Mon., 11/1 Wed., 11/3 Thurs. 11/4 Fri., 11/5	**7.4-.5** Proof and Examples **7.6-.8** Generalized Variables & Classical Hamiltonian (Recommend 7.9 if you've had Phys 332) **8.1-.2** 2-Body Central Forces, Relative Coordinates.	**HW7**
Mon. 11/8 Wed., 11/10 Thurs. 11/11 Fri., 11/12	**Review** **Exam 2** (Ch 5-7) **8.3-.4** Equations of Motion for 2-Body Central Force	**Project Outline**

Last time we worked a few examples making use of the fact that natural systems have the minimum "action", so

$\mathcal{L} = T - U$

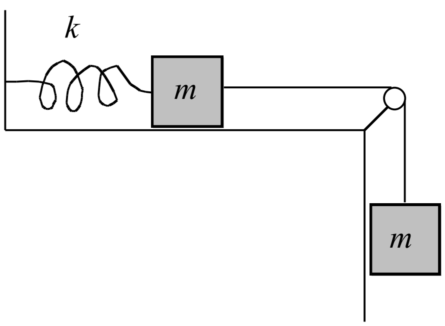
Regardless of what coordinates we express it in terms of

$\mathcal{L} \equiv T(t, q_i(t), \dot{q}_1(t), \ldots q_N(t), \dot{q}_N(t)) - U(t, q_i(t), \dot{q}_1(t), \ldots q_N(t), \dot{q}_N(t))$

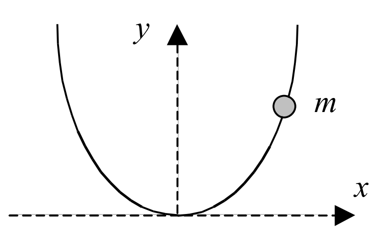
Must satisfy

$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0$ for all the individual coordinates.

Example #1:



Say Exercise #2: A bead of mass m slides without friction along a wire bent into a parabola, $y = ax^2$, where the y axis points upward.



Example #3: Suppose a particle is confined to move on top of a hemisphere of radius R. Pick a good set of generalized coordinates and express the height and speed of the particle in terms of them.

Along the way, we experienced the advantages of the Lagrangian approach:

- The Lagrangian (& energies) are scalars, instead of vectors like force and acceleration.
- It is easier to use whatever coordinates are "natural" with this method.
- It provides a somewhat "systematic" approach to getting equations of motion. Write down the energies in terms of the chosen coordinates and take some derivatives of $\mathcal{L} = T - U$.

Drawbacks of the Lagrangian approach:

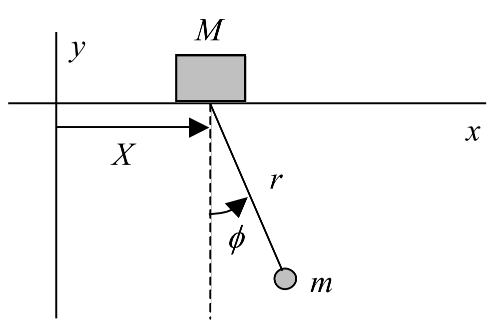
- Friction/drag can't be included easily.
- You don't get the same mechanistic understanding that dealing with forces and torques gives

For today, you were to read the general proof that the 'action integral' is indeed minimized (or at least 'stationary') along the correct path and that implies that the integrand (the Lagrangian) must satisfy Lagrange's equation. The argument is very similar to what we used in Ch. 6 to develop Lagrange's equation in the first place – assume a *wrong* path that has an error of amplitude ε, and then reason out that, when ε is 0 you indeed get no error in the integral.

The discussion in the book is fairly clear, so probably more important than our going over it is our getting lots of practice *using* the Lagrange approach.

Now, we'll continue getting more practice.

<u>I do: Example #3</u>: (Ex. 10.8 of F&C 5th ed.) Find the equations of motion for the system shown below. The block slides without friction along the *x* axis and the pendulum swings in the *xy* plane on a massless rod of length *r*.



