Fri.10/29	**7.2-.3** Lagrange's with Constrained	
Mon., 11/1 Wed., 11/3 Thurs. 11/4	**7.4-.5** Proof and Examples **7.6-.8** Generalized Variables & Classical Hamiltonian (Recommend 7.9 if you've had Phys 332)	**HW7**

Last time we learned that the integral of the magnitude of momentum over the path that a system really takes is minimal, that is, the principle of "least action" holds.

$$S = \int_i^f p\,ds$$

Through a little bit of math, we found that this statement is equivalent to saying that *time* integral of the difference between kinetic and potential energy, i.e., the "Lagrangian", is minimized,

$$\mathcal{L} = T - U$$

Regardless of what coordinates we express it in terms of

$$\mathcal{L} \equiv T(t, q_i(t), \dot{q}_1(t), \ldots q_N(t), \dot{q}_N(t)) - U(t, q_i(t), \dot{q}_1(t), \ldots q_N(t), \dot{q}_N(t))$$

$$\int \mathcal{L}(t, q_i(t), \dot{q}_1(t), ..q_N(t), \dot{q}_N(t))dt$$

Of course, that's equivalent to saying that the Lagrangian satisfies the differential equations

$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0$ for all the individual coordinates.

We showed that at least for Cartesian coordinates *that* in turn is equivalent to Newton's 2nd law.

As was our practice in Chapter 6, we focused much more on this differential equation which the integrand satisfies than on the integral.

We worked through a few familiar and interesting problems. They could all be characterized as "unconstrained" in that there weren't any restraining forces that, say, forced the system to only move on a surface or along a curve.

This time we will consider *constrained* systems.

Constrained Motion:

Often, it does not take $3N$ parameters to describe the motion of N particles because they are subject to constraints. For example, it only takes 2 parameters to describe the motion of a particle sliding on a plane.

Suppose you are trying to describe the motion of N particles. The parameters $q_1, q_2, \cdots, q_n$ ($n \le 3N$) are a set of *generalized coordinates* if the position of each particle can be expressed in terms of $q_1, q_2, \cdots, q_n$ and possibly the time t:

$$\vec{r}_\alpha = \vec{r}_\alpha(q_1, q_2, \cdots, q_n, t) \quad \alpha = 1, \cdots, N.$$

It will also be possible to express each of the generalized coordinates in terms of the positions of the particles and possibly the time t:

$$q_i = q_i(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_N, t) \quad i = 1, \cdots, n.$$

The number of *degrees of freedom* for a system is equal to $3N$ minus the number of constraints (see Example #1 below). If the minimum number of generalized coordinates required to completely describe a system is equal the number of degrees of freedom, the system is *holonomic*. That is the easier type to handle, so we won't consider nonholonomic systems.

An example of a nonholonomic system is a ball rolling without slipping on a flat surface. The ball has two degrees of freedom, but it takes more than two coordinates to specify the orientation of the ball. The ball can take different paths between two points, so it can end up with different points at the top. More coordinates are needed to specify which point is at the top of the ball.

The Lagrangian can be written as a function of the generalized coordinates, their time derivatives, and time:

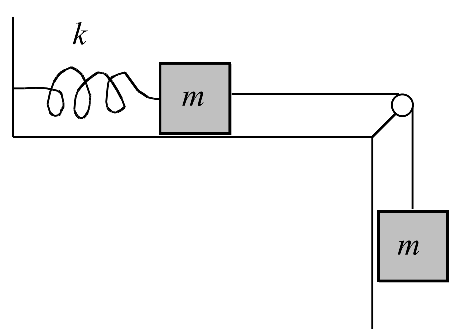
$$\mathcal{L} = \mathcal{L}(q_1, q_2, \cdots, q_n, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n, t)$$

and there will be a Lagrange equation associated with each generalized coordinate:

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \quad i = 1, \cdots, n.$$

The equations of motion found using the Lagrangian approach are equivalent to the ones found using the Newtonian approach. However, the Lagrangian method is much simpler for many problems. It also allows the use of whatever coordinates are convenient for describing a system.

