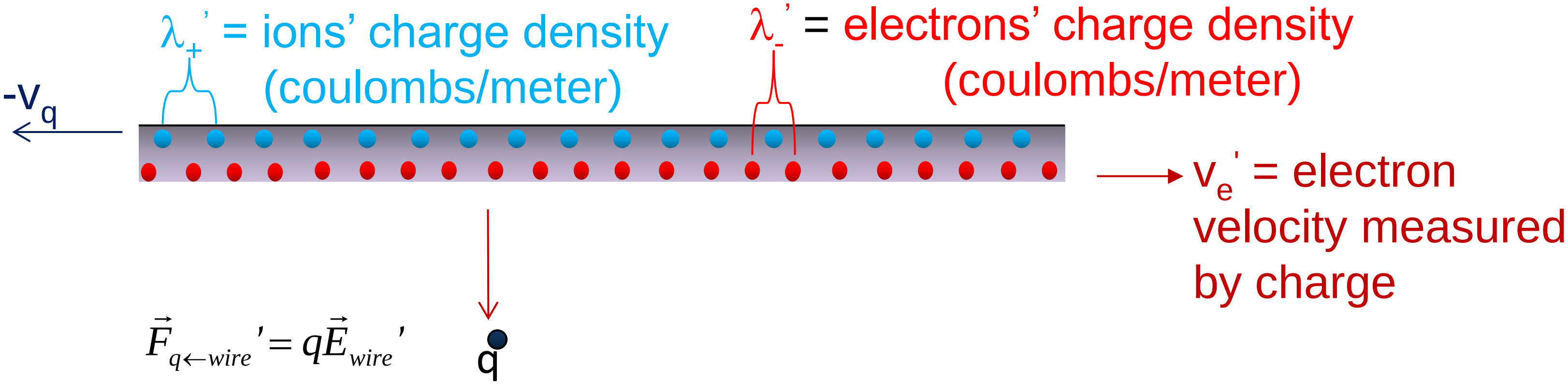
Wed. Thurs. Fri.	(C 17) 5.1.3 Lorentz Force Law: currents (C 17) 5.2 Biot-Savart Law	HW6

Transition / Transformation from E to M

Charge's frame:

$$\lambda_+' = \frac{\gamma_{q'} e}{\Delta x_{lab}} \qquad \lambda_-' = \frac{-\gamma_{e'} e}{\gamma_e \Delta x_{lab}} = \frac{-\gamma_{e'}}{\gamma_e \gamma_{q'}} \lambda_+'$$



$$\vec{F}_{q\leftarrow wire}' = q\vec{E}_{wire}'$$

$$\vec{E}_{wire}' = \vec{E}_+' + \vec{E}_-' = \tfrac{1}{4\pi\varepsilon_o}\frac{2\lambda_+'}{r} + \tfrac{1}{4\pi\varepsilon_o}\frac{2\lambda_-'}{r}$$

$$= \tfrac{1}{4\pi\varepsilon_o}\frac{2\lambda_+\gamma_{q'}}{r}\left(1 - \frac{\gamma_{e'}}{\gamma_e\gamma_{q'}}\right)$$

$$E_{wire}' = \tfrac{1}{4\pi\varepsilon_o}\frac{2\lambda_+\gamma_{q'}}{r}\frac{v_e v_q}{c^2}$$

$$E_{wire}' = \tfrac{\mu_o}{4\pi}\frac{2I\gamma_{q'}}{r}v_q$$ Define $I = \lambda_+ v_e$ and $\mu_o \equiv \tfrac{1}{\varepsilon_o c^2}$

$$\vec{F}_{q\leftarrow wire}' = qv_q \tfrac{\mu_o}{4\pi}\frac{2I}{r}\gamma_{q'}$$

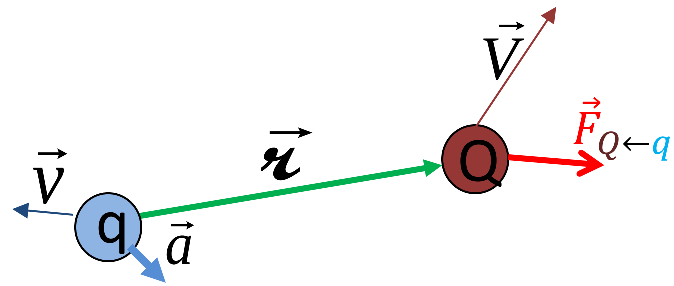
Positive charge moving *with* current attracts (moving opposite current repels)

$$\vec{F}_{q\leftarrow wire} = qv_q \tfrac{\mu_o}{4\pi}\frac{2I}{r} \qquad \vec{F}_{q\leftarrow wire} = qv_q B\hat{s}$$

(radially out from wire)

What would happen to a charge initially moving radially inside the hole of a current loop?

