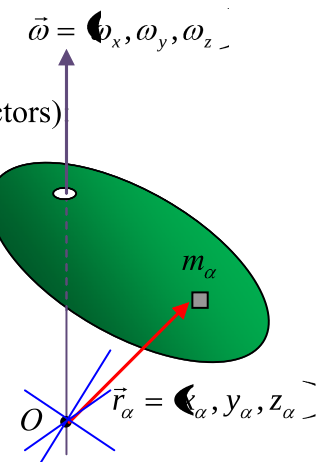
Phys 331: 10.3-.4 Rotating about any Axis, Inertia Tensor, Principle Axes

Mon., 12/3	**10.3-.4** Rotation about any Axis, Inertia Tensor Principle Axes	
Tues. 12/4		**HW10a** (10.6-.22)
Wed., 12/5	**10.5-.6** Finding Principle Axes, Precession	
Thurs. 12/6		**HW10b** (10.36, 10.39)
Fri., 12/7	**10.7-.8** Euler's Equations	

10.3 Rotation about Any Axis; the Inertia Tensor:

Suppose a body is spinning about an arbitrary axis (not necessarily z) through any point on the body. The body's angular momentum relative to an origin on the axis is:



$$\vec{L} = \sum m_\alpha \vec{r}_\alpha \times \vec{v}_\alpha = \sum m_\alpha \vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha),$$

where $\vec{r}_\alpha$ is the position of the mass m_α. Use the relation (note that both sides are vectors)

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}),$$

or for a component ($i = x,\ y,$ or z):

$$[\vec{A} \times (\vec{B} \times \vec{C})] = B_i(\vec{A} \cdot \vec{C}) - C_i(\vec{A} \cdot \vec{B}).$$

Each term of the sum for $\vec{L}$ involves a position $\vec{r} = (x, y, z)$ and the angular velocity $\vec{\omega} = (\omega_x, \omega_y, \omega_z)$. The x component of one term in the sum is:

$$[\vec{r} \times (\vec{\omega} \times \vec{r})]_x = \omega_x(x^2 + y^2 + z^2) - x(x\omega_x + y\omega_y + z\omega_z) = (y^2 + z^2)\omega_x - xy\omega_y - xz\omega_z.$$

Similarly, we get (check these for yourself):

$$[\vec{r} \times (\vec{\omega} \times \vec{r})]_y = -yx\omega_x + (x^2 + z^2)\omega_y - yz\omega_z,$$

$$[\vec{r} \times (\vec{\omega} \times \vec{r})]_z = -zx\omega_x - zy\omega_y + (x^2 + y^2)\omega_z.$$

The components of the angular momentum can be written as:

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z,$$

$$L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z,$$

$$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z,$$

where:

$$I_{xx} = \sum m_\alpha (y_\alpha^2 + z_\alpha^2),$$

$$I_{xy} = I_{yx} = -\sum m_\alpha x_\alpha y_\alpha,$$

and so on for the other I's. Of course, these expressions can be turned into integrals for solid objects.

The I's can be written as a 3 × 3 matrix called the *inertia tensor* (note the double arrow symbol):

$$\overleftrightarrow{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}.$$

If we think of the angular momentum and angular velocity vectors as 3 × 1 matrices:

$$\vec{L} = \begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} \quad \text{and} \quad \vec{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix},$$

then the relationship between the angular momentum and the angular velocity can be written compactly as:

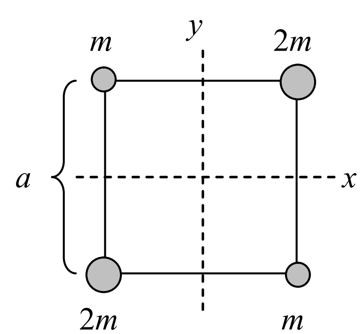
$$\vec{L} = \overleftrightarrow{I}\vec{\omega}.$$

There are only six elements of the moment of inertia to be calculated because:

$$I_{ij} = I_{ji}.$$

The *transpose* of a matrix A, labeled $\tilde{A}$, is found by reflecting the matrix across its main diagonal (the elements A_{ii}). The inertia tensor is equal to its own transpose, $I = \tilde{I}$, which means it is a *symmetric* matrix.

<u>Example #1</u>: The four point masses below are connected by massless, rigid rods of length a. Find the inertia tensor using the axes shown (the z axis points out of the page).



In the sums below, the contribution from the upper left mass is first and the others follow in clockwise order.