<u>Example #1</u>: How many generalized coordinates are needed to describe the system pictured below? (How many degrees of freedom does the system have?)



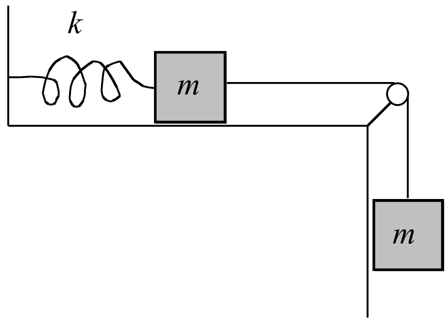
Just <u>one</u>. The distance from the wall of the block on the horizontal surface or the stretch of the spring are good choices.

The number of degrees of freedom for the two particle is:

DOF=3(2) - 2(objects move in plane) - 2(objects move along lines) - 1(linked together) = 1.

How about working it through.

Exercise #1: Two equal masses are constrained by the spring-and-pulley system shown below. Assume that there is no friction and that the pulley is massless. Let x be the distance the spring has stretched.



(a) Write the Lagrangian for the system in terms of x.

The kinetic energy of each block is the same, so the total KE is:

$$T = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m\dot{x}^2 = m\dot{x}^2 .$$

The potential energy is:

$$U = U_{spr} + U_{grav} = \tfrac{1}{2}kx^2 - mgx ,$$

and the Lagrangian is:

$$\mathcal{L} = T - U = m\dot{x}^2 - \tfrac{1}{2}kx^2 + mgx .$$

(b) Find the equation of motion of the system.

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} ,$$

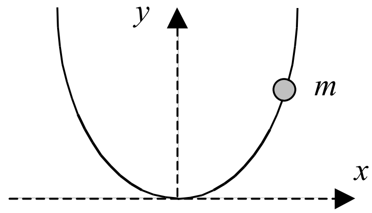
$$-kx + mg = \frac{d}{dt}\left(2m\dot{x}\right),$$

$$2m\ddot{x} + kx - mg = 0 .$$

From the previous chapter, that would have the solution

$$x(t) = A\cos(\omega t - \delta) + \frac{mg}{k} \quad \text{where } \omega = \sqrt{\frac{2m}{k}}$$

Say <u>Exercise #2</u>: A bead of mass *m* slides without friction along a wire bent into a parabola, $y = ax^2$, where the *y* axis points upward.



(a) Write the Lagrangian for the system in terms of *x*.

They Do:

The kinetic energy is:

$$T = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right),$$

but:

$$\dot{y} = \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{d\left(ax^2\right)}{dx}\dot{x} = 2ax\dot{x},$$

so:

$$T = \tfrac{1}{2}m\left[\dot{x}^2 + \left(2ax\dot{x}\right)^2\right] = \tfrac{1}{2}m\dot{x}^2\left(1 + 4a^2x^2\right).$$

The potential energy is:

$$U = mgy = mgax^2,$$

so the Lagrangian is:

$$\mathcal{L} = T - U = \tfrac{1}{2}m\dot{x}^2\left(1 + 4a^2x^2\right) - mgax^2.$$

(b) Find the equations of motion of the system.

I do:

$$\frac{\partial\mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}},$$

$$4ma^2\dot{x}^2x - 2mgax = \frac{d}{dt}\left[m\dot{x}\left(1 + 4a^2x^2\right)\right] = m\left[\ddot{x}\left(1 + 4a^2x^2\right) + \dot{x}\left(8a^2x\dot{x}\right)\right],$$

$$\ddot{x}\left(1 + 4a^2x^2\right) + 4a^2\dot{x}^2x + 2gax = 0.$$

I'll confess to *not* being terribly eager to find an analytical solution for this expression. However, it's perfectly reasonable to use this in a computational simulation.

$$\ddot{x} = -\frac{4a^2\dot{x}^2x + 2gax}{\left(1 + 4a^2x^2\right)}$$

$$\dot{x} \Leftarrow \dot{x} + \ddot{x} * \Delta t$$

$$x \Leftarrow x + \dot{x} * \Delta t$$

$$y = ax^2$$

Try this for

a = 2

$x_0 = 1$

g = 9.8

$v_{xo}=0$

Example #2: Suppose a particle is confined to move on top of a hemisphere of radius R. Pick a good set of generalized coordinates and express the height and speed of the particle in terms of them.