Force between moving charges



$$\vec{u} \equiv c\hat{\mathscr{r}} - \vec{v}$$

$$\vec{F}_{Q\leftarrow q} = \frac{qQ}{4\pi\varepsilon_o}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\left\{[(c^2 - v^2)\vec{u} + \vec{\mathscr{r}} \times (\vec{u} \times \vec{a})] + \frac{\vec{V}}{c} \times \left[\hat{\mathscr{r}} \times [(c^2 - v^2)\vec{u} + \vec{\mathscr{r}} \times (\vec{u} \times \vec{a})]\right]\right\}$$

Electric

Depends on *observer's* perception of *source* charge's velocity and acceleration

Magnetic

Also depends on *observer's* perception of *recipient* charge's velocity

Magnetic force

$$\vec{F}_{Q\leftarrow q.mag} = Q\vec{V} \times \frac{q}{4\pi c\varepsilon_o}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\left\{\left[\hat{\mathscr{r}} \times [(c^2 - v^2)\vec{u} + \vec{\mathscr{r}} \times (\vec{u} \times \vec{a})]\right]\right\}$$

Magnetic Field, B

$$\vec{F}_{Q\leftarrow q.mag} = Q\vec{V} \times \vec{B}$$

Qualitatively rationalize cross product from charge's perspective

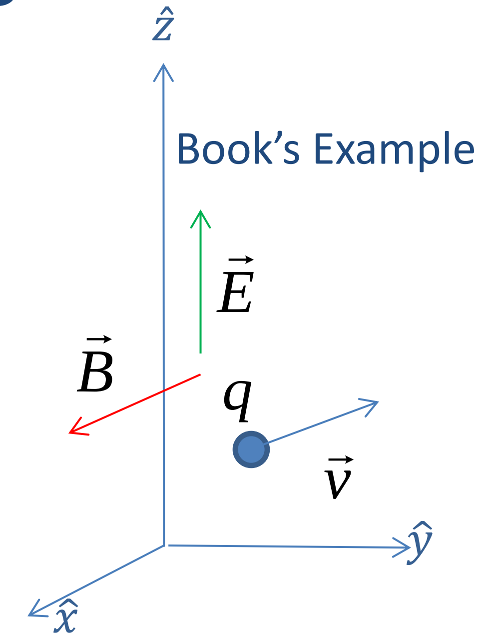
Cyclotron Motion in a Uniform Magnetic Field

$$\frac{d\vec{p}}{dt} = \vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

$$m\frac{d\vec{v}}{dt} = q\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

In general

$$m\begin{Bmatrix} dv_x/dt \\ dv_y/dt \\ dv_z/dt \end{Bmatrix} = q\left(\begin{Bmatrix} E_x \\ E_y \\ E_z \end{Bmatrix} + \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ v_x & v_y & v_z \\ B_x & B_y & B_z \end{vmatrix}\right) = q\begin{Bmatrix} E_x + v_yB_z - v_zB_y \\ E_y + v_zB_x - v_xB_z \\ E_z + v_xB_y - v_yB_x \end{Bmatrix}$$



$$\frac{dv_y}{dt} = \frac{qB}{m}v_z \qquad \frac{dv_z}{dt} = \frac{qE}{m} - \frac{qB}{m}v_y$$

Take next derivative

$$\frac{d^2v_y}{dt^2} = \frac{qB}{m}\frac{dv_z}{dt} \qquad \frac{d^2v_z}{dt^2} = -\frac{qB}{m}\frac{dv_y}{dt}$$

