Wed., 10/31 Thurs. 11/1 Fri., 11/2	**7.6-.8** Generalized Variables & Classical Hamiltonian (Recommend 7.9 if you've had Phys 332) **8.1-.2** 2-Body Central Forces, Relative Coordinates.	**HW7c** (7."53", 7."54") **Project Outline**
Mon. 11/5 Wed., 11/7	**Review** **Exam 2** (Ch 5-7)	

Today , we'll work some more examples, but first we'll look at two very important ideas

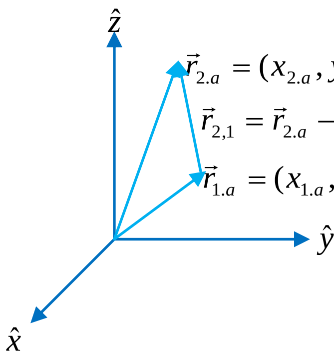
Conservation Laws and Noether's theorem

Hamilton's Equation.

Admittedly, we're not going to do much with these, but the first is interesting in its own right and the second gets a lot of play in Quantum.

Conservation of Momentum and Translational Invariance

Imagine you're watching something play out, I don't know, bodies gravitationally interacting. In order to get quantitative about modeling what you see, the first thing you do is define a coordinate system. Then you can specify the positions and motions of the particles relative to that origin and the coordinate directions.



$$\vec{r}_{2.a} = (x_{2.a}, y_{2.a}, z_{2.a}) \quad \vec{v}_2 = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

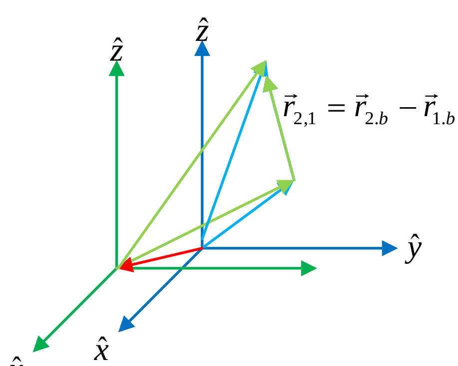
$$\vec{r}_{2,1} = \vec{r}_{2.a} - \vec{r}_{1.a}$$

$$\vec{r}_{1.a} = (x_{1.a}, y_{1.a}, z_{1.a}) \quad \vec{v}_1 = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

Now, the kinetic energy of the system depends upon the speeds and the potential energy *of an isolated system* depends on the relative separation of the particles.

$U(\vec{r}_{2,1})$, $T(v_1, v_2)$

Then again, you could have chosen a different location for your origin, and that wouldn't change very much about your description of the system's behavior – the two gravitationally-interacting objects would still be just as far apart from each other and they'd have the same velocities, they'd just have different locations relative to the origin.



$$\vec{r}_1 = (x_1, y_1, z_1) = (x_1 + \varepsilon_x, y_1 + \varepsilon_y, z_1 + \varepsilon_z)$$

$$\vec{v} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

$$\vec{r}_2 = (x_2, y_2, z_2) = (x_2 + \varepsilon_x, y_2 + \varepsilon_y, z_2 + \varepsilon_z)$$

$$\vec{v} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

So,

$$U(\vec{r}_{2,1}),\ T(v_1, v_2)$$

Obviously then

$$\mathcal{L}_b - \mathcal{L}_a = 0$$

Considering a displacement just in one direction at a time, x for instance,

$$\boldsymbol{d\mathcal{L}} = dx_1 \frac{\partial \mathcal{L}}{\partial x_1} + dx_2 \frac{\partial \mathcal{L}}{\partial x_2} = 0$$

$$\boldsymbol{d\mathcal{L}} = \varepsilon_x \frac{\partial \mathcal{L}}{\partial x_1} + \varepsilon_x \frac{\partial \mathcal{L}}{\partial x_2} = 0$$

$$\boldsymbol{d\mathcal{L}} = \varepsilon_x \left(\frac{\partial \mathcal{L}}{\partial x_1} + \frac{\partial \mathcal{L}}{\partial x_2} \right) = 0$$

$$\left(\frac{\partial \mathcal{L}}{\partial x_1} + \frac{\partial \mathcal{L}}{\partial x_2} \right) = 0$$

