Wed. 10/2 Fri. 10/4	**3.2** Images **T4** Relaxation Method **3.4.1-.4.2** Multipole Expansion	
Mon. 10/7 Wed. 10/9 Thurs 10/10	**3.4.3-.4.4** Multipole Expansion *(C 17)* **12.1.1-.1.2, 12.3.1** E to B; **5.1.1-.1.2** Lorentz Force Law: fields and forces	HW4

Materials

Corner mirror (demonstrate Pr. 19)

Announcements

Last Times

This Time

Using the Image Charge technique
Another approach all together – Relaxation
Begin Multi-Pole Expansion

Summary

Methods of Finding Solutions

We are going to discuss three methods of solving Laplace's equation:

(1) *Method of Images* – replace a problem with a simpler equivalent one (based on corollary of the first uniqueness theorem)

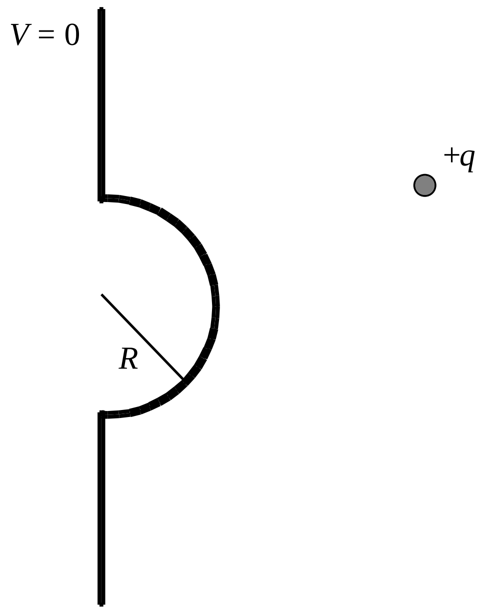
(2) *Relaxation Method* – a computational method based on the potential at a point being the average of the values at the same distance (more about Next Time).

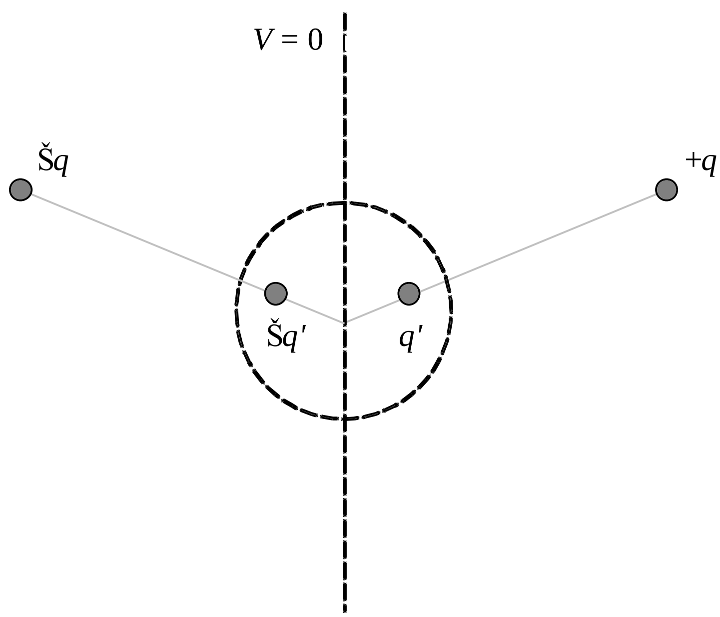
(3) *Multipole Expansion* – a method for getting approximate answers for *V* far from a charge distribution

Method of Images.

Examples/Exercises:

What is the equivalent image charge configuration? (a hemispherical bump on a plane)





Summary

Multipole Expansion

Maybe the place to begin this discussion is with the irrefutable mathematical statement

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta')$$

Where P_n is the n^th^ Legendre polynomial.

$$P_0 = 1$$

$$P_1(u) = u$$

$$P_2(u) = (3u^2 - 1)/2$$

$$\ldots$$

Griffiths demonstrated that this was the case; we'll pick up from there.