The Cartesian coordinates of the pendulum bob are:

$$x = X + r\sin\phi \quad \text{and} \quad y = -r\cos\phi .$$

The derivatives of these are:

$$\dot{x} = \dot{X} + r\dot{\phi}\cos\phi \quad \text{and} \quad \dot{y} = r\dot{\phi}\sin\phi .$$

The kinetic energy of the system is:

$$T = \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}m\left[\left(\dot{X} + r\dot{\phi}\cos\phi\right)^2 + \left(r\dot{\phi}\sin\phi\right)^2\right],$$

$$T = \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}m\left[\dot{X}^2 + \left(r\dot{\phi}\right)^2 + 2\dot{X}r\dot{\phi}\cos\phi\right].$$

The potential energy is $U = -mgr\cos\phi$, so the Lagrangian is:

$$\mathcal{L} = T - U = \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}m\left[\dot{X}^2 + (r\dot{\phi})^2 + 2\dot{X}r\dot{\phi}\cos\phi\right] + mgr\cos\phi .$$

The equations of motion are:

$$\frac{\partial\mathcal{L}}{\partial X} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{X}} \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial\phi} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}},$$

which give:

$$0 = \frac{d}{dt}\left[M\dot{X} + m(\dot{X} + r\dot{\phi}\cos\phi)\right],$$

and:

$$-m\dot{X}r\dot{\phi}\sin\phi - mgr\sin\phi = \frac{d}{dt}\left[m(r^2\dot{\phi} + \dot{X}r\cos\phi)\right] = mr^2\ddot{\phi} + m\ddot{X}r\cos\phi - m\dot{X}r\dot{\phi}\sin\phi .$$

The first equation means that the total momentum in the *x* direction is conserved:

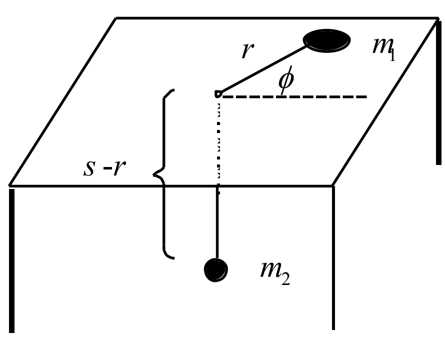
$$P_x = M\dot{X} + m(\dot{X} + r\dot{\phi}\cos\phi) = \text{constant}.$$

The second equation simplifies to:

$$\ddot{\phi} + \frac{\ddot{X}}{r}\cos\phi + \frac{g}{r}\sin\phi = 0 .$$

If $M >> m$, the upper mass will barely move ($\ddot{X} \approx 0$) and this reduces to the equation for a simple pendulum. These two coupled differential equations are difficult to solve, but they are easy to derive using the Lagrangian approach.

<u>They do: Exercise #4</u>: Here's a real classic: suppose a string of length *s* connects a puck of mass m_1 on a frictionless table and an object with mass m_2 through a hole (see the figure below).



(a) Write the Lagrangian for the system in terms of the puck's polar coordinates.

The kinetic energy for the puck is:

$$T_1 = \tfrac{1}{2}m_1(\dot{r}^2 + r^2\dot{\phi}^2),$$

and for the other mass is:

$$T_2 = \tfrac{1}{2} m_2 (-\dot{r})^2 = \tfrac{1}{2} m_2 \dot{r}^2 .$$

define the gravitational potential energy to be zero at the level of the table, so:

$$U = -m_2 g (s - r).$$

The Lagrangian is:

$$\mathcal{L} = (T_1 + T_2) - U = \tfrac{1}{2} m_1 (\dot{r}^2 + r^2 \dot{\phi}^2) + \tfrac{1}{2} m_2 \dot{r}^2 + m_2 g (s - r).$$

(b) Find the equations of motion for the system.