Cross substitute

$$\frac{d^2v_y}{dt^2} = \frac{qB}{m}\left(\frac{qE}{m} - \frac{qB}{m}v_y\right) \qquad \frac{d^2v_z}{dt^2} = -\left(\frac{qB}{m}\right)^2 v_z$$

$$\frac{d^2v_y}{dt^2} = \frac{q^2BE}{m^2} - \left(\frac{qB}{m}\right)^2 v_y$$

Guess Solution Forms

$$v_z(t) = C_1\cos(\omega t) + C_2\sin(\omega t)$$

$$v_y(t) = C_3\cos(\omega t) + C_4\sin(\omega t) + \frac{E}{B}$$

Plug in

$$\frac{dv_y}{dt} = \frac{qB}{m}v_z$$

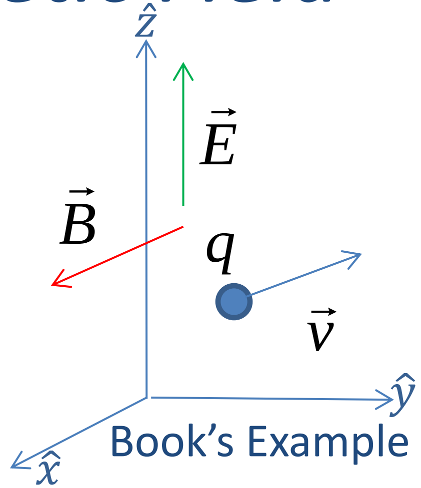
$$\omega\left(-C_3\sin(\omega t) + C_4\cos(\omega t)\right) = \frac{qB}{m}\left(C_1\cos(\omega t) + C_2\sin(\omega t)\right)$$

Compare terms and conclude

$$\omega = \frac{qB}{m} \qquad -C_3 = C_2 \qquad C_4 = C_1$$

Cyclotron Motion in a Uniform Magnetic Field

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$



$$v_z(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t) \qquad v_y(t) = C_1 \sin(\omega t) - C_2 \cos(\omega t) + \frac{E}{B}$$

where $\omega = \frac{qB}{m}$

For position components, integrate

$$z(t) = \frac{C_1}{\omega} \sin(\omega t) - \frac{C_2}{\omega} \cos(\omega t) + C_5 \qquad y(t) = -\frac{C_1}{\omega} \cos(\omega t) - \frac{C_2}{\omega} \sin(\omega t) + \frac{E}{B} t + C_6$$

Impose Initial Conditions

Start at origin

$$z(0) = -\frac{C_2}{\omega} + C_5 = 0 \qquad y(0) = -\frac{C_1}{\omega} + C_6 = 0$$

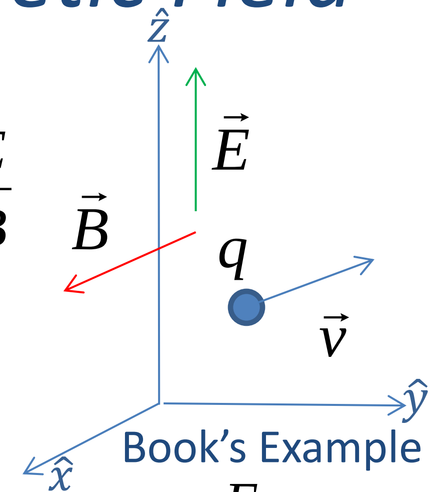
$$C_5 = \frac{C_2}{\omega} \qquad C_6 = \frac{C_1}{\omega}$$

$$z(t) = \frac{C_1}{\omega} \sin(\omega t) + \frac{C_2}{\omega} \left(1 - \cos(\omega t)\right) \qquad y(t) = \frac{C_1}{\omega} \left(1 - \cos(\omega t)\right) - \frac{C_2}{\omega} \sin(\omega t) + \frac{E}{B} t$$

Cyclotron Motion in a Uniform Magnetic Field

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

$$v_z(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t) \qquad v_y(t) = C_1 \sin(\omega t) - C_2 \cos(\omega t) + \frac{E}{B}$$



where $\omega = \frac{qB}{m}$

Impose Initial Conditions

Initial position: (0,0)

$$z(t) = \frac{C_1}{\omega} \sin(\omega t) + \frac{C_2}{\omega}\left(1 - \cos(\omega t)\right) \qquad y(t) = \frac{C_1}{\omega}\left(1 - \cos(\omega t)\right) - \frac{C_2}{\omega}\sin(\omega t) + \frac{E}{B} t$$

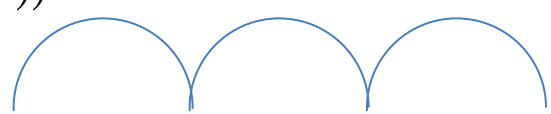
Initial Velocity: $\vec{v}(0) = 0$

$$v_z(0) = C_1 \cos(0) + C_2 \sin(0) \qquad v_y(0) = C_1 \sin(0) - C_2 \cos(0) + \frac{E}{B}$$

$$v_z(0) = C_1 = 0 \qquad v_y(0) = -C_2 + \frac{E}{B} = 0 \qquad C_2 = \frac{E}{B}$$