So, what the heck is that good for? Well,

a) The potential due to a set of point charges involves terms with $\frac{1}{\mathscr{r}}$ for each source. So, if you want the potential due to a series of sources, it's $\frac{1}{4\pi\varepsilon_o}\sum_i \frac{q_i}{\mathscr{r}_i}$. So it certainly looks *relevant.*

b) As for useful, each $\mathscr{r}_i = \sqrt{r^2 + r'^2 - 2rr'\cos\theta_{r\to r'}}$. So, they all have something in common, that r (vector pointing to the observation location) and it would be nice to be able to 'factor' that out, but it's kind of locked inside the sqrt. *This* expression allows you to factor it out.

   a. $V(r) = \frac{1}{4\pi\varepsilon_o}\sum_i \frac{q_i}{\mathscr{r}_i} = \frac{1}{4\pi\varepsilon_o}\sum_i \frac{q_i}{r}\sum_{n=0}^{\infty}\left(\frac{r_i'}{r}\right)^n P_n(\cos\theta_i') = \frac{1}{4\pi\varepsilon_o}\sum_{n=0}^{\infty} r^{-(n+1)}\sum_i r_i'^n P_n(\cos\theta_i')q_i$

   b. Or for a continuous distribution, we're looking at the integral

   $V(r) = \frac{1}{4\pi\varepsilon_o}\sum_{n=0}^{\infty} r^{-(n+1)}\int r_i'^n P_n(\cos\theta_i')\rho(r')d\tau'$

c) I know, you're saying, 'that doesn't look like much of a *simplification* to me!' But, much like a Taylor Series, you seldom use the whole darn sum, instead, you use only enough terms to get an answer that's 'good enough.' In particular, such series are handy when the variable of expansion is very small so you only need the first few terms. In this case, the variable of expansion is (mostly) $\frac{r_i'}{r}$.

a. When the distance (from the origin) to the observation location is much greater than the distance (from the origin) to any of the sources, then you only need to hold the first few terms.
b. **Warning – relative to the origin.** Pay special attention to the parenthetical (from the origin). Mathematically speaking, you have to watch out for that. The different terms in the series will evaluate to different values depending on where your origin is.
c. **Useful origin.** Often, these expansions are used to answer the question of 'how does this charge distribution look from very far away.' What we usually mean by that is 'when the distance *from the charges* to the observation location is much greater than *the distance between the charges.* To answer that question, you want the origin to be *inside the charge distribution.*

The first four Terms

Analogy to Taylor Expansion. When you approximate a mathematical function with a Taylor Series,

$$F(x) = F(x_o) + \left.\frac{dF}{dx}\right|_o (x - x_o) + \tfrac{1}{2}\left.\frac{d^2F}{dx^2}\right|_o (x - x_o)^2 + \tfrac{1}{6}\left.\frac{d^3F}{dx^3}\right|_o (x - x_o)^3 + \ldots$$

You say 'if the point of evaluation is close enough to xo, then the function's value is just that at x_o – it's a constant; but if you back out a little, then it's like that constant plus a little stretch up a line; you back out a little further still, it's like that constant plus a stretch along the line plus a little arc up a parabola,…

Now for the Multipole Expansion. Similarly, if you're quite far from the sources, you say, it's like a point charge way out there; if you get a little closer, you can see that there's some slight polarization – a little more charge on this end than the other – so it's like a point charge + a dipole; you get a little closer and you an resolve 'it's like a point charge + a dipole + a quadrupole,… Just like the first few terms of a Taylor Series Expansion are graphically simple building blocks, the first few terms of a Multipole Series Expansion are like (differential forms of) reasonably simple charge distributions: The monopole, the Dipole, the Quadrupole, the Octopole.

Monopole: far *enough* from a (non-neutral) charge distribution, the voltage looks like that of a point charge

$$V \approx \frac{1}{4\pi\varepsilon_0}\frac{Q}{r},$$

where the total charge is

$$Q = \sum_i q_i \rightarrow \int \rho(\vec{r}\,')d\tau'.$$

We saw many examples of the electric field going to the limit of what it would be for all of the charge treated like a point charge.

What if $Q = 0$? A simple example is the physical dipole – equal and opposite charges ($\pm q$) separated by a distance d. In that case,

$$V \approx \frac{1}{4\pi\varepsilon_0}\frac{qd\cos\theta}{r^2}.$$

QuickTime™ and a
TIFF (Uncompressed) decompressor
are needed to see this picture.

Different arrangements of charges have potentials that fall off more quickly as the distance gets large.

