Mon., 11/26	**9.6-.7** Fictional Forces: Centrifugal & Coriolis	
Tues. 11/27		**HW9b** (9.14, 9.24)
Wed., 11/28	**9.8-.9** Free Fall & Coriolis, Foucault Pendulum	
Thurs. 11/29		**HW9c** (9.25, 9.27)
Fri., 11/30	**10.1-.2** Center of Mass & Rotation about a Fixed Axis	

Equipment:

- Globe
- Ball with coordinate axes
- Turntable with paper taped to it

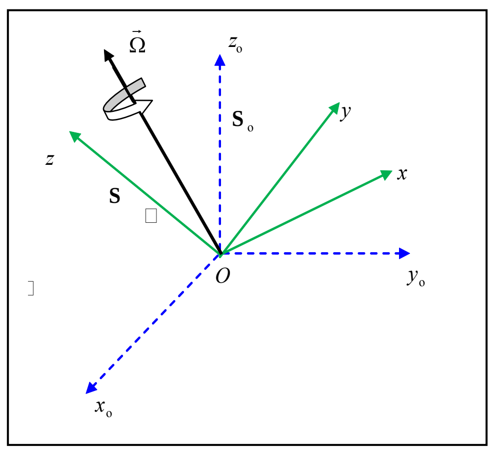
Non-inertial Frames: Rotating

Last time we learned that when one frame is rotating relative to the other, say, the Earth, relative to the 'fixed stars', then velocity and acceleration measurements made in the two frames are related by

$\dot{\vec{r}} = \dot{\vec{r}}_o - \vec{V}_f(\vec{r})$ where $\vec{V}_f(\vec{r}) = \vec{\Omega}\times\vec{r}$, i.e., $\vec{V}_f(\vec{r}) = r_{axis}\Omega\hat{\phi}$

And

$\ddot{\vec{r}} = \ddot{\vec{r}}_o - \left(\vec{A}_f(\vec{r}) + \vec{A}_{Corr}(\dot{\vec{r}})\right)$ where $\vec{A}_f = \vec{A}_{cent} = \left(\vec{r}\times\vec{\Omega}\right)\times\vec{\Omega} = \left(-r_{axis}\dot{\phi}^2\right)\hat{r}_{axis}$ and $\vec{A}_{corr} = 2\dot{\vec{r}}\times\vec{\Omega}$



Of course, Newton's 2nd Law applies only in an inertial frame

$$\ddot{\vec{r}}_o = \frac{\vec{F}_{net}}{m}$$

So,

$$\ddot{\vec r} = \frac{\vec F_{net}}{m} - \left(\vec A_f(\vec r) + \vec A_{Corr}(\dot{\vec r})\right)$$

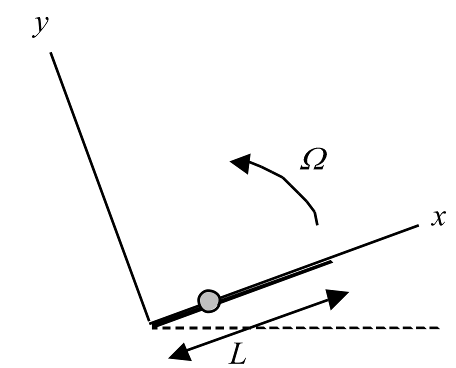
Fictitious Inertial / Frame Force

$$m\ddot{\vec r} = \vec F_{net} + \vec F_{frame}$$

$\vec F_{frame} \equiv -m\left(\vec A_f(\vec r) + \vec A_{Corr}(\dot{\vec r})\right) = \vec F_{centripetal} + \vec F_{coriolis}$ where

$\vec F_{cent} = -m\vec A_{cent} = m\left(\vec\Omega\times\vec r\right)\times\vec\Omega = \left(mr_{axis}\dot\phi^2\right)\hat r_{axis}$ and $\vec F_{corr} = -m\vec A_{corr} = m2\vec\Omega\times\dot{\vec r}$

Example: (Ex. 5.8 F&C) A rod of length L is rotating at a constant rate of Ω in a horizontal plane. A bead starts just off the axis of rotation, a distance L_o, at rest (with respect to the rod). Ignore friction. How long will it take the bead to reach the other end of the rod?