The spherical polar coordinates θ and ϕ with the z axis upward and the origin at the center of the sphere is a natural choice. The Cartesian coordinates of the particle are:

$$x = R\sin\theta\cos\phi, \quad y = R\sin\theta\sin\phi, \quad \text{and} \quad z = R\cos\theta$$

The z component is the height. The components of the particle's velocity are (remember that R is a constant):

$$\dot{x} = R\left(\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi\right), \quad \dot{y} = R\left(\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi\right) \quad \text{and} \quad \dot{z} = -R\dot{\theta}\sin\theta .$$

The speed squared of the particle is:

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 ,$$

$$v^2 = R^2\left[\left(\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi\right)^2 + \left(\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi\right)^2 + \left(-\dot{\theta}\sin\theta\right)^2\right].$$

The cross terms of the first two squares cancel, so:

$$v^2 = R^2\left[\left(\dot{\theta}^2\cos^2\theta\cos^2\phi + \dot{\phi}^2\sin^2\theta\sin^2\phi\right) + \left(\dot{\theta}^2\cos^2\theta\sin^2\phi + \dot{\phi}^2\sin^2\theta\cos^2\phi\right) + \dot{\theta}^2\sin^2\theta\right],$$

$$v^2 = R^2\left[\dot{\theta}^2\cos^2\theta + \dot{\phi}^2\sin^2\phi + \dot{\theta}^2\sin^2\theta\right],$$

$$v^2 = R^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right),$$

$$v = R\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right)^{1/2}.$$

Say this takes place on Earth, so there's a gravitational potential of $U = mgz = mgR(1-\cos\theta)$ relative to the top of the sphere. And we'll say we start with the object on the top of the sphere.

Then

$$\mathcal{L} = T - U = \tfrac{1}{2}mv^2 - mgz = m\left(\tfrac{1}{2}R^2\left(\dot\theta^2 + \dot\phi^2\sin^2\theta\right) - gR\cos\theta\right)$$

They do: Which must satisfy

$$\frac{\partial\mathcal{L}}{\partial\phi} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\phi}$$

$$\frac{\partial m\left(\tfrac{1}{2}R^2\left(\dot\theta^2 + \dot\phi^2\sin^2\theta\right) - gR\cos\theta\right)}{\partial\phi} = \frac{d}{dt}\frac{\partial m\left(\tfrac{1}{2}R^2\left(\dot\theta^2 + \dot\phi^2\sin^2\theta\right) - gR\cos\theta\right)}{\partial\dot\phi}$$

$$0 = \frac{d}{dt}\left(\dot\phi\sin^2\theta\right)$$

So, $\dot\phi\sin^2\theta = C$

Now, if we start with the thing on the top of the sphere, then θ = 0 so sinθ = 0 so C = 0. Of course, the point of C's being a *constant* is that it stays the same value, 0, as the object slides. Naturally, θ will change and so will sinθ, so the only way for C to remain 0 is for $\dot\phi = 0$. That means the thing just slides straight down.

Now the Lagrangian must also satisify

$$\frac{\partial\mathcal{L}}{\partial\theta} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\theta}$$

$$\frac{\partial m\left(\tfrac{1}{2}R^2\left(\dot\theta^2 + \dot\phi^2\sin^2\theta\right) - gR\cos\theta\right)}{\partial\theta} = \frac{d}{dt}\frac{\partial m\left(\tfrac{1}{2}R^2\left(\dot\theta^2 + \dot\phi^2\sin^2\theta\right) - gR\cos\theta\right)}{\partial\dot\theta}$$

$$R\dot\phi^2\sin\theta\cos\theta + g\sin\theta = R\frac{d}{dt}\dot\theta$$

But we just reasoned that $\dot\phi = 0$, so we're left with the old familiar

$$g\sin\theta = R\ddot\theta$$

As for an inverted pendulum.