Using the fact that $\frac{\partial \mathcal{L}}{\partial x_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i}$ and, for our individual particles, $\frac{\partial \mathcal{L}}{\partial \dot{x}_i} = \frac{\partial \mathcal{T}}{\partial \dot{x}_i} = m\dot{x}_i = p_{i.x}$

We have that

$$\left(\frac{d}{dt} p_{x.1} + \frac{d}{dt} p_{x.2} \right) = 0$$

$$\frac{d}{dt}\left(p_{x.1} + p_{x.2} \right) = 0$$

So the total momentum is conserved.

So, that a system's momentum is conserved (no net external force) is equivalent to saying that the lagrangian is unaffected by translations through space.

An obvious counter example is when something is acted upon the gravitational force of the Earth, and we take the Earth to be outside the system.

Rotation and Angular Momentum. We could of course make a similar argument about *rotating* from one frame to another. The Lagrangian's being unaffected by this is equivalent to what being conserved? *Angular* momentum. To show this, all you really have to do is replace *x* with θ, and redo the work above.

Translation through time.

Now, is there anything special if the lagrangian doesn't change with time?

Let's look at the time derivative of the lagrangian.

$$\frac{d\mathcal{L}}{dt} = \frac{d\mathcal{L}(q_1, \dot{q}_1, q_2, \dot{q}_2, \ldots t)}{dt} = \frac{dq_1}{dt}\frac{\partial \mathcal{L}}{\partial q_1} + \frac{d\dot{q}_1}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1} + \ldots \frac{\partial \mathcal{L}}{\partial t}$$

$$\text{But } \frac{\partial \mathcal{L}}{\partial q_1} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_1}\right)$$

So, what we have is

$$\frac{d\mathcal{L}}{dt} = \dot{q}_1\left(\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_1}\right)\right) + \left(\frac{d}{dt}(\dot{q}_1)\right)\frac{\partial \mathcal{L}}{\partial \dot{q}_1} + \ldots \frac{\partial \mathcal{L}}{\partial t}$$

The reason I write it out this way is so we can recognize the product rule in action.

$$\frac{d}{dt}\left(\dot{q}_1\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_1}\right)\right) = \dot{q}_1\left(\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_1}\right)\right) + \left(\frac{d}{dt}(\dot{q}_1)\right)\frac{\partial \mathcal{L}}{\partial \dot{q}_1}$$

So,

$$\frac{d\mathcal{L}}{dt} = \frac{d}{dt}\left(\dot{q}_1\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_1}\right)\right) + \ldots \frac{\partial \mathcal{L}}{\partial t}$$

Now, if our coordinates are Cartesian, then

$$p_1 \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_1}$$

Or, as the book suggests, more generally we can talk about this derivative as a "generalized" momentum. For example, if q is an *angle* then p is the *angular* momentum.

So, using that shorthand, we have

$$\frac{d\mathcal{L}}{dt} = \frac{d}{dt}(\dot{q}_1 p_1 + \ldots) + \frac{\partial \mathcal{L}}{\partial t}$$

Rephrased, we have

$$\frac{d}{dt}\left(\sum_i \dot{q}_i p_i - \mathcal{L}\right) = -\frac{\partial \mathcal{L}}{\partial t}$$

This doesn't look too incredible all by itself, *but* if the Lagrangian doesn't *have* any explicit time dependence (recall, none of the examples we've considered have had any), then the right-hand side is zero, meaning the argument must have *no (explicit or otherwise)* time dependence – it's a constant and conserved quantity.

$$\frac{d}{dt}\left(\sum_i \dot{q}_i p_i - \mathcal{L}\right) = 0 \text{ if } \frac{\partial \mathcal{L}}{\partial t} = 0$$

So in those cases we have a conserved quantity, and that quantity is called…

$$\mathcal{H} \equiv \sum_i \dot{q}_i p_i - \mathcal{L}$$

The Hamiltonian.