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}},$$

$$m_1 r \dot{\phi}^2 - m_2 g = \frac{d}{dt}(m_1 \dot{r} + m_2 \dot{r}) \quad \text{and} \quad 0 = \frac{d}{dt}(m_1 r^2 \dot{\phi}),$$

$$m_1 r \dot{\phi}^2 - m_2 g = (m_1 + m_2)\ddot{r} \quad \text{and} \quad m_1 r^2 \dot{\phi} = \ell = \text{constant}.$$

The second equation is equivalent to conservation of angular momentum. The first equation can be rewritten as:

$$(m_1 + m_2)\ddot{r} = \frac{\ell^2}{m_1 r^3} - m_2 g .$$

Look at some Qualitative cases:

Whether the radius grows or shrinks depends on whether the orbital term is greater or less than the gravitational term. The balancing point, where there is no radial acceleration is when

$$\frac{\ell^2}{m_1 r^3} = m_2 g$$

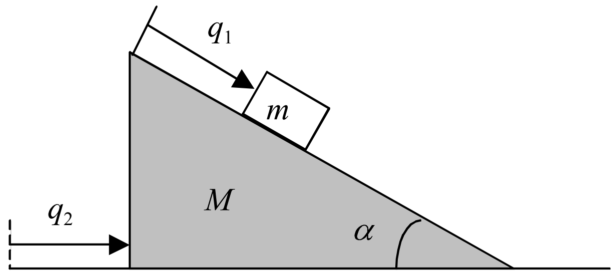
It may not look too familiar in that form, but rephrasing it as

$$m_1 r \dot{\theta}^2 = F_{radial}$$

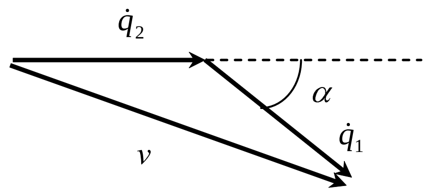
$$m_1 \frac{v_{\tan gential}^2}{r} = F_{radial}$$

May look familiar – when the radial force experienced by m_1 is equal to mv^2/r, the thing executes a *circular* orbit, so r = constant.

Example #3: (Ex. 7.5.) A small block is sliding along a plane that can slide without friction. The coordinate q_1 is clearly not a Cartesian coordinate.



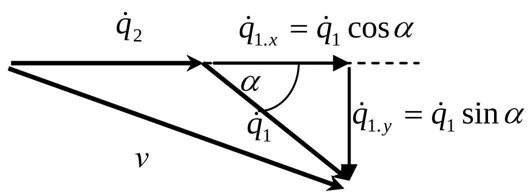
The velocity of the block relative to the ground (inertial frame) is the velocity of the plane plus the velocity of the block relative to the plane (see the diagram below).



By the law of cosines, the size of the block's speed squared is:

$$v^2 = \dot{q}_1^2 + \dot{q}_2^2 + 2\dot{q}_1\dot{q}_2 \cos\alpha \, .$$

Alternatively, one can break it down this way and get the same result:



The kinetic energy of the system is:

$$T = \tfrac{1}{2} M\dot{q}_2^2 + \tfrac{1}{2} m \left(\dot{q}_1^2 + \dot{q}_2^2 + 2\dot{q}_1\dot{q}_2 \cos\alpha \right),$$

and the potential energy is:

$$U = -mgq_1 \sin\alpha .$$

The Lagrangian is:

$$\mathcal{L} = T - U = \tfrac{1}{2} M\dot{q}_2^2 + \tfrac{1}{2} m \left(\dot{q}_1^2 + \dot{q}_2^2 + 2\dot{q}_1\dot{q}_2 \cos\alpha \right) + mgq_1 \sin\alpha \, .$$

The equations of motion are:

$$\frac{\partial \mathcal{L}}{\partial q_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1} \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial q_2} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_2},$$

which give:

$$0 = \frac{d}{dt}\left[M\dot{q}_2 + m\left(\dot{q}_2 + \dot{q}_1 \cos\alpha \right) \right] \text{ and } \quad mg \sin\alpha = \frac{d}{dt}\left(m\dot{q}_1 + m\dot{q}_2 \cos\alpha \right),$$

$$M\ddot{q}_2 + m\left(\ddot{q}_2 + \ddot{q}_1\cos\alpha\right) = 0 \quad \text{and} \quad \ddot{q}_1 + \ddot{q}_2\cos\alpha = g\sin\alpha\,.$$