I do:

What if you *don't* start it at the top and you *do* give it an initial rotation around, then you have

$$\ddot\theta = \dot\phi^2\sin\theta\cos\theta + \frac{g}{R}\sin\theta \quad \text{And} \quad \begin{aligned}\ddot\phi\sin^2\theta &= -2\dot\phi\dot\theta\sin\theta\cos\theta \\ \ddot\phi &= -2\dot\phi\dot\theta\cot\theta\end{aligned}$$

Again, nothing I crave solving analytically, but something that's quite reasonable to handle computationally.

$$\dot{\theta} \Leftarrow \dot{\theta} + \ddot{\theta}\Delta t \quad \text{and} \quad \dot{\phi} \Leftarrow \dot{\phi} + \ddot{\phi}\Delta t$$
$$\theta \Leftarrow \theta + \dot{\theta}\Delta t \qquad\qquad \phi \Leftarrow \phi + \dot{\phi}\Delta t$$

Then

$$\vec{r} = R\langle \cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta \rangle$$

***You* code this.**

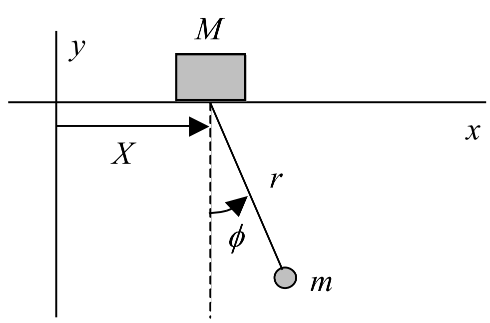
Advantages of the Lagrangian approach:

- The Lagrangian (& energies) are scalars, instead of vectors like force and acceleration.
- It is easier to use whatever coordinates are "natural" with this method.
- It provides a somewhat "systematic" approach to getting equations of motion. Write down the energies in terms of the chosen coordinates and take some derivatives of $\mathcal{L} = T - U$.

Drawbacks of the Lagrangian approach:

- Friction/drag can't be included easily.
- You don't get the same "physical" understanding that dealing with forces and torques gives

Example #3: (Ex. 10.8 of F&C 5th ed.) Find the equations of motion for the system shown below. The block slides without friction along the *x* axis and the pendulum swings in the *xy* plane on a massless rod of length *r*.



The Cartesian coordinates of the pendulum bob are:

$$x = X + r\sin\phi \quad \text{and} \quad y = -r\cos\phi .$$

The derivatives of these are:

$$\dot{x} = \dot{X} + r\dot{\phi}\cos\phi \quad \text{and} \quad \dot{y} = r\dot{\phi}\sin\phi .$$

The kinetic energy of the system is:

$$T = \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}m\left[\left(\dot{X} + r\dot{\phi}\cos\phi\right)^2 + \left(r\dot{\phi}\sin\phi\right)^2\right],$$

$$T = \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}m\left[\dot{X}^2 + \left(r\dot{\phi}\right)^2 + 2\dot{X}r\dot{\phi}\cos\phi\right].$$

The potential energy is $U = -mgr\cos\phi$, so the Lagrangian is:

$$\mathcal{L} = T - U = \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}m\left[\dot{X}^2 + \left(r\dot{\phi}\right)^2 + 2\dot{X}r\dot{\phi}\cos\phi\right] + mgr\cos\phi .$$

The equations of motion are:

$$\frac{\partial\mathcal{L}}{\partial X} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{X}} \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial\phi} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}},$$

which give:

$$0 = \frac{d}{dt}\left[M\dot{X} + m\left(\dot{X} + r\dot{\phi}\cos\phi\right)\right],$$

and:

$$-m\dot{X}r\dot{\phi}\sin\phi - mgr\sin\phi = \frac{d}{dt}\left[m\left(r^2\dot{\phi} + \dot{X}r\cos\phi\right)\right] = mr^2\ddot{\phi} + m\ddot{X}r\cos\phi - m\dot{X}r\dot{\phi}\sin\phi .$$