Clearly, if we're dealing in Cartesian coordinates, the q's are x,y,z for the different particles and the p's are p_x, p_y, p_z., then

$$\mathcal{H} \equiv \sum_i \dot{x}_i p_{xi} - (T - U) = \sum_i m\dot{x}_i^{\,2} - (T - U) = 2T - (T - U) = T + U = E$$

The book proves more generally that, as long as your generalized coordinate system is not changing with respect to Cartesian coordinates, then you get this.

You may well wonder what's so significant about the Hamiltonian if it just total energy? For that matter, you could have asked 'what's so significant about the Lagrangian if it's just the difference between kinetic and potential energy.' The significance is not the thing itself, but what we can do with it. The Lagrangian satisfies

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

The Hamiltonian satisfies

$$-\frac{\partial \mathcal{H}}{\partial q_i} = \dot{p}_i \text{ and } \frac{\partial \mathcal{H}}{\partial p_i} = \dot{q}_i$$

Sometimes this *pair* of simpler equations are easier to work with than Lagrange's equation.

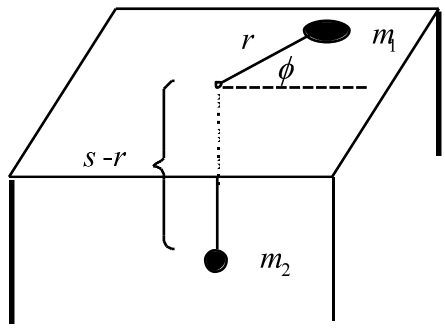
Examples

Okay, time for more work on the examples we'd begun last time.

Practice with Constrained systems.

Now, we'll continue getting more practice. First, I'll give the set up for a handful of problems, and then I'll give the solutions, so you can try working them yourself, but also peak at the solutions.

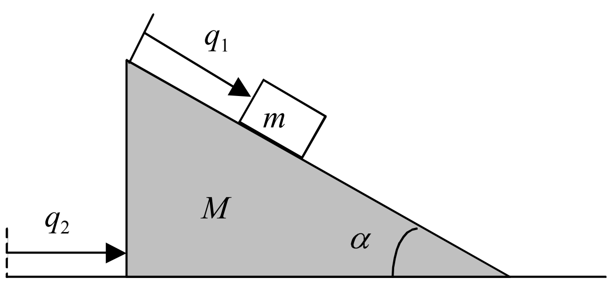
Here's a real classic: suppose a string of length s connects a puck of mass m_1 on a frictionless table and an object with mass m_2 through a hole (see the figure below).



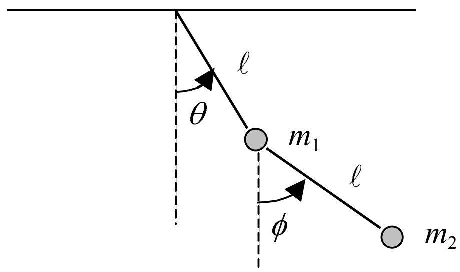
(a) Write the Lagrangian for the system in terms of the puck's polar coordinates.

(b) Find the equations of motion for the system.

(**Book does Ex. 7.5.**) A small block is sliding along a plane that can slide without friction. The coordinate q_1 is clearly not a Cartesian coordinate.