The first equation gives:

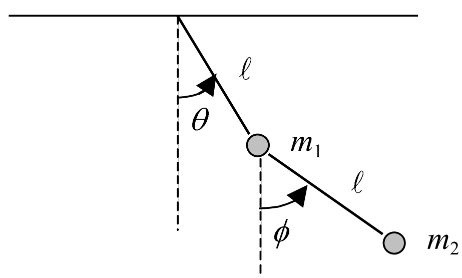
$$\ddot{q}_2 = \frac{-m}{M+m}\ddot{q}_1\cos\alpha\,.$$

These equations can be solved (you can check the algebra) for $\ddot{q}_1$ and $\ddot{q}_2$ to get:

$$\ddot{q}_1 = \frac{g\sin\alpha}{1-\dfrac{m\cos^2\alpha}{m+M}} \quad \text{and} \quad \ddot{q}_2 = \frac{-g\sin\alpha\cos\alpha}{\dfrac{m+M}{m}-\cos^2\alpha}\,.$$

The right hand sides of these equations are constant, so these are easy to solve. The acceleration $\ddot{q}_1$ of the block relative to the plane and the acceleration $\ddot{q}_2$ of the plane are constant. Try getting these equations using Newton's second law!

Exercise #5: Suppose a double pendulum consists of two rigid, massless rods connecting two masses. The masses are not equal, but the lengths of the rods are. The pendulum only swings in one vertical plane.



(a) Write the Lagrangian for the system in terms of the angles shown above.

Take the top pivot point as the origin and choose positive x to the right and positive y upward. The components of the position of m_1 are:

$$x_1 = \ell\sin\theta \quad \text{and} \quad y_1 = -\ell\cos\theta,$$

so the components of its velocity are,

$$\dot{x}_1 = \ell\dot{\theta}\cos\theta \quad \text{and} \quad \dot{y}_1 = \ell\dot{\theta}\sin\theta\,.$$

The components of the position of m_2 are:

$$x_2 = \ell\sin\theta + \ell\sin\phi \quad \text{and} \quad y_2 = -\ell\cos\theta - \ell\cos\phi,$$

$$x_2 = \ell(\sin\theta + \sin\phi) \quad \text{and} \quad y_2 = -\ell(\cos\theta + \cos\phi),$$

so the components of its velocity are,

$$\dot{x}_2 = \ell\left(\dot{\theta}\cos\theta + \dot{\phi}\cos\phi\right) \quad \text{and} \quad \dot{y}_2 = \ell\left(\dot{\theta}\sin\theta + \dot{\phi}\sin\phi\right).$$

The kinetic energy is:

$$T = \tfrac{1}{2} m_1 \left(\dot{x}_1^2 + \dot{y}_1^2\right) + \tfrac{1}{2} m_2 \left(\dot{x}_2^2 + \dot{y}_2^2\right),$$

$$T = \tfrac{1}{2} m_1 \ell^2 \dot{\theta}^2 + \tfrac{1}{2} m_2 \ell^2 \left[\left(\dot{\theta}\cos\theta + \dot{\phi}\cos\phi\right)^2 + \left(\dot{\theta}\sin\theta + \dot{\phi}\sin\phi\right)^2\right],$$

$$T = \tfrac{1}{2} m_1 \ell^2 \dot{\theta}^2 + \tfrac{1}{2} m_2 \ell^2 \left[\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\left(\cos\theta\cos\phi + \sin\theta\sin\phi\right)\right],$$

$$T = \tfrac{1}{2} m_1 \ell^2 \dot{\theta}^2 + \tfrac{1}{2} m_2 \ell^2 \left[\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\theta - \phi)\right],$$

$$T = \tfrac{1}{2} (m_1 + m_2)\ell^2 \dot{\theta}^2 + \tfrac{1}{2} m_2 \ell^2 \left[\dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\theta - \phi)\right].$$