The first equation means that the total momentum in the x direction is conserved:

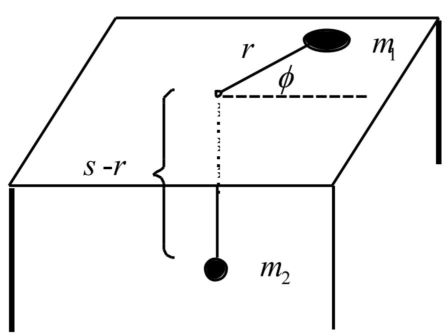
$$P_x = M\dot{X} + m\left(\dot{X} + r\dot{\phi}\cos\phi\right) = \text{constant}.$$

The second equation simplifies to:

$$\ddot{\phi} + \frac{\ddot{X}}{r}\cos\phi + \frac{g}{r}\sin\phi = 0 .$$

If $M >> m$, the upper mass will barely more ($\ddot{X} \approx 0$) and this reduces to the equation for a simple pendulum. These two coupled differential equations are difficult to solve, but they are easy to derive using the Lagrangian approach.

Exercise #4: Suppose a string of length s connects a puck of mass m_1 on a frictionless table and an object with mass m_2 through a hole (see the figure below).



(a) Write the Lagrangian for the system in terms of the puck's polar coordinates.

The kinetic energy for the puck is:

$$T_1 = \tfrac{1}{2} m_1 \left(\dot{r}^2 + r^2 \dot{\phi}^2\right),$$

and for the other mass is:

$$T_2 = \tfrac{1}{2} m_2 \left(-\dot{r}\right)^2 = \tfrac{1}{2} m_2 \dot{r}^2 .$$

The define the gravitational potential energy to be zero at the level of the table, so:

$$U = -m_2 g (s - r).$$

The Lagrangian is:

$$\mathcal{L} = \left(T_1 + T_2\right) - U = \tfrac{1}{2} m_1 \left(\dot{r}^2 + r^2 \dot{\phi}^2\right) + \tfrac{1}{2} m_2 \dot{r}^2 + m_2 g \left(s - r\right).$$

(b) Find the equations of motion for the system.

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}},$$

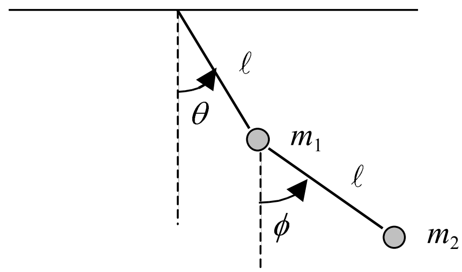
$$m_1 r \dot{\phi}^2 - m_2 g = \frac{d}{dt}\left(m_1 \dot{r} + m_2 \dot{r}\right) \quad \text{and} \quad 0 = \frac{d}{dt}\left(m_1 r^2 \dot{\phi}\right),$$

$$m_1 r \dot{\phi}^2 - m_2 g = \left(m_1 + m_2\right)\ddot{r} \quad \text{and} \quad m_1 r^2 \dot{\phi} = \ell = \text{constant}.$$

The second equation is equivalent to conservation of angular momentum. The first equation can be rewritten as:

$$\left(m_1 + m_2\right)\ddot{r} = \frac{\ell}{m_1 r^3} - m_2 g .$$

Exercise #5: Suppose a double pendulum consists of two rigid, massless rods connecting two masses. The masses are not equal, but the lengths of the rods are. The pendulum only swings in one vertical plane.



(a) Write the Lagrangian for the system in terms of the angles shown above.