QuickTime™ and a
TIFF (Uncompressed) decompressor
are needed to see this picture.

$$V(\vec{r}) = V_{\text{mon}}(\vec{r}) + V_{\text{dip}}(\vec{r}) + V_{\text{quad}}(\vec{r}) + \ldots$$

The first two terms are:

$$V_{\text{mon}}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\sum_i q_i$$

$$V_{\text{dip}}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\sum_i q_i r_i'\cos\theta' = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$$

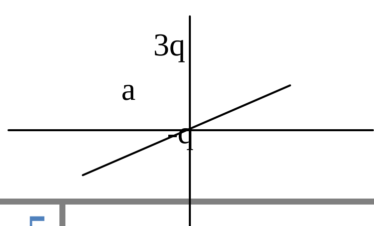
where the dipole moment is defined using the property $\vec{r}'\cdot\hat{r} = r'\cos\theta'$:

$$\vec{p} = \sum_i q_i \vec{r}_i'.$$

Warning: These position vectors are relative to an origin. Just like a "moment of inertia" exactly what you get depends on the point you're measuring against (in that case, the axis of rotation.) For the series to converge the fastest, you want the origin to be in the center of charge, so all *r's* are as small as they can be.

Examples/Exercises:

Problem 3.30(a) (EXAMPLE): Find the first two terms in the mulitpole expansion for the figure shown below.



The total charge is $Q = 2q$, so

$$V_{\text{mon}}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{2q}{r}.$$

The dipole moment is

$$\vec{p} = \sum_i q_i \vec{r}_i' = (-q)(0) + (3q)(a\hat{z}) = 3qa\,\hat{z},$$

so

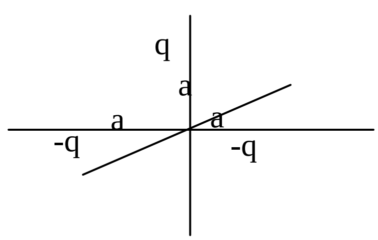
$$V_{\text{dip}}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{3qa}{r^2}\hat{z}\cdot\hat{r} = \frac{1}{4\pi\varepsilon_0}\frac{3qa}{r^2}\cos\theta.$$

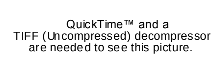
In the last step, use $\hat{z}\cdot\hat{r} = \cos\theta$ (draw the vector $\vec{r}$ in the diagram above).

Q: What if we move the origin up, half-way between the charges?

As with a Taylor series; though the sum of all terms may be insensitive to where your reference point, the exact contribution of each term is quite sensitive.

Problem 3.32 (EXERCISE): Find the first two terms in the mulitpole expansion for the figure shown below.





The total charge is $Q = -q$, so

$$V_{\text{mon}}(\vec{r}) = -\frac{1}{4\pi\varepsilon_0}\frac{q}{r}.$$

The dipole moment is

$$\vec{p} = \sum_i q_i \vec{r}_i' = (-q)(-a\hat{y}) + (-q)(a\hat{y}) + q(a\hat{z}) = qa\,\hat{z},$$

so

$$V_{\text{dip}}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{qa}{r^2}\hat{z}\cdot\hat{r} = \frac{1}{4\pi\varepsilon_0}\frac{qa}{r^2}\cos\theta.$$

Relaxation Method

Go over the handout – download the example from the course webpage

Have students do the exercises – bring Python program with solutions (email to self)

[]

Have students go through exercises on the tutorial about the *relaxation method*.

1. Give hints about how to set `NX` and `NY`. For example:

$$NX = \left(\frac{x_{\max} - x_{\min}}{d} \right) + 1$$

   because the expression in brackets gives the number of "gaps," but there is one more row or column in the matrix.

2. Page 4 gives the lines to produce `X` and `Y` grids. Don't forget to put the line: "`from pylab import *`" at the beginning of the program.

3. This is not essential. It just speeds things up.

"I didn't understand the example that started on p. 152 so I hope we can go over that. Also can we talk about the concept/meaning of the dipole moment?"
Sam Hide response Post a response
Admin

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I would also be interested to know more about the dipole moment. Especially why it is useful to us if we separate it from r and whether or not it is of any use to us if we are not using it to calculate Vdip.

Ben Kid

"I'm having a hard time following the mathematics in the examples. Could we spend some time discussing this?"
Spencer Post a response
Admin

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"Does the dipole moment have any real significance other than to provide a more succinct def. of potential. And, perhaps it was the integration of the binomial expansion that got me, but was that carried out to apply to every type of possible -pole?"
Rachael Hach Hide responses Post a response
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Or what I mean is, could it apply to a quadrapole or octopole, if, that is in fact a real thing?

Rachael Hach

I'd also appreciate talking more about the conceptual meaning of section 3.4.2.

Casey McGrath

"It was unclear to me how Griffiths reasoning about how we know that the potential of the different configurations fall off. I get his reasoning for the monopole and the dipole falling off, but I don't get it for the quadrupole or octopole."
Jessica Post a response
Admin