So.

$$v_z(t) = \frac{E}{B}\sin(\omega t) \qquad v_y(t) = \frac{E}{B}\left(1 - \cos(\omega t)\right)$$

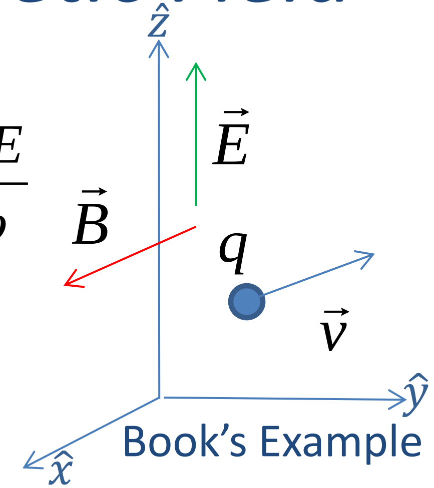


$$z(t) = \frac{E}{\omega B}\left(1 - \cos(\omega t)\right) \qquad y(t) = \frac{E}{\omega B}\left(\omega t - \sin(\omega t)\right)$$

Cyclotron Motion in a Uniform Magnetic Field

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

$$v_z(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t) \qquad v_y(t) = C_1 \sin(\omega t) - C_2 \cos(\omega t) + \frac{mE}{\omega}$$



where $\omega = \frac{qB}{m}$

Impose Initial Conditions

Initial position: (0,0)

$$z(t) = \frac{C_1}{\omega} \sin(\omega t) + \frac{C_2}{\omega}\left(1 - \cos(\omega t)\right) \qquad y(t) = \frac{C_1}{\omega}\left(1 - \cos(\omega t)\right) - \frac{C_2}{\omega} \sin(\omega t) + \frac{mE}{\omega} t$$

Initial Velocity: $\vec{v}(0) = (E / B)\hat{y}$

$$v_y(0) = C_1 \sin(0) - C_2 \cos(0) + \frac{mE}{\omega}$$

$$v_z(0) = C_1 \cos(0) + C_2 \sin(0)$$

$$v_y(0) = -C_2 + \frac{mE}{\omega} = \frac{E}{B}$$

$$v_z(0) = C_1 = 0$$

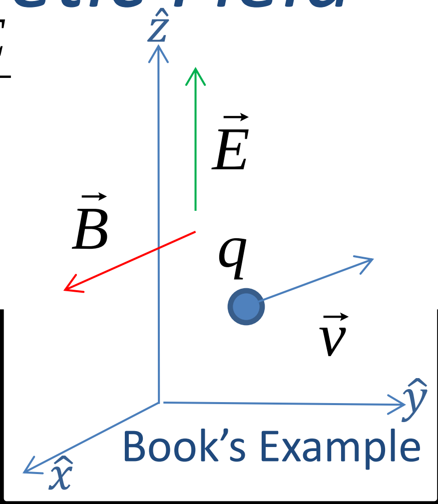
$$C_2 = \frac{mE}{\omega} - \frac{E}{B}$$

Cyclotron Motion in a Uniform Magnetic Field

$$v_z(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t) \qquad v_y(t) = C_1 \sin(\omega t) - C_2 \cos(\omega t) + \frac{mE}{\omega}$$

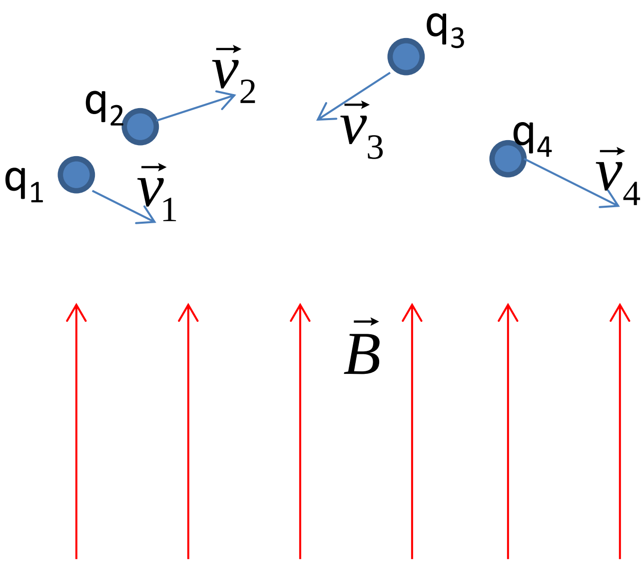
where $\omega = \frac{qB}{m}$

Exercise: Say $\vec{v}(0) = (E/B)(\hat{y} + \hat{z})$ and $\vec{r}(0) = 0$



[redacted]

