Wed.10/27 Fri.10/29	**7.1** Lagrange's Equations with Unconstrained **7.2-.3** Lagrange's with Constrained	
Mon., 11/1 Wed., 11/3 Thurs. 11/4 Fri., 11/5	**7.4-.5** Proof and Examples **7.6-.8** Generalized Variables & Classical Hamiltonian (Recommend 7.9 if you've had Phys 332) **8.1-.2** 2-Body Central Forces, Relative Coordinates.	**HW7**

Note: email out EulerCromer.py code (again) and let them know that they'll be modifying it for the last problem in the HW.

So, last chapter we learning how to find the equation of a path for which *some* property is minimized – the length, the potential energy, the time under one velocity condition or another. The trick was setting up an integral for that property, then imposing that the *integrand* must satisfy Lagrange's equation.

Minimized $S = \int f(x, y(x), y'(x))dx$ means $\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$.

Where these variables don't need to be the Cartesian coordinates, they could be r and ϕ for example.

$S = \int f(r, \phi(r), \phi'(r))dr$ then $\frac{\partial f}{\partial \phi} - \frac{d}{dr}\frac{\partial f}{\partial \phi'} = 0$

Last time we learned that this could be generalized to situations in which it's more convenient to parameterize the spatial variables in terms of time.

Minimized $S = \int f(t, x(t), \dot{x}(t).y(t), \dot{y}(t), z(t), \dot{z}(t))dt$ means $\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}} = 0$, $\frac{\partial f}{\partial y} - \frac{d}{dt}\frac{\partial f}{\partial \dot{y}} = 0$, $\frac{\partial f}{\partial z} - \frac{d}{dt}\frac{\partial f}{\partial \dot{z}} = 0$

Or, perhaps we're dealing with spherical coordinates so

$S = \int f(t, r(t), \dot{r}(t).\phi(t), \dot{\phi}(t), \theta(t), \dot{\theta}(t))dt$ with $\frac{\partial f}{\partial r} - \frac{d}{dt}\frac{\partial f}{\partial \dot{r}} = 0$ $\frac{\partial f}{\partial \phi} - \frac{d}{dt}\frac{\partial f}{\partial \dot{\phi}} = 0$, $\frac{\partial f}{\partial \theta} - \frac{d}{dt}\frac{\partial f}{\partial \dot{\theta}} = 0$

Heck, whatever the variables may be and however many of them there may be, let's call them q_i, we'd have

$S = \int f(t, q_i(t), \dot{q}_1(t).q_2(t), \dot{q}_2(t), ..q_N(t), \dot{q}_N(t))dt$ with $\frac{\partial f}{\partial q_i} - \frac{d}{dt}\frac{\partial f}{\partial \dot{q}_i} = 0$ being satisfied for each of whatever the coordinates may be.

Up to this point, we've been considering minimizing some property of a path through space, but step back and look at this mathematical relationship as just that – a mathematical truism, then it's equally applicable to any integral that can be phrased this way.

Now, Historically, it was postulated as an article of faith that the "action" of an object followed the path through space which minimized its "action", that is, $S=\int_i^f p\,ds$. Let's see where that leads and if we can than justify this claim.

$$S=\int_i^f p\,ds=\int_i^f p\frac{ds}{dt}dt=\int_i^f pv\,dt$$, classically (at slow speeds) p = mv, so

$$S=\int_i^f mvv\,dt=\int_i^f mv^2dt=\int_i^f 2\left(\tfrac{1}{2}mv^2\right)dt=\int_i^f 2T\,dt$$

Now, E = T+U so 2T = T-U + E

$$S=\int_i^f (T-U+E)\,dt=\int_i^f (T-U)\,dt+\int_i^f E\,dt$$

Then, following the path of minimum "action" is the same as following the path for which these integrals are minimized. If we imagine as a given the initial conditions (times, locations, and speeds) and the time interval over which we're looking, then, for an system that has no work done on it (only restraining forces and forces that related to potential energies*)*, E is constant, so the second integral is just some constant. That means that minimizing the "action" is essentially up to minimizing the first integral.