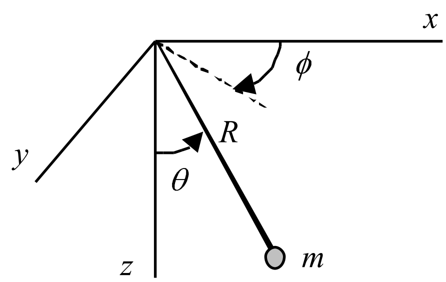
Exercise #5: (book sets up but doesn't do) Suppose a double pendulum consists of two rigid, massless rods connecting two masses. The masses are not equal, but the lengths of the rods are. The pendulum only swings in one vertical plane.



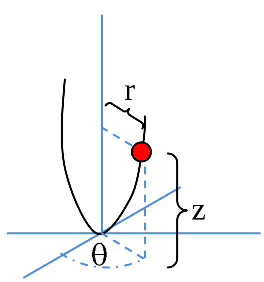
(a) Write the Lagrangian for the system in terms of the angles shown above.

(b) Find the equations of motion of the system.

(Ex. 10.9 of F&C 5th ed.) Spherical pendulum – Find the equations of motion using spherical polar coordinates. Note that the length is fixed. This is the same problem as we worked when the ball was confined to the surface of a sphere except now we define z down in the direction of the gravitational force.

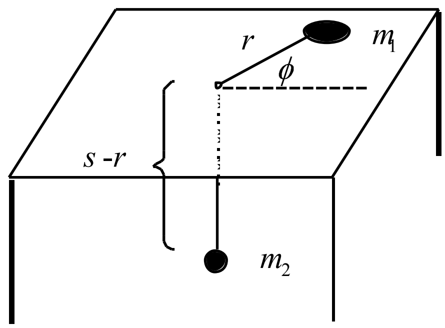


Example #2: (Ex. 7.7 of Thornton) Frictionless bead on a *spinning* parabolic wire $z = cr^2$ rotating with an angular velocity $\dot{\theta} = \omega$. What is the condition on the angular frequency to get the bead to rotate at a fixed location on the wire?



Solutions

Here's a real classic: suppose a string of length *s* connects a puck of mass m_1 on a frictionless table and an object with mass m_2 through a hole (see the figure below).



(a) Write the Lagrangian for the system in terms of the puck's polar coordinates.

The kinetic energy for the puck is:

$$T_1 = \tfrac{1}{2} m_1 \left(\dot{r}^2 + r^2\dot{\phi}^2\right),$$

and for the other mass is:

$$T_2 = \tfrac{1}{2} m_2 \left(-\dot{r}\right)^2 = \tfrac{1}{2} m_2 \dot{r}^2 .$$

define the gravitational potential energy to be zero at the level of the table, so:

$$U = -m_2 g(s-r).$$

The Lagrangian is:

$$\mathcal{L} = \left(T_1 + T_2\right) - U = \tfrac{1}{2} m_1 \left(\dot{r}^2 + r^2\dot{\phi}^2\right) + \tfrac{1}{2} m_2 \dot{r}^2 + m_2 g\left(s-r\right).$$

(b) Find the equations of motion for the system.

$$\frac{\partial\mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial\phi} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}},$$

$$m_1 r\dot{\phi}^2 - m_2 g = \frac{d}{dt}\left(m_1\dot{r} + m_2\dot{r}\right) \quad \text{and} \quad 0 = \frac{d}{dt}\left(m_1 r^2\dot{\phi}\right),$$

$$m_1 r\dot{\phi}^2 - m_2 g = \left(m_1 + m_2\right)\ddot{r} \quad \text{and} \quad m_1 r^2\dot{\phi} = \ell = \text{constant}.$$

The second equation is equivalent to conservation of angular momentum. The first equation can be rewritten as:

$$\left(m_1 + m_2\right)\ddot{r} = \frac{\ell^2}{m_1 r^3} - m_2 g .$$

Look at some Qualitative cases:

Whether the radius grows or shrinks depends on whether the orbital term is greater or less than the gravitational term. The balancing point, where there is no radial acceleration is when

$$\frac{\ell^2}{m_1 r^3} = m_2 g$$

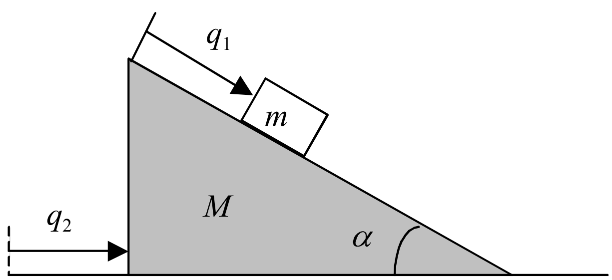
It may not look too familiar in that form, but rephrasing it as

$$m_1 r\dot{\theta}^2 = F_{radial}$$

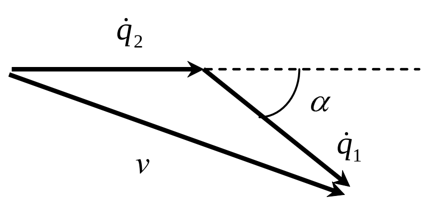
$$m_1 \frac{v_{\tan gential}^2}{r} = F_{radial}$$

May look familiar – when the radial force experienced by m_1 is equal to mv^2/r, the thing executes a *circular* orbit, so r = constant.

(**Book does Ex. 7.5.**) A small block is sliding along a plane that can slide without friction. The coordinate q_1 is clearly <u>not</u> a Cartesian coordinate.



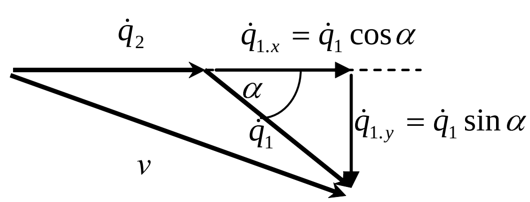
The velocity of the block relative to the ground (inertial frame) is the velocity of the plane plus the velocity of the block relative to the plane (see the diagram below).



By the law of cosines, the size of the block's speed squared is:

$$v^2 = \dot{q}_1^2 + \dot{q}_2^2 + 2\dot{q}_1\dot{q}_2 \cos\alpha\ .$$

Alternatively, one can break it down this way and get the same result:



The kinetic energy of the system is:

$$T = \tfrac{1}{2} M\dot{q}_2^2 + \tfrac{1}{2} m\left(\dot{q}_1^2 + \dot{q}_2^2 + 2\dot{q}_1\dot{q}_2\cos\alpha\right),$$

and the potential energy is:

$$U = -mgq_1 \sin\alpha .$$

The Lagrangian is:

$$\mathcal{L} = T - U = \tfrac{1}{2} M\dot{q}_2^2 + \tfrac{1}{2} m\left(\dot{q}_1^2 + \dot{q}_2^2 + 2\dot{q}_1\dot{q}_2\cos\alpha\right) + mgq_1 \sin\alpha .$$

The equations of motion are:

$$\frac{\partial\mathcal{L}}{\partial q_1} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial \dot{q}_1} \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial q_2} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial \dot{q}_2},$$

which give:

$$0 = \frac{d}{dt}\left[M\dot{q}_2 + m\left(\dot{q}_2 + \dot{q}_1\cos\alpha\right)\right] \quad \text{and} \quad mg\sin\alpha = \frac{d}{dt}\left(m\dot{q}_1 + m\dot{q}_2\cos\alpha\right),$$

$$M\ddot{q}_2 + m\left(\ddot{q}_2 + \ddot{q}_1\cos\alpha\right) = 0 \quad \text{and} \quad \ddot{q}_1 + \ddot{q}_2\cos\alpha = g\sin\alpha .$$

The first equation gives:

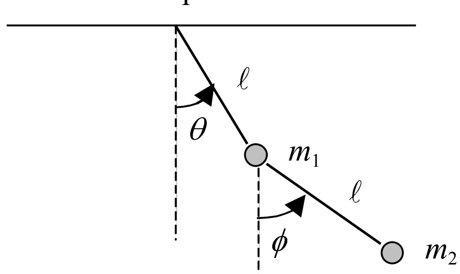
$$\ddot{q}_2 = \frac{-m}{M+m}\ddot{q}_1\cos\alpha .$$

These equations can be solved (you can check the algebra) for $\ddot{q}_1$ and $\ddot{q}_2$ to get:

$$\ddot{q}_1 = \frac{g\sin\alpha}{1 - \dfrac{m\cos^2\alpha}{m+M}} \quad \text{and} \quad \ddot{q}_2 = \frac{-g\sin\alpha\cos\alpha}{\dfrac{m+M}{m} - \cos^2\alpha}.$$

The right hand sides of these equations are constant, so these are easy to solve. The acceleration $\ddot{q}_1$ of the block relative to the plane and the acceleration $\ddot{q}_2$ of the plane are constant. Try getting these equations using Newton's second law!

Exercise #5: (book sets up but doesn't do) Suppose a double pendulum consists of two rigid, massless rods connecting two masses. The masses are not equal, but the lengths of the rods are. The pendulum only swings in one vertical plane.



(a) Write the Lagrangian for the system in terms of the angles shown above.

Take the top pivot point as the origin and choose positive x to the right and positive y upward. The components of the position of m_1 are:

$$x_1 = \ell \sin\theta \quad \text{and} \quad y_1 = -\ell \cos\theta,$$

so the components of its velocity are,

$$\dot{x}_1 = \ell\dot{\theta}\cos\theta \quad \text{and} \quad \dot{y}_1 = \ell\dot{\theta}\sin\theta .$$

The components of the position of m_2 are:

$$x_2 = \ell\sin\theta + \ell\sin\phi \quad \text{and} \quad y_2 = -\ell\cos\theta - \ell\cos\phi,$$

$$x_2 = \ell(\sin\theta + \sin\phi) \quad \text{and} \quad y_2 = -\ell(\cos\theta + \cos\phi),$$

so the components of its velocity are,

$$\dot{x}_2 = \ell\left(\dot{\theta}\cos\theta + \dot{\phi}\cos\phi\right) \quad \text{and} \quad \dot{y}_2 = \ell\left(\dot{\theta}\sin\theta + \dot{\phi}\sin\phi\right).$$

The kinetic energy is:

$$T = \tfrac{1}{2}m_1\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \tfrac{1}{2}m_2\left(\dot{x}_2^2 + \dot{y}_2^2\right),$$

$$T = \tfrac{1}{2}m_1\ell^2\dot{\theta}^2 + \tfrac{1}{2}m_2\ell^2\left[\left(\dot{\theta}\cos\theta + \dot{\phi}\cos\phi\right)^2 + \left(\dot{\theta}\sin\theta + \dot{\phi}\sin\phi\right)^2\right],$$

$$T = \tfrac{1}{2}m_1\ell^2\dot{\theta}^2 + \tfrac{1}{2}m_2\ell^2\left[\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\left(\cos\theta\cos\phi + \sin\theta\sin\phi\right)\right],$$

$$T = \tfrac{1}{2}m_1\ell^2\dot{\theta}^2 + \tfrac{1}{2}m_2\ell^2\left[\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\theta - \phi)\right],$$

$$T = \tfrac{1}{2}(m_1 + m_2)\ell^2\dot{\theta}^2 + \tfrac{1}{2}m_2\ell^2\left[\dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\theta - \phi)\right].$$

The potential energy is:

$$U = m_1 g y_1 + m_2 g y_2,$$

$$U = -(m_1 + m_2)g\ell\cos\theta - m_2 g\ell\cos\phi,$$

so the Lagrangian is:

$$\mathcal{L} = \tfrac{1}{2}(m_1 + m_2)\ell^2\dot{\theta}^2 + \tfrac{1}{2}m_2\ell^2\dot{\phi}^2 + m_2\ell^2\dot{\theta}\dot{\phi}\cos(\theta - \phi) + (m_1 + m_2)g\ell\cos\theta + m_2 g\ell\cos\phi .$$

(b) Find the equations of motion of the system.