The potential energy is:

$$U = m_1 g y_1 + m_2 g y_2,$$

$$U = -(m_1 + m_2) g\ell\cos\theta - m_2 g\ell\cos\phi,$$

so the Lagrangian is:

$$\mathcal{L} = \tfrac{1}{2}(m_1 + m_2)\ell^2\dot{\theta}^2 + \tfrac{1}{2} m_2 \ell^2 \dot{\phi}^2 + m_2 \ell^2 \dot{\theta}\dot{\phi}\cos(\theta - \phi) + (m_1 + m_2) g\ell\cos\theta + m_2 g\ell\cos\phi\ .$$

(b) Find the equations of motion of the system.

The equation associated with θ is:

$$\frac{\partial\mathcal{L}}{\partial\theta} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}},$$

$$-m_2\ell^2\dot{\theta}\dot{\phi}\sin(\theta - \phi) - (m_1 + m_2) g\ell\sin\theta = \frac{d}{dt}\left[(m_1 + m_2)\ell^2\dot{\theta} + m_2\ell^2\dot{\phi}\cos(\theta - \phi)\right],$$

$$-m_2\ell^2\dot{\theta}\dot{\phi}\sin(\theta - \phi) - (m_1 + m_2) g\ell\sin\theta = (m_1 + m_2)\ell^2\ddot{\theta} + m_2\ell^2\left[\ddot{\phi}\cos(\theta - \phi) - \dot{\phi}(\dot{\theta} - \dot{\phi})\sin(\theta - \phi)\right]$$

The equation associated with ϕ is:

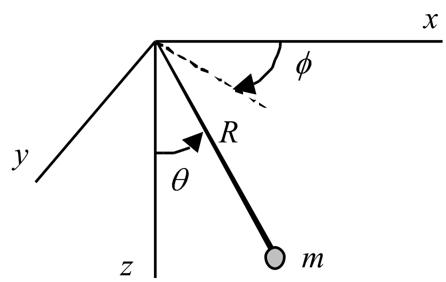
$$\frac{\partial\mathcal{L}}{\partial\phi} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}},$$

$$m_2\ell^2\dot{\theta}\dot{\phi}\sin(\theta - \phi) - m_2 g\ell\sin\phi = \frac{d}{dt}\left[m_2\ell^2\left(\dot{\phi} + \dot{\theta}\cos(\theta - \phi)\right)\right],$$

$$m_2\ell^2\dot{\theta}\dot{\phi}\sin(\theta - \phi) - m_2 g\ell\sin\phi = m_2\ell^2\left[\ddot{\phi} + \ddot{\theta}\cos(\theta - \phi) - \dot{\theta}(\dot{\theta} - \dot{\phi})\sin(\theta - \phi)\right].$$

These equations are hideously complicated, but it is not difficult to get using the Lagrangian approach, if you are careful. They would be tremendously difficult to get using the Newtonian approach!

Example #1: (Ex. 10.9 of F&C 5th ed.) Spherical pendulum – Find the equations of motion using spherical polar coordinates. Note that the length is fixed. This is the same problem as we worked when the ball was confined to the surface of a sphere except now we define z down in the direction of the gravitational force.



The position of the bob in Cartesian coordinates is:

$$x = R\sin\theta\cos\phi, \quad y = R\sin\theta\sin\phi, \quad \text{and} \quad z = R\cos\theta,$$

The easiest way to get the speed squared is as follows:

$$d\vec{r} = dr\hat{r} + r\sin\theta d\phi\hat{\phi} + rd\theta\hat{\theta}$$

So

$$\frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r\sin\theta\frac{d\phi}{dt}\hat{\phi} + r\frac{d\theta}{dt}\hat{\theta}$$

Then

$$v^2 = \left(\frac{dr}{dt}\right)^2 + \left(r\sin\theta\frac{d\phi}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2$$