Use coordinates that rotate with the rod so that the x axis follows the rod, so the equation of motion is:

$$m\ddot{\vec r} = \vec F + 2m\dot{\vec r}\times\vec\Omega + m\left(\vec\Omega\times\vec r\right)\times\vec\Omega .$$

The angular momentum is $\vec\Omega = \Omega\hat z$. The bead is constrained to move along the x axis and the rod can only exert a normal force in the y direction since there is no friction, so:

$$m\ddot x\hat x = N_y\hat y + N_z\hat z - mg\hat z + 2m\dot x\hat x\times\left(\Omega\hat z\right) + m\left(\Omega\hat z\times x\hat x\right)\times\Omega\hat z ,$$

$$m\ddot x\hat x = N_y\hat y + N_z\hat z - mg\hat z - 2m\dot x\Omega\hat y + m\Omega^2 x\hat x .$$

The three component equations are:

$$m\ddot x = m\Omega^2 x , \qquad N_y = 2m\dot x\Omega , \qquad N_z = mg.$$

The second equation relates the size of the horizontal component of the normal force to the Coriolis force. The solution to the first equation is:

$$x(t) = Ae^{\Omega t} + Be^{-\Omega t} ,$$

so:

$$\dot{x}(t) = A\Omega e^{\Omega t} - \Omega B e^{-\Omega t}.$$

The initial conditions are $x = L_o$ and $\dot{x}(0) = 0$, so:

$$A + B = L_o \quad \text{and} \quad (A - B)\Omega = 0.$$

The second is $A = B$, so adding the conditions gives $A = L_o / 2$. The position is:

$$x(t) = L_o \left(\frac{e^{\Omega t} + e^{-\Omega t}}{2} \right) = L_o \cosh(\Omega t).$$

Let T be the time when the bead reaches the other end, so:

$$x(T) = L = L_o \cosh(\Omega T),$$

$$T = \frac{1}{\Omega} \cosh^{-1}\left(\frac{L}{L_o} \right),$$

Note that the units are right because the angular velocity has units of radians (unitless) over time.

Today and tomorrow, we'll look at some effects of the centrifugal force:

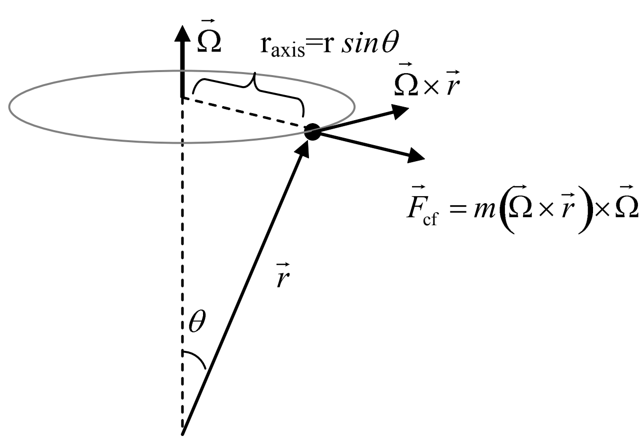
$$\vec{F}_{\mathrm{cf}} = m\left(\vec{\Omega} \times \vec{r}\right) \times \vec{\Omega},$$

and the Coriolis force:

$$\vec{F}_{\mathrm{cor}} = 2m\dot{\vec{r}} \times \vec{\Omega}.$$

9.6 Centrifugal Force:

Suppose a mass has a fixed position $\vec{r}$ in a rotating coordinate system with angular velocity $\vec{\Omega}$. What is the centrifugal force in this case? Label the angle between $\vec{r}$ and $\vec{\Omega}$ as θ. The diagram below will be helpful.