Take the top pivot point as the origin and choose positive *x* to the right and positive *y* upward. The components of the position of m_1 are:

$$x_1 = \ell \sin\theta \quad \text{and} \quad y_1 = -\ell \cos\theta,$$

so the components of its velocity are,

$$\dot{x}_1 = \ell\dot{\theta}\cos\theta \quad \text{and} \quad \dot{y}_1 = \ell\dot{\theta}\sin\theta .$$

The components of the position of m_2 are:

$$x_2 = \ell\sin\theta + \ell\sin\phi \quad \text{and} \quad y_2 = -\ell\cos\theta - \ell\cos\phi,$$

$$x_2 = \ell(\sin\theta + \sin\phi) \quad \text{and} \quad y_2 = -\ell(\cos\theta + \cos\phi),$$

so the components of its velocity are,

$$\dot{x}_2 = \ell(\dot{\theta}\cos\theta + \dot{\phi}\cos\phi) \quad \text{and} \quad \dot{y}_2 = \ell(\dot{\theta}\sin\theta + \dot{\phi}\sin\phi).$$

The kinetic energy is:

$$T = \tfrac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \tfrac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2),$$

$$T = \tfrac{1}{2}m_1\ell^2\dot{\theta}^2 + \tfrac{1}{2}m_2\ell^2\left[(\dot{\theta}\cos\theta + \dot{\phi}\cos\phi)^2 + (\dot{\theta}\sin\theta + \dot{\phi}\sin\phi)^2\right],$$

$$T = \tfrac{1}{2}m_1\ell^2\dot{\theta}^2 + \tfrac{1}{2}m_2\ell^2\left[\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}(\cos\theta\cos\phi + \sin\theta\sin\phi)\right],$$

$$T = \tfrac{1}{2}m_1\ell^2\dot{\theta}^2 + \tfrac{1}{2}m_2\ell^2\left[\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\theta - \phi)\right],$$

$$T = \tfrac{1}{2}(m_1 + m_2)\ell^2\dot{\theta}^2 + \tfrac{1}{2}m_2\ell^2\left[\dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\theta - \phi)\right].$$

The potential energy is:

$$U = m_1 g y_1 + m_2 g y_2,$$

$$U = -(m_1 + m_2)g\ell\cos\theta - m_2 g\ell\cos\phi,$$

so the Lagrangian is:

$$\mathcal{L} = \tfrac{1}{2}(m_1 + m_2)\ell^2\dot{\theta}^2 + \tfrac{1}{2}m_2\ell^2\dot{\phi}^2 + m_2\ell^2\dot{\theta}\dot{\phi}\cos(\theta - \phi) + (m_1 + m_2)g\ell\cos\theta + m_2 g\ell\cos\phi .$$

(b) Find the equations of motion of the system.

The equation associated with θ is:

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}},$$

$$-m_2\ell^2\dot{\theta}\dot{\phi}\sin(\theta-\phi) - (m_1+m_2)g\ell\sin\theta = \frac{d}{dt}\left[(m_1+m_2)\ell^2\dot{\theta} + m_2\ell^2\dot{\phi}\cos(\theta-\phi)\right],$$

$$-m_2\ell^2\dot{\theta}\dot{\phi}\sin(\theta-\phi) - (m_1+m_2)g\ell\sin\theta = (m_1+m_2)\ell^2\ddot{\theta} + m_2\ell^2\left[\ddot{\phi}\cos(\theta-\phi) - \dot{\phi}(\dot{\theta}-\dot{\phi})\sin(\theta-\phi)\right]$$

The equation associated with ϕ is:

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}},$$

$$m_2\ell^2\dot{\theta}\dot{\phi}\sin(\theta-\phi) - m_2 g\ell\sin\phi = \frac{d}{dt}\left[m_2\ell^2\left(\dot{\phi} + \dot{\theta}\cos(\theta-\phi)\right)\right],$$

$$m_2\ell^2\dot{\theta}\dot{\phi}\sin(\theta-\phi) - m_2 g\ell\sin\phi = m_2\ell^2\left[\ddot{\phi} + \ddot{\theta}\cos(\theta-\phi) - \dot{\theta}(\dot{\theta}-\dot{\phi})\sin(\theta-\phi)\right].$$

These equations are hideously complicated, but it is not difficult to get using the Lagrangian approach, if you are careful. They would be tremendously difficult to get using the Newtonian approach!

Next two classes:

- Monday – More Examples of Lagrange's Equations