Magnetic Force *on* Charge Distribution



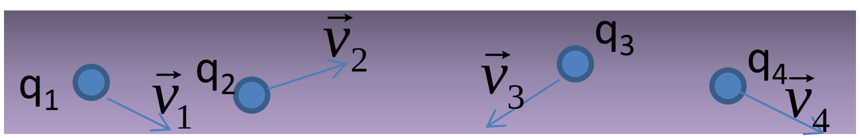
$$\vec{F} = q_1\vec{v}_1 \times \vec{B} + q_2\vec{v}_2 \times \vec{B} + q_3\vec{v}_3 \times \vec{B} + q_4\vec{v}_4 \times \vec{B} + \ldots$$

$$\vec{F} = \left(\sum_i q_i\vec{v}_i\right) \times \vec{B}$$

Q: Sure, you *can* calculate this, but when is it physically meaningful? **A:** When all moving charges are confined to an object: wire, surface, volume… so the magnetic force on *them* is essentially the magnetic force on the object as a whole.

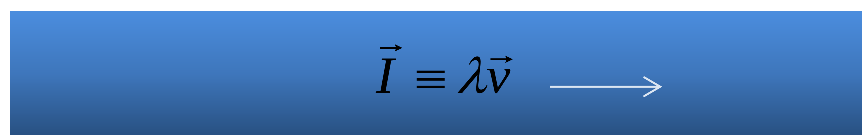
1-D collection of moving charges: wire

Discrete



$$\vec{F} = \left(\sum_i q_i\vec{v}_i\right) \times \vec{B}$$

Continuous



$$\vec{F} \Rightarrow \int dq\vec{v} \times \vec{B} \quad = \int dl \frac{dq}{dl}\vec{v} \times \vec{B}$$

$$= \int dl\lambda\vec{v} \times \vec{B} \quad = \int dl\vec{I} \times \vec{B}$$

Current

$$\vec{I} \equiv \lambda\vec{v}$$

$$\vec{I} = \frac{dq}{dl}\frac{d\vec{l}}{dt}$$

$$I = \frac{dq}{dt}$$

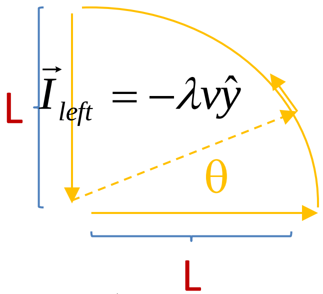
For charge flow confined to infinitesimally-thin wire must flow *along* wire, so $\hat{v} = d\hat{l}$ and

$$\vec{F} = \int dl\vec{I} \times \vec{B} = \int d\vec{l}\, I \times \vec{B}$$

Note: *I* and *B* can vary along (be functions of) length.

Example: Force on arced, constant-current-carrying wire in uniform field

a) If field B points out of the board (z direction), what are the forces on each leg, and what's the net force on the structure?



$$\vec{I}_{left} = -\lambda v\hat{y}$$

$$\vec{I}_{bottom} = \lambda v\hat{x}$$

I=λ*v*

$$\vec{I}_{arc} = \lambda v\left(-\sin\theta\hat{x} + \cos\theta\hat{y}\right)$$

$$\vec{F}_{mag} = \int dl\vec{I} \times \vec{B}$$

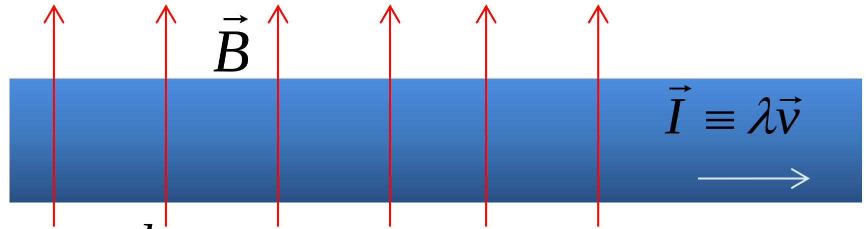
b) If field B points right across the board (x direction), what are the forces on each leg, and what's the net force on the structure?

Exercise:

a) A square loop of side a lying in the yz plane and centered on the origin, carrying current I counterclockwise when viewed down the x axis. $\vec{B} = kz\hat{x}$. What is the force?