The size of the first cross product is:

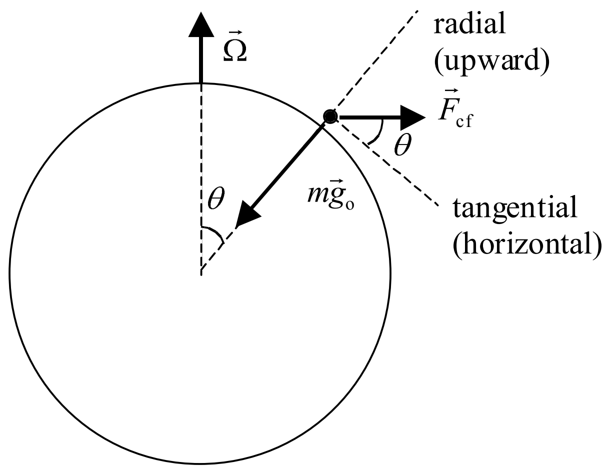
$$\left| \vec{\Omega} \times \vec{r} \right| = \Omega r \sin\theta.$$

It is tangent to the path of the mass (circle) as it rotates. The result is also perpendicular to the angular velocity: $\left(\vec{\Omega}\times\vec{r}\right)\perp\vec{\Omega}$. That means that the size of the centrifugal force is:

$$F_{\text{cf}} = \left|m\left(\vec{\Omega}\times\vec{r}\right)\times\vec{\Omega}\right| = m\Omega^2 r\sin\theta.$$

The centrifugal force $\vec{F}_{\text{cf}}$ is perpendicular to $\vec{\Omega}$ and points *away from the axis of rotation*.

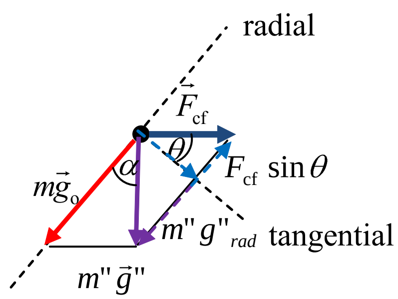
Suppose we are describing an object near the surface of the earth. The size of the centrifugal force is larger near the equator. As described by an observer on the earth, the direction of the centrifugal force (how much N-S and how much In-Out) will depend on the location. We will use the angle θ from the angular momentum (the North Pole). This is known as the *colatitude* and is 90° minus the latitude. See the diagram below.



From the perspective of someone rotating along with the rotating frame, there's an effective gravitational force:

$$\vec{F}_{\text{eff.g}} = \vec{F}_{\text{grav}} + \vec{F}_{\text{cf}}.$$

Define $\vec{g}_{\text{o}}$ as the gravitational acceleration that would be felt if there was <u>no</u> rotation. We will now use "$\vec{g}$" for the *effective* gravitational acceleration in the rotating frame. As the book points out, while we can conceptually distinguish the real gravitational from the centrifugal, we can't experimentally distinguish them. The individual and effective forces are shown below.



The effective gravitational acceleration in the rotating frame is given by:

$$"\vec{g}" = \vec{g}_{\text{o}} + \frac{\vec{F}_{\text{cf}}}{m}.$$

We can split this into radial / in-out and tangential components / north-south. The size of the radial component is:

$$|g_{\text{rad}}| = \left|-g_{\text{o}} + F_{cf}\sin\theta\right|$$

$$|g_{\text{rad}}| = g_{\text{o}} - \Omega^2 R\sin^2\theta,$$

(sign flip when dropping the absolute value signs since g_o will generally be larger than the centrifugal contribution)

At the poles $(\theta = 0 \text{ or } \pi)$, $|g_{\text{rad}}| = g_o$ and at the equator $(\theta = \pi/2)$ it is less by:

$$\Omega^2 R = \left(7.3\times 10^{-5}\ \text{s}\right)^2 \left(6.4\times 10^6\ \text{m}\right) = 0.034\ \text{m/s}^2$$

