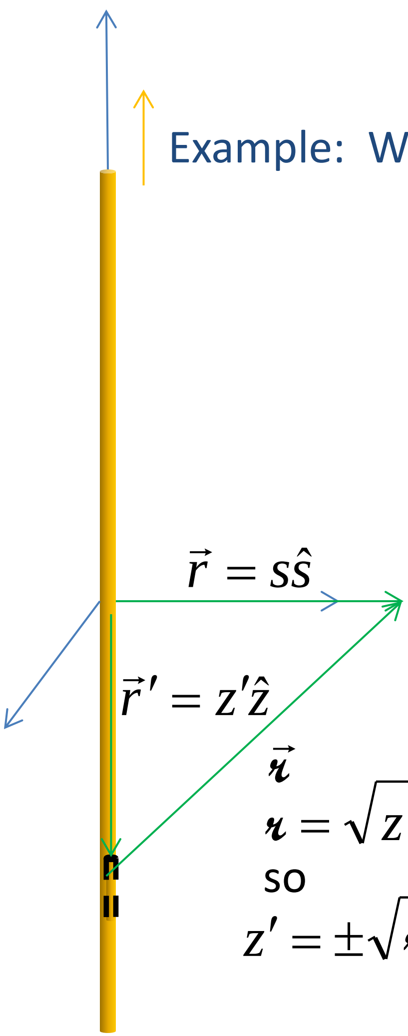
$$\vec{A}(\vec{r},t) = \begin{cases} -\frac{\mu_o}{4\pi} kt\left(\ln\left(\frac{1+\sqrt{1-\left(\frac{s}{ct}\right)^2}}{1-\sqrt{1-\left(\frac{s}{ct}\right)^2}}\right) - 2\sqrt{1-\left(\frac{s}{ct}\right)^2}\right)\hat{z} & \text{for } s < ct \\ 0 & \text{for } s > ct \end{cases}$$

Example: What are B and E?

$$\vec{B}(\vec{r},t) = \begin{cases} -\frac{\mu_o}{4\pi c} 2k\sqrt{\left(\frac{ct}{s}\right)^2 - 1}\;\hat{\phi} & \text{for } s < ct \\ 0 & \text{for } s > ct \end{cases}$$

$$\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}$$

All but one factor of t is bound up in (s/ct), so same thing, times –(s/t), in z direction, and a term for the one lone t



$$\mathscr{r} = \sqrt{z'^2 + s^2}$$

so

$$z' = \pm\sqrt{\mathscr{r}^2 - s^2}$$

$$\vec{E}(\vec{r},t) = \left(\frac{\mu_o}{4\pi} k\left(\ln\left(\frac{1+\sqrt{1-\left(\frac{s}{ct}\right)^2}}{1-\sqrt{1-\left(\frac{s}{ct}\right)^2}}\right) + 2\sqrt{1-\left(\frac{s}{ct}\right)^2}\right) + \frac{\mu_o}{4\pi} 2k\sqrt{1-\left(\frac{s}{ct}\right)^2}\right)\hat{z}$$

$$\vec{E}(\vec{r},t) = \begin{cases} \frac{\mu_o}{4\pi} k\ln\left(\frac{1+\sqrt{1-\left(\frac{s}{ct}\right)^2}}{1-\sqrt{1-\left(\frac{s}{ct}\right)^2}}\right)\hat{z} & \text{for } s < ct \\ 0 & \text{for } s > ct \end{cases}$$

Continuous Source Distribution

$$\vec{A}(\vec{r},t) = -\frac{\mu_o}{4\pi}\int \frac{\vec{J}(\vec{r}\,',t_r)}{\mathscr{r}}\,d\tau' \quad \text{where} \quad t_r \equiv t - \frac{\mathscr{r}}{c}$$

Exercise: find the Vector potential for a wire that momentarily had a burst of current.