I Do:

$$I_{xx} = \sum m_\alpha \left(y_\alpha^2 + z_\alpha^2\right) = ma^2\left[\left(\tfrac{1}{2}\right)^2 + 2\left(\tfrac{1}{2}\right)^2 + \left(-\tfrac{1}{2}\right)^2 + 2\left(-\tfrac{1}{2}\right)^2\right] = \tfrac{3}{2}ma^2,$$

$$I_{yy} = \sum m_\alpha \left(x_\alpha^2 + z_\alpha^2\right) = ma^2\left[\left(-\tfrac{1}{2}\right)^2 + 2\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2 + 2\left(-\tfrac{1}{2}\right)^2\right] = \tfrac{3}{2}ma^2,$$

They Do:

$$I_{zz} = \sum m_\alpha \left(x_\alpha^2 + y_\alpha^2\right) = ma^2\left[\left(\left(-\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2\right) + 2\left(\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2\right) + \left(\left(\tfrac{1}{2}\right)^2 + \left(-\tfrac{1}{2}\right)^2\right) + 2\left(\left(-\tfrac{1}{2}\right)^2 + \left(-\tfrac{1}{2}\right)^2\right)\right]$$
$$= 3ma^2,$$

$$I_{xy} = I_{yx} = -\sum m_\alpha x_\alpha y_\alpha = -ma^2\left[\left(-\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) + 2\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) + \left(-\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) + 2\left(-\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right)\right] = -\tfrac{1}{2}ma^2,$$

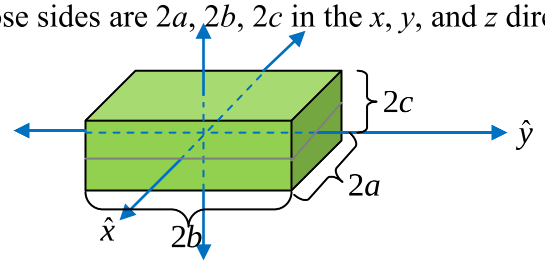
$$I_{xz} = I_{zx} = -\sum m_\alpha x_\alpha z_\alpha = -ma^2(0+0+0) = 0,$$

$$I_{yz} = I_{zy} = -\sum m_\alpha y_\alpha z_\alpha = -ma^2(0+0+0) = 0.$$

The inertia tensor is:

$$\overleftrightarrow{I} = \begin{bmatrix} 3/2 & -1/2 & 0 \\ -1/2 & 3/2 & 0 \\ 0 & 0 & 3 \end{bmatrix} ma^2.$$

<u>Example #2</u>: (Prob 10.25) Find the inertia tensor with respect to the CM of a uniform cuboid (a rectangular brick shape) whose sides are $2a$, $2b$, $2c$ in the x, y, and z directions and whose mass is M.



The density of the cuboid is:

$$\rho = \frac{M}{(2a)(2b)(2c)} = \frac{M}{8abc}.$$

Calculate the elements of the inertia tensor by dividing the cuboid into small pieces with dimensions dx, dy, and dz. The mass of each piece is $\rho\, dx\, dy\, dz$. The sums become integrals, so:

$$I_{xx} = \sum m_\alpha \left(y_\alpha^2 + z_\alpha^2\right) \to \int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a} \left(y^2 + z^2\right)\rho\, dx\, dy\, dz,$$

$$I_{xx} = \frac{M}{8abc}(2a)\int_{-c}^{c}\int_{-b}^{b}\left(y^2+z^2\right)dy\, dz = \frac{M}{4bc}\left[(2c)\int_{-b}^{b} y^2 dy + (2b)\int_{-c}^{c} z^2 dz\right],$$

$$I_{xx} = \frac{M}{4bc}\left\{(2c)\left[\frac{y^3}{3}\right]_{-b}^{b} + (2b)\left[\frac{z^3}{3}\right]_{-c}^{c}\right\} = \frac{M}{4bc}\left(\frac{4cb^3}{3} + \frac{4bc^3}{3}\right),$$

$$I_{xx} = \tfrac{1}{3}M\left(b^2 + c^2\right).$$

Similarly, the other diagonal terms are:

$$I_{yy} = \tfrac{1}{3}M\left(a^2 + c^2\right) \quad \text{and} \quad I_{zz} = \tfrac{1}{3}M\left(a^2 + b^2\right).$$

The object has reflection symmetry across the $z = 0$ plane, so the products of inertia involving z:

$$I_{xz} = I_{zx} = -\sum m_\alpha x_\alpha z_\alpha = 0\,,$$

$$I_{yz} = I_{zy} = -\sum m_\alpha y_\alpha z_\alpha = 0\,,$$

because for every mass at (x,y,z) there is an equal mass at $(x,y,-z)$. Similarly, the remaining products of inertia involving y, $I_{xy} = I_{yx}$, are zero because there is reflection symmetry for the $y = 0$ axis. The inertia tensor is:

$$\overleftrightarrow{I} = \tfrac{1}{3} M \begin{bmatrix} b^2 + c^2 & 0 & 0 \\ 0 & a^2 + c^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{bmatrix}.$$