The tangential component is:

$$g_{\text{tang}} = F_{cf}\cos\theta$$
$$g_{\text{tang}} = \Omega^2 R \sin\theta\cos\theta \, .$$

The angle between $\vec{g}$ and the radial direction ($\vec{g}_o$) is always small, so (in radians):

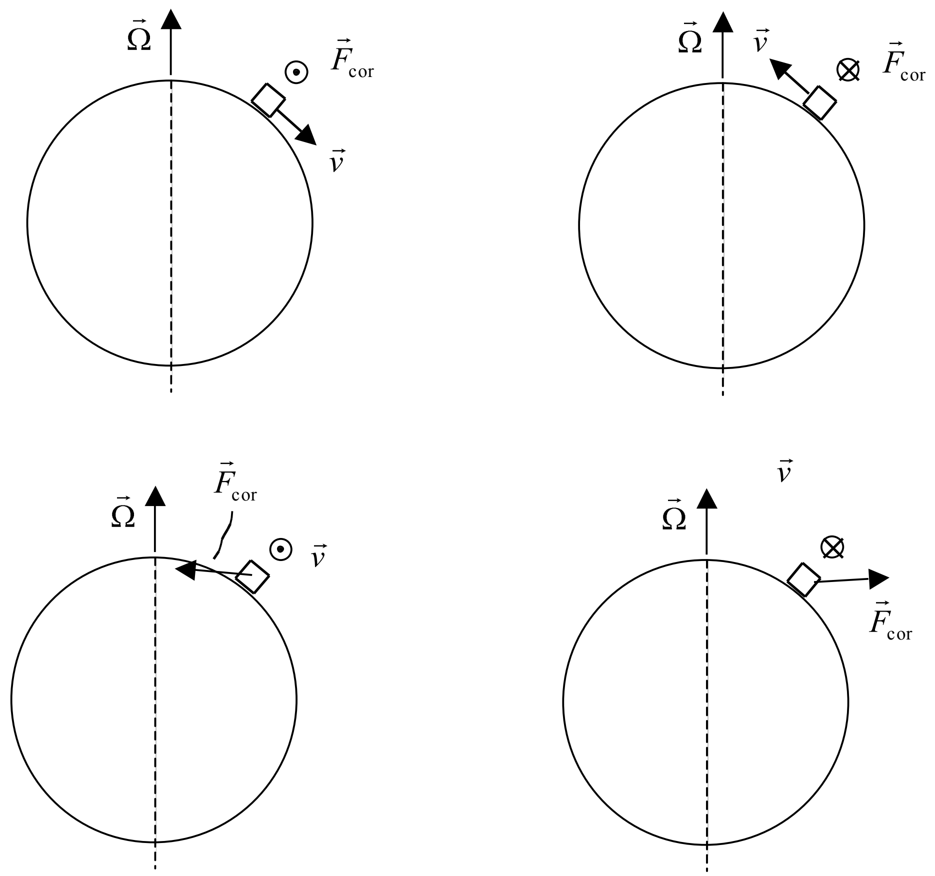
$$\alpha = \tan^{-1}\left(\frac{g_{\text{tang}}}{g_{\text{rad}}}\right) \approx \frac{g_{\text{tang}}}{g_{\text{rad}}} \approx \frac{\Omega^2 R\sin\theta\cos\theta}{g_o} \, .$$

A plumb line (a bob on a string) will hang at *this* angle relative to $\vec{g}_o$ // radially inward in equilibrium in the rotating frame. The angle is largest at $\theta = 45^\circ$ where:

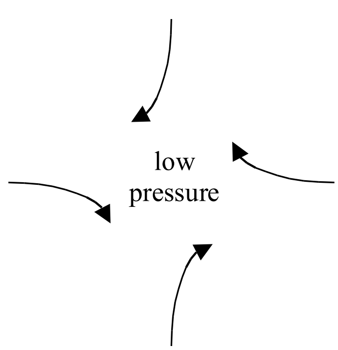
$$\alpha \approx \frac{\Omega^2 R}{2g_o} = \frac{0.034\ \text{m/s}^2}{2\left(9.8\ \text{m/s}^2\right)} \approx 0.0017\ \text{rad} \approx 0.1^\circ .$$

9.7 Coriolis Force:

Let’s consider an object moving close to the surface of the earth. The Coriolis force depends on what direction an object is going (velocity) relative to the angular velocity. Note that what it means to go “north” depends on the location on earth!



Combining all of the pictures, we can explain why hurricanes tend to rotate counterclockwise in the <u>Northern</u> Hemisphere. If air is moving inward toward an area of low pressure, it is deflected by the Coriolis force in the way shown below (viewed from above).

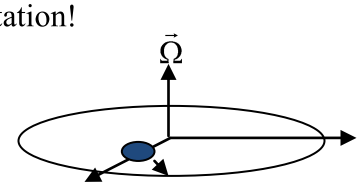


The opposite rotation results in the Southern Hemisphere. This effect is too small to determine the way water rotates as it flows down the drain (e.g. like when the Simpson’s visit Australia).

Both Forces:

To get a little practice with a relatively simple situation, let's consider the motion of a frictionless puck on a horizontal, rotating turntable. Compared to the spinning of the turntable on its own axis, the spinning of the room (sitting on the face of a spinning Earth) is negligible, so we'll treat the room as an inertial frame. Of course, in the inertial frame the puck will simply move in a straight line because there is no net force. A (noninertial) rotating observer may observe more complicated motions which will be explained by the centrifugal and the Coriolis forces.

Example #: Prob. 9.20 (background for 9.24) Suppose a frictionless puck moves on a horizontal turntable rotating counterclockwise (viewed from above) at an angular speed Ω. Write down the equations of motion for the puck in the rotating system if the puck starts at an initial position $\vec{r}_{\mathrm{i}} = (x_{\mathrm{i}}, 0)$ with an initial velocity $\vec{v}_{\mathrm{i}} = (v_{x\mathrm{i}}, v_{y\mathrm{i}})$ as measured in the rotating frame. Ignore Earth's rotation!



In an inertial frame there is no net force on the puck, so

$m\ddot{\vec{r}}_o = \vec{F}_{\mathrm{net}} = 0$ and *we'd* see the puck moving with constant velocity / in a straight line.

How would it look to a little bug ridding on the turntable? In that noninertial frame that rotates with the turntable, Newton's second law is:

$$m\ddot{\vec{r}} = \vec{F}_{\mathrm{cf}} + \vec{F}_{\mathrm{cor}} = m(\vec{\Omega}\times\vec{r})\times\vec{\Omega} + 2m\dot{\vec{r}}\times\vec{\Omega}.$$

Taking the angular velocity of the turntable to be $\vec{\Omega} = (0,0,\Omega)$, the position is $\vec{r} = (x,y,0)$. Calculate the cross products:

$$\text{I do:}\quad \vec{\Omega}\times\vec{r} = \det\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \Omega \\ x & y & 0 \end{vmatrix} = (-\Omega y, \Omega x, 0),$$

$$\text{They do:}\quad (\vec{\Omega}\times\vec{r})\times\vec{\Omega} = \det\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\Omega y & \Omega x & 0 \\ 0 & 0 & \Omega \end{vmatrix} = (\Omega^2 x, \Omega^2 y, 0),$$

$$\textbf{They do:}\quad \dot{\vec{r}}\times\vec{\Omega} = \det\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dot{x} & \dot{y} & 0 \\ 0 & 0 & \Omega \end{vmatrix} = (\Omega\dot{y}, -\Omega\dot{x}, 0).$$