With r=R constant,

$$v^2 = \left(R\sin\theta\dot{\phi}\right)^2 + \left(R\dot{\theta}\right)^2$$

The Lagrangian is:

$$\mathcal{L} = T - U = \tfrac{1}{2}mv^2 - \left(-mgz\right) = \tfrac{1}{2}mR^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + mgR\cos\theta,$$

so the equations of motion are:

$$\frac{\partial\mathcal{L}}{\partial\theta} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial\phi} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}},$$

$$mR^2\dot{\phi}^2\sin\theta\cos\theta - mgR\sin\theta = \frac{d}{dt}\left(mR^2\dot{\theta}\right) \quad \text{and} \quad 0 = \frac{d}{dt}\left(mR^2\dot{\phi}\sin^2\theta\right).$$

These can be rewritten as:

$$\ddot{\theta} = \dot{\phi}^2\sin\theta\cos\theta - \frac{g}{R}\sin\theta \quad \text{and} \quad mR^2\dot{\phi}\sin^2\theta = \text{constant}.$$

The second equation means that the angular momentum of the pendulum is conserved.

If $\phi = \text{constant}$, then $\dot{\phi} = 0$ and the pendulum swings in a plane. The equation of motion reduces to the familiar (hopefully):

$$\ddot{\theta} = -\frac{g}{R}\sin\theta .$$

<u>Example #2</u>: (Ex. 7.7 of Thornton) Frictionless bead on a *spinning* parabolic wire $z = cr^2$ rotating with an angular velocity $\dot{\theta} = \omega$. What is the condition on the angular frequency to get the bead to rotate at a fixed location on the wire?

In cylindrical polar coordinates, the velocity is:

$$\vec{v} = \dot{r}\hat{r} + \dot{z}\hat{z} + r\dot{\theta}\hat{\theta} ,$$

so:

$$T = \tfrac{1}{2}m\vec{v}^2 = \tfrac{1}{2}m\left(\vec{v}\cdot\vec{v}\right) = \tfrac{1}{2}m\left[\dot{r}^2 + \dot{z}^2 + \left(r\dot{\theta}\right)^2\right].$$

This system has one degree of freedom, where the bead is along the wire. Use *r* to describe this (could also use *z*). Since $z = cr^2$, the dervivative is $\dot{z} = 2cr\dot{r}$ and:

$$T = \tfrac{1}{2}m\left(\dot{r}^2 + 4c^2r^2\dot{r}^2 + r^2\omega^2\right).$$

The potential energy is $U = mgz = mgcr^2$, so the Lagrangian is:

$$\mathcal{L} = T - U = \tfrac{1}{2}m\left(\dot{r}^2 + 4c^2r^2\dot{r}^2 + r^2\omega^2\right) - mgcr^2 .$$

The derivatives of the Lagrangian are:

$$\frac{\partial\mathcal{L}}{\partial r} = \frac{m}{2}\left(8cr\dot{r}^2 + 2r\omega^2\right) - 2mgcr ,$$

$$\frac{\partial\mathcal{L}}{\partial\dot{r}} = \frac{m}{2}\left(2\dot{r} + 8c^2r^2\dot{r}\right),$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} = \frac{m}{2}\left(2\ddot{r} + 16c^2r\dot{r}^2 + 8c^2r^2\ddot{r}\right).$$

The equation of motion is:

$$\frac{\partial\mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} ,$$

$$\frac{m}{2}\left(8cr\dot{r}^2 + 2r\omega^2\right) - 2mgcr = \frac{m}{2}\left(2\ddot{r} + 16c^2r\dot{r}^2 + 8c^2r^2\ddot{r}\right).$$

This can be rewritten as:

$$\ddot{r}\left(1 + 4c^2r^2\right) + \dot{r}^2\left(4c^2r\right) + r\left(2gc - \omega^2\right) = 0 ,$$

which is a complicated, <u>nonlinear</u> ($\dot{r}$ is squared) differential equation.

If the bead rotates with $r = R = \text{constant}$, then $\dot{r} = \ddot{r} = 0$ so:

$$R\left(2gc - \omega^2\right) = 0 ,$$

and

$$\omega = \sqrt{2gc}\ .$$

Next two classes:

- Monday – More Examples of Lagrange's Equations