Defined piecewise through time

$$I(t) = q_o\delta(t - t_b)$$

$$\vec{A}(\vec{r},t) = -\frac{\mu_o}{4\pi}\int_{-\infty}^{\infty} \frac{I(\vec{r}\,',t_r)}{\mathscr{r}}\,d\vec{l}\,'$$

So, at some time, t_b, the current will blink on and off again. The observer will first notice the middle blink, then just either side of the middle, then a little further out,…

$$\vec{A}(\vec{r},t) = -\frac{\mu_o}{4\pi}\int_{-\infty}^{\infty} \frac{q_o\delta(t_r - t_b)}{\mathscr{r}}\,dz'\hat{z}$$

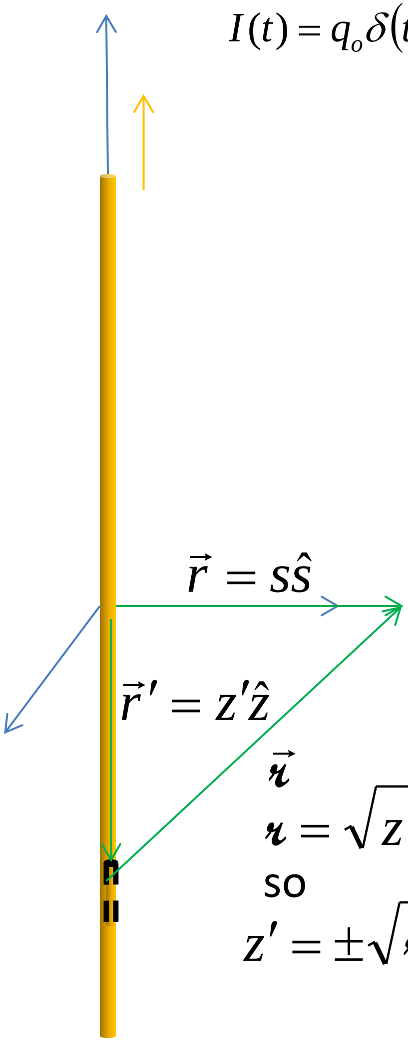
So, we get contribution to our integral only when

$$t_b = t_r = t - \frac{\mathscr{r}}{c}$$

$$\mathscr{r} = c(t - t_b)$$

Which is true at *two* locations at any moment t:

$$z' = \pm\sqrt{(c(t - t_b))^2 - s^2}$$



$$\mathscr{r} = \sqrt{z'^2 + s^2}$$

so

$$z' = \pm\sqrt{\mathscr{r}^2 - s^2}$$

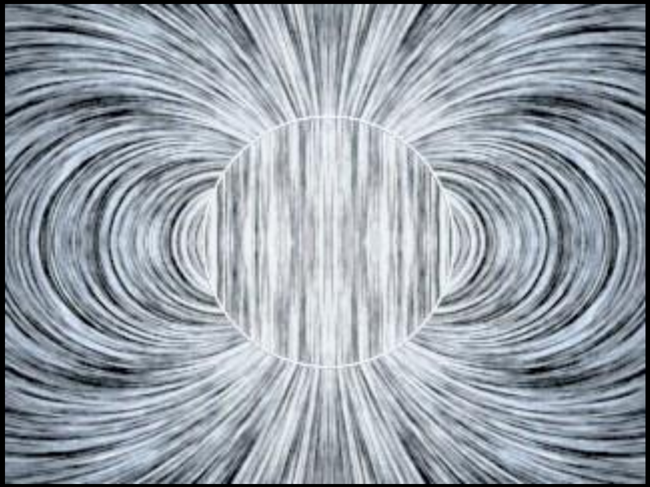
We could rephrase the delta function as being a spike at these two locations, or we could observe the integral is 'even' and then wave our hands

$$\vec{A}(\vec{r},t) = -\frac{\mu_o}{4\pi}2\int_{0}^{\infty} \frac{q_o\delta(t_r - t_b)}{\mathscr{r}}\,dz'\hat{z} = -\frac{\mu_o}{2\pi}\frac{q_o}{c(t - t_b)}\hat{z}$$

Continuous Source Distribution

$$\vec{A}(\vec{r},t) = -\frac{\mu_o}{4\pi}\int \frac{\vec{J}(\vec{r}\,',t_r)}{\mathscr{r}}\,d\tau' \quad \text{where} \quad t_r \equiv t - \frac{\mathscr{r}}{c}$$

Charged sphere spinning up from rest



http://web.mit.edu/viz/spin/ choose slow spin up – time evolving magnetic field for a sphere of charge spinning up

Wed.	10.1 - .2.1 Potential Formulation Lunch with UCR Engr – 12:20 – 1:00	
Fri.	10.2 Continuous Distributions	
Mon.	10.3 Point Charges	HW11