The equation associated with θ is:

$$\frac{\partial\mathcal{L}}{\partial\theta} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}},$$

$$-m_2\ell^2\dot{\theta}\dot{\phi}\sin(\theta - \phi) - (m_1 + m_2)g\ell\sin\theta = \frac{d}{dt}\left[(m_1 + m_2)\ell^2\dot{\theta} + m_2\ell^2\dot{\phi}\cos(\theta - \phi)\right],$$

$$-m_2\ell^2\dot{\theta}\dot{\phi}\sin(\theta - \phi) - (m_1 + m_2)g\ell\sin\theta = (m_1 + m_2)\ell^2\ddot{\theta} + m_2\ell^2\left[\ddot{\phi}\cos(\theta - \phi) - \dot{\phi}(\dot{\theta} - \dot{\phi})\sin(\theta - \phi)\right]$$

The equation associated with ϕ is:

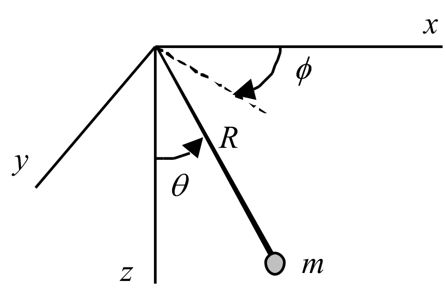
$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}},$$

$$m_2 \ell^2 \dot{\theta}\dot{\phi} \sin(\theta - \phi) - m_2 g \ell \sin\phi = \frac{d}{dt}\left[m_2 \ell^2 \left(\dot{\phi} + \dot{\theta}\cos(\theta - \phi)\right)\right],$$

$$m_2 \ell^2 \dot{\theta}\dot{\phi} \sin(\theta - \phi) - m_2 g \ell \sin\phi = m_2 \ell^2 \left[\ddot{\phi} + \ddot{\theta}\cos(\theta - \phi) - \dot{\theta}(\dot{\theta} - \dot{\phi})\sin(\theta - \phi)\right].$$

These equations are hideously complicated, but it is not difficult to get using the Lagrangian approach, if you are careful. They would be tremendously difficult to get using the Newtonian approach!

(Ex. 10.9 of F&C 5th ed.) Spherical pendulum – Find the equations of motion using spherical polar coordinates. Note that the length is fixed. This is the same problem as we worked when the ball was confined to the surface of a sphere except now we define z down in the direction of the gravitational force.



The position of the bob in Cartesian coordinates is:

$$x = R\sin\theta\cos\phi, \quad y = R\sin\theta\sin\phi, \quad \text{and} \quad z = R\cos\theta,$$

The easiest way to get the speed squared is as follows:

$$d\vec{r} = dr\hat{r} + r\sin\theta d\phi\hat{\phi} + rd\theta\hat{\theta}$$

So

$$\frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r\sin\theta\frac{d\phi}{dt}\hat{\phi} + r\frac{d\theta}{dt}\hat{\theta}$$

Then

$$v^2 = \left(\frac{dr}{dt}\right)^2 + \left(r\sin\theta\frac{d\phi}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2$$