So the equation of motion gives (dividing out the mass):

$$(\ddot{x},\ddot{y},0)=(\Omega^2 x,\Omega^2 y,0)+(2\Omega\dot{y},-2\Omega\dot{x},0),$$

or the equations for the x and y components are:

$$\ddot{x}=\Omega^2 x+2\Omega\dot{y} \quad \text{and} \quad \ddot{y}=\Omega^2 y-2\Omega\dot{x}\,.$$

A trick for solving both of these coupled differential equations at the same time (see Sect. 2.7) is to define $\eta=x+iy$. If we add i times the $\ddot{y}$-equation to the $\ddot{x}$-equation, we get:

$$\ddot{x}+i\ddot{y}=\Omega^2(x+iy)+2\Omega(-i\dot{x}+\dot{y})=\Omega^2(x+iy)-2i\Omega(\dot{x}+i\dot{y}),$$

$$\ddot{\eta}=\Omega^2\eta-2i\Omega\dot{\eta}\,.$$

This looks an awful lot like the damped harmonic oscillator (aside from that factor of i and the lack of a negative sign on the linear term). So we can guess the basic form of the solution that we'd guessed in that case.

Since this is a linear, differential equation, guess the solution $\eta=e^{-i\alpha t}$, which gives the auxiliary equation:

They do:

$$-\alpha^2=\Omega^2-2\Omega\alpha,$$

$$\Omega^2-2\Omega\alpha+\alpha^2=(\Omega-\alpha)^2=0.$$

This implies that $\alpha=\Omega$. There is only one solution for α, so we need a second solution (the differential equation is second order). This sounds a lot like the problem with the critically damped simple harmonic oscillator. So we've got a good chance that a similar solution will work. Just as with critical damping, you can check that in addition to $e^{-i\Omega t}$, $te^{-i\Omega t}$ is a solution, so the general solution is:

$$\eta(t)=e^{-i\Omega t}(C_1+C_2 t),$$

where the coefficients may be complex.

Impose Initial Conditions

$\vec{r}_{\rm i}=(x_{\rm i},0,0)$ and $\dot{\vec{r}}_{\rm i}=(v_{x\rm i},v_{y\rm i},0)$ or $\eta_{\rm i}=x_{\rm i}$ and $\dot{\eta}_{\rm i}=v_{x\rm i}+iv_{y\rm i}$.

The first condition implies that

$$C_1=x_{\rm i}$$

and the derivative is:

$$\dot{\eta}(t)=-i\Omega e^{-i\Omega t}(x_i+C_2 t)+C_2 e^{-i\Omega t}\,,$$

So the second condition implies:

$$\dot{\eta}(0)=C_2-i\Omega x_i=v_{x\rm o}+iv_{y\rm o},$$

$$C_2=v_{x\rm i}+i(v_{y\rm i}+\Omega x_i).$$

This gives:

$$\eta(t) = e^{-i\Omega t}\left[x_{\mathrm{i}} + v_{x\mathrm{i}}t + i\left(v_{y\mathrm{i}} + \Omega x_{\mathrm{i}}\right)t\right] = \left(\cos\Omega t - i\sin\Omega t\right)\left[x_{\mathrm{i}} + v_{x\mathrm{i}}t + i\left(v_{y\mathrm{i}} + \Omega x_{\mathrm{i}}\right)t\right].$$

The real part of η is $x(t)$ and the imaginary part is $y(t)$, which gives (Eq. 9.72):

They do: $x(t) = \left(x_{\mathrm{i}} + v_{x\mathrm{i}}t\right)\cos\Omega t + \left(v_{y\mathrm{i}} + \Omega x_{\mathrm{i}}\right)t\sin\Omega t$,

$$y(t) = -\left(x_{\mathrm{i}} + v_{x\mathrm{i}}t\right)\sin\Omega t + \left(v_{y\mathrm{i}} + \Omega x_{\mathrm{i}}\right)t\sin\Omega t\ .$$

You will explore (computationally) the behavior of the motion for different initial velocities in the homework (Prob. 9.24).