Magnetic Force *on* Charge Distribution

1-D collection of moving charges: wire

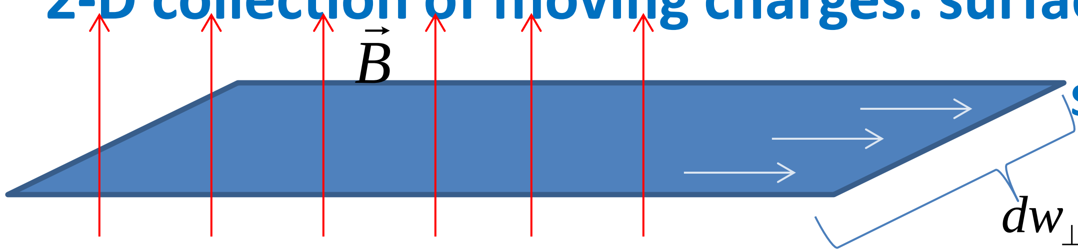


$$\vec{F} \Rightarrow \int dq\vec{v}\times\vec{B} = \int dl\,\frac{dq}{dl}\vec{v}\times\vec{B} = \int dl\lambda\vec{v}\times\vec{B} = \int dl\vec{I}\times\vec{B} \qquad \vec{I} \equiv \lambda\vec{v}$$

For charge flow confined to infinitesimally-thin wire must flow *along* wire, so $\hat{v} = d\hat{l}$ **and** $\vec{F} = \int dl\vec{I}\times\vec{B} = \int d\vec{l}\,I\times\vec{B}$

Note: *I* and *B* can vary along (be functions of) length.

2-D collection of moving charges: surface



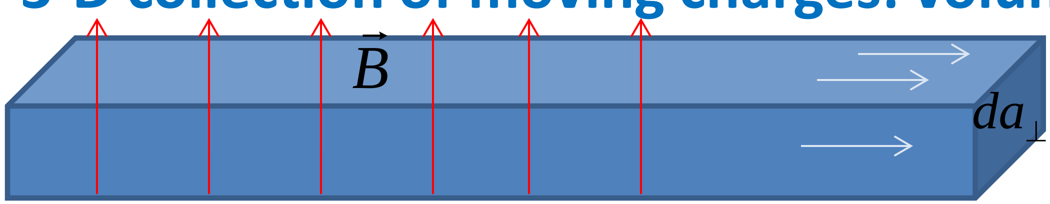
Surface Current Density

$$\vec{K} \equiv \sigma\vec{v}$$

$$\vec{K} = \frac{d\vec{I}}{dw_\perp}$$

$$\vec{F} = \int dq\vec{v}\times\vec{B} = \int da\,\frac{dq}{da}\vec{v}\times\vec{B} = \int da\sigma\vec{v}\times\vec{B} = \int da\vec{K}\times\vec{B}$$

3-D collection of moving charges: volume



Volume Current Density

$$\vec{J} \equiv \rho\vec{v}$$

$$\vec{J} = \frac{d\vec{I}}{da_\perp}$$

$$\vec{F} = \int dq\vec{v}\times\vec{B} = \int d\tau\,\frac{dq}{d\tau}\vec{v}\times\vec{B} = \int d\tau\rho\vec{v}\times\vec{B} = \int d\tau\vec{J}\times\vec{B}$$

Exercise:

a) Current I flows down wire of radius a. If it's uniformly distributed over the wire's cylindrical surface, then what is the surface current density *K*?

b) Current I flows down wire of radius a. If it's distributed throughout the volume such that $J \propto \frac{1}{s}$, what would be the expression for the volume charge density?

Charge Continuity Equation

3-D collection of moving charges

Volume Current Density

$$\vec{J} \equiv \rho \vec{v}$$

$$\vec{J} = \frac{d\vec{I}}{da_\perp}$$

Current crossing through closed surface is rate of change of charge in enclosed volume:

$$\oint \vec{J} \cdot d\vec{a} = I = -\frac{dq}{dt}$$

Fundamental theorem for Divergences

$$\int \left(\vec{\nabla} \cdot \vec{J}\right) d\tau = I = -\frac{d}{dt} \int \rho d\tau$$

Equating Integrands

$$\vec{\nabla} \cdot \vec{J} = -\frac{d\rho}{dt}$$

A divergence of current means a depletion of charge

Wed. Thurs. Fri.	(C 17) 5.1.3 Lorentz Force Law: currents (C 17) 5.2 Biot-Savart Law	HW6