With r=R constant,

$$v^2 = (R\sin\theta\dot{\phi})^2 + (R\dot{\theta})^2$$

The Lagrangian is:

$$\mathcal{L} = T - U = \tfrac{1}{2}mv^2 - (-mgz) = \tfrac{1}{2}mR^2\left(\dot\theta^2 + \dot\phi^2\sin^2\theta\right) + mgR\cos\theta\,,$$

so the equations of motion are:

$$\frac{\partial\mathcal{L}}{\partial\theta} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\theta} \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial\phi} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\phi}\,,$$

$$mR^2\dot\phi^2\sin\theta\cos\theta - mgR\sin\theta = \frac{d}{dt}\left(mR^2\dot\theta\right) \quad \text{and} \quad 0 = \frac{d}{dt}\left(mR^2\dot\phi\sin^2\theta\right).$$

These can be rewritten as:

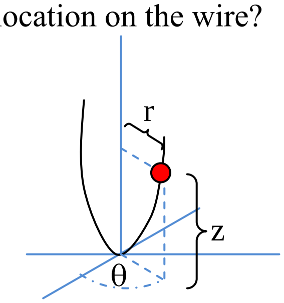
$$\ddot\theta = \dot\phi^2\sin\theta\cos\theta - \frac{g}{R}\sin\theta \quad \text{and} \quad mR^2\dot\phi\sin^2\theta = \text{constant.}$$

The second equation means that the angular momentum of the pendulum is conserved.

If $\phi = \text{constant}$, then $\dot\phi = 0$ and the pendulum swings in a plane. The equation of motion reduces to the familiar (hopefully):

$$\ddot\theta = -\frac{g}{R}\sin\theta\,.$$

Example #2: (Ex. 7.7 of Thornton) Frictionless bead on a *spinning* parabolic wire $z = cr^2$ rotating with an angular velocity $\dot\theta = \omega$. What is the condition on the angular frequency to get the bead to rotate at a fixed location on the wire?



In cylindrical polar coordinates, the velocity is:

$$\vec v = \dot r\hat r + \dot z\hat z + r\dot\theta\hat\theta\,,$$

so:

$$T = \tfrac{1}{2}m\vec v^2 = \tfrac{1}{2}m\left(\vec v\cdot\vec v\right) = \tfrac{1}{2}m\left[\dot r^2 + \dot z^2 + \left(r\dot\theta\right)^2\right].$$

This system has one degree of freedom, where the bead is along the wire. Use *r* to describe this (could also use *z*). Since $z = cr^2$, the derivative is $\dot z = 2cr\dot r$ and:

$$T = \tfrac{1}{2}m\left(\dot r^2 + 4c^2r^2\dot r^2 + r^2\omega^2\right).$$

The potential energy is $U = mgz = mgcr^2$, so the Lagrangian is:

$$\mathcal{L} = T - U = \tfrac{1}{2}m\left(\dot r^2 + 4c^2r^2\dot r^2 + r^2\omega^2\right) - mgcr^2\,.$$

The derivatives of the Lagrangian are:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{m}{2}\left(8cr\dot{r}^2 + 2r\omega^2\right) - 2mgcr\,,$$

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{m}{2}\left(2\dot{r} + 8c^2r^2\dot{r}\right),$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{m}{2}\left(2\ddot{r} + 16c^2r\dot{r}^2 + 8c^2r^2\ddot{r}\right).$$

The equation of motion is:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}}\,,$$

$$\frac{m}{2}\left(8cr\dot{r}^2 + 2r\omega^2\right) - 2mgcr = \frac{m}{2}\left(2\ddot{r} + 16c^2r\dot{r}^2 + 8c^2r^2\ddot{r}\right).$$

This can be rewritten as:

$$\ddot{r}\left(1 + 4c^2r^2\right) + \dot{r}^2\left(4c^2r\right) + r\left(2gc - \omega^2\right) = 0\,,$$

which is a complicated, <u>nonlinear</u> ($\dot{r}$ is squared) differential equation.

If the bead rotates with $r = R = \text{constant}$, then $\dot{r} = \ddot{r} = 0$ so:

$$R\left(2gc - \omega^2\right) = 0,$$

And $\omega = \sqrt{2gc}\,.$