Exercise: (Thornton 11-13) Find the inertia tensor for an object with:

$$m_1 = 3m \quad \text{at} \quad (b,0,b),$$

$$m_2 = 4m \quad \text{at} \quad (b,b,-b),$$

$$m_3 = 2m \quad \text{at} \quad (-b,b,0).$$

Writing the contributions from the masses in the order given above:

$$I_{xx} = \sum m_\alpha \left(y_\alpha^2 + z_\alpha^2\right) = mb^2 \left[3(0+1) + 4(1+1) + 2(1+0)\right] = 13mb^2,$$

$$I_{yy} = \sum m_\alpha \left(x_\alpha^2 + z_\alpha^2\right) = mb^2 \left[3(1+1) + 4(1+1) + 2(1+0)\right] = 16mb^2,$$

$$I_{zz} = \sum m_\alpha \left(x_\alpha^2 + y_\alpha^2\right) = mb^2 \left[3(1+0) + 4(1+1) + 2(1+1)\right] = 15mb^2,$$

$$I_{xy} = I_{yx} = -\sum m_\alpha x_\alpha y_\alpha = -mb^2 \left[3(1)(0) + 4(1)(1) + 2(-1)(1)\right] = -2mb^2,$$

$$I_{xz} = I_{zx} = -\sum m_\alpha x_\alpha z_\alpha = -mb^2 \left[3(1)(1) + 4(1)(-1) + 2(-1)(0)\right] = mb^2,$$

$$I_{yz} = I_{zy} = -\sum m_\alpha y_\alpha z_\alpha = -mb^2 \left[3(0)(1) + 4(1)(-1) + 2(1)(0)\right] = 4mb^2.$$

The inertia tensor is:

$$\overleftrightarrow{I} = \begin{bmatrix} 13 & -2 & 1 \\ -2 & 16 & 4 \\ 1 & 4 & 15 \end{bmatrix} mb^2.$$

10.4 Principal Axes of Inertia:

While the inertia tensor is generally of the form

$$\vec{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

For some chosen set of coordinate axes, $\hat{x}, \hat{y}, \hat{z}$;

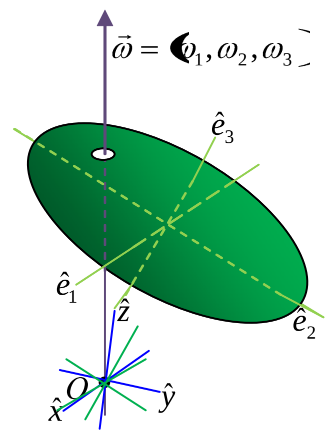
Theorem - For any rigid body and any point O, there are three perpendicular principal axes through O. In other words, you can choose perpendicular axes so that the inertia tensor is diagonal.

(We will not prove this!)

We call this set of exes the *principle* axes of the object. For obviously symmetric objects, it's these are the obvious axes of symmetry. But even for less obviously symmetric objects, a coordinate axis exists, call them $\hat{e}_1, \hat{e}_2, \hat{e}_3$, so that expressed in terms of *this* set of axes (rather than an arbitrary x, y, and z), the inertia tensor is diagonal:

Personally, I'd then write the inertia tensor as

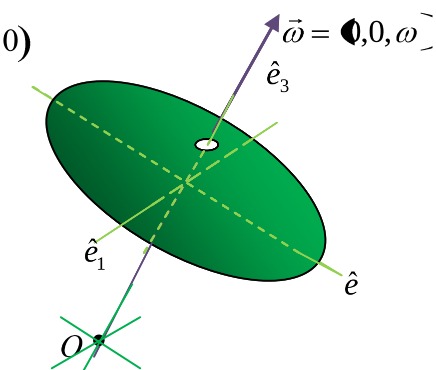
$$\vec{I} = \begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix}$$



But the book uses λ instead:

$$\vec{I} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix},$$

For rotation purely about one of these axes, say $\hat{e}_2$, then $\vec{\omega} = (0, \omega, 0)$



$$\vec{L} = \vec{I}\vec{\omega} = \begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix} \begin{bmatrix} 0 \\ \omega_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ I_{22}\omega \\ 0 \end{bmatrix}$$

That is to say, $\vec{L}$ is parallel to $\vec{\omega}$ so:

$$\vec{L} = I_{22}\vec{\omega},$$

which is the very simple relation we're familiar with from Phys 231 or even earlier in this course.

Next time, we'll learn how to find the principal axes. The next two times, we will also learn why principal axes are important (how they are used in calculations).