Then the integral we're interested in has the form

$$\int T(t,q_i(t),\dot{q}_1(t),..q_N(t),\dot{q}_N(t))-U(t,q_i(t),\dot{q}_1(t),..q_N(t),\dot{q}_N(t))dt$$

The integrand, the difference between the kinetic and potential energy, is called its "Lagrangian."

$$\mathcal{L}\equiv T(t,q_i(t),\dot{q}_1(t),...q_N(t),\dot{q}_N(t))-U(t,q_i(t),\dot{q}_1(t),...q_N(t),\dot{q}_N(t))$$

Now, if this were your desire, to find the way the system would evolve as to minimize its "action" then you'd say that the integrand, the "Lagrangian" must satisfy Lagrange's equation for each degree of freedom:

$$\frac{\partial\mathcal{L}}{\partial q_i}-\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_i}=0 \text{ or } \frac{\partial\mathcal{L}}{\partial q_i}=\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_i}$$

I'll first demonstrate that this is plausible, but a more general (probably not airtight) proof will wait until later in the chapter.

Single Unconstrained Particle: We will begin by considering a single particle that is *unconstrained* (there is no explicit restriction on its motion). An example of a constraint is specifying that a particle moves on the surface of a sphere. The *Lagrangian (function)* is defined as:

$$\mathcal{L} = T - U,$$

where the kinetic energy is:

$$T = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\dot{\vec{r}}^2 = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right),$$

and the potential energy is:

$$U = U(\vec{r}) = U(x,y,z).$$

The Lagrangian ($\mathcal{L} = T - U$) is not the same as the total energy of a system ($E = T + U$)! Don't get the sign wrong or your results will be wrong.

The following partial derivatives of the Lagrangian are related to the force:

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial x} &= \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} \\
\frac{\partial(T-U)}{\partial x} &= \frac{d}{dt}\frac{\partial(T-U)}{\partial\dot{x}} \\
-\frac{\partial U}{\partial x} &= \frac{d}{dt}(m\dot{x}) \\
F_x &= \frac{dp_x}{dt}
\end{aligned},$$

Which, of course is definitively true! Similarly for the other three components. Then, tracing this logic back up, yes, it's true that

$$\frac{\partial\mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}}$$

Or

$$\int_i^f (T-U)\,dt$$

Is minimized for the trajectories of particles, and thus

$$S = \int_i^f p\,ds$$

The "action" is minimized.

Though, as we were in the last chapter, we'll more often be interested in applying Lagrange's equation to the integrand, the "Lagrangian."

Let's use this.

Example 1: A particle in 2-D with gravity in Cartesian Coordinates.

Choose the y axis to be upward. The Lagrangian in Cartesian coordinates is:

$$\mathcal{L} = T - U = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right) - \left(mgy\right).$$

The equations of motion are:

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \qquad \text{and} \qquad \frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}},$$

$$0 = \frac{d}{dt}\left(m\dot{x}\right) = m\ddot{x} \quad \text{and} \qquad -mg = \frac{d}{dt}\left(m\dot{y}\right) = m\ddot{y}.$$

These are exactly what you would get using Newton's second law.

Ask them to set-up problem 7.3

$$\mathcal{L} = T - U = \tfrac{1}{2} m\dot{x}^2 + \tfrac{1}{2} m\dot{y}^2 - \tfrac{1}{2} k\left(x^2 + y^2\right)$$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}$$

$$\frac{\partial\left(\tfrac{1}{2} m\dot{x}^2 + \tfrac{1}{2} m\dot{y}^2 - \tfrac{1}{2} k\left(x^2 + y^2\right)\right)}{\partial x} = \frac{d}{dt}\frac{\partial\left(\tfrac{1}{2} m\dot{x}^2 + \tfrac{1}{2} m\dot{y}^2 - \tfrac{1}{2} k\left(x^2 + y^2\right)\right)}{\partial \dot{x}}$$

$$-kx = \frac{d}{dt} m\dot{x} = m\ddot{x}$$

Similarly,

$$-ky = m\ddot{y}$$

Generalized Coordinates:

While the equivalence of this 'lest action principle' and Newton's 2nd law is evident in Cartesian coordinates, you can of course re-express x, y, z in terms of polar or spherical coordinates, or even some other convenient variables. So, the action integral can be written in terms of other coordinates. These do not have to be spherical polar or cylindrical polar coordinates. We'll just vaguely say use our *generalized coordinates* and labeled $q_1,\ q_2,\ q_3$. They must have the property that they uniquely specify the position, $\vec{r}$:

$$\vec{r} = \vec{r}\left(q_1, q_2, q_3\right)$$

And, for that matter, vice versa:

$$q_i = q_i\left(\vec{r}\right) \quad i = 1,2,3.$$

The Lagrangian can be written in terms of the generalized coordinates and their derivatives:

$$\mathcal{L} = \mathcal{L}\left(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3\right),$$

and the action integral is:

$$S = \int_{t_1}^{t_2} \mathcal{L}\left(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3\right) dt .$$

Using Hamilton's principle, the equations of motion (Euler-Lagrange equations) are:

$$\frac{\partial \mathcal{L}}{\partial q_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1}, \qquad \frac{\partial \mathcal{L}}{\partial q_2} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_2}, \qquad \text{and} \qquad \frac{\partial \mathcal{L}}{\partial q_3} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_3} .$$

The derivatives of the Lagrangian are not necessarily components of the force and the momentum. However, they are similar so we will make the definitions:

$$\frac{\partial \mathcal{L}}{\partial q_1} = \left(\textit{generalized force}\ \text{associated with the}\ i^{th}\ \text{coordinate}\right),$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_1} = \left(\textit{generalized momentum}\ \text{associated with the}\ i^{th}\ \text{coordinate}\right).$$

The latter play an important role in Hamiltonian mechanics (Ch. 13).

<u>Example 2</u>: A particle in 2-D in Polar Coordinates.

The Lagrangian in Cartesian coordinates is:

$$\mathcal{L} = T - U = \tfrac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right) - U\left(x, y\right).$$

The transformations from polar coordinates are:

$$x = r\cos\phi \quad \text{and} \quad y = r\sin\phi,$$

so the derivatives of the Cartesian coordinates are:

$$\dot{x} = \dot{r}\cos\phi - r\dot{\phi}\sin\phi \quad \text{and} \quad \dot{y} = \dot{r}\sin\phi + r\dot{\phi}\cos\phi .$$

The Lagrangian can be rewritten as:

$$\mathcal{L} = \tfrac{1}{2} m \left[\left(\dot{r}\cos\phi - r\dot{\phi}\sin\phi\right)^2 + \left(\dot{r}\sin\phi + r\dot{\phi}\cos\phi\right)^2\right] - U\left(r, \phi\right) = \tfrac{1}{2} m \left(\dot{r}^2 + r^2\dot{\phi}^2\right) - U\left(r, \phi\right).$$

The equation of motion associated with r is:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}},$$

$$m r\dot{\phi}^2 - \frac{\partial U}{\partial r} = \frac{d}{dt}\left(m\dot{r}\right) = m\ddot{r} .$$

This can be rewritten to show that it is equivalent to the radial component of Newton's second law:

$$F_r = -\frac{\partial U}{\partial r} = m\left(\ddot{r} - r\dot{\phi}^2\right) = m a_r .$$

The equation of motion associated with ϕ is:

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}},$$

$$-\frac{\partial U}{\partial \phi} = \frac{d}{dt}\left(mr^2\dot{\phi}\right).$$

The tangential (ϕ) component of the force is:

$$F_\phi = -\frac{1}{r}\frac{\partial U}{\partial \phi}.$$

The torque is $\Gamma = rF_\phi$ and the angular momentum is $L = mr^2\dot{\phi}$, so the second equation of motion is equivalent to:

$$\Gamma = \frac{dL}{dt}.$$

Several Unconstrained Particles:

What we have done can be generalized for multiple particles. The Lagrangian for N particles in Cartesian coordinates is:

$$\mathcal{L}\left(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_N, \dot{\vec{r}}_1, \dot{\vec{r}}_2, \cdots, \dot{\vec{r}}_N\right) = \tfrac{1}{2}m_1\dot{\vec{r}}_1^{\,2} + \tfrac{1}{2}m_2\dot{\vec{r}}_2^{\,2} + \cdots + \tfrac{1}{2}m_N\dot{\vec{r}}_N^{\,2} - U\left(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_N\right),$$

and the $3N$ equations of motion are:

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_1}, \quad \frac{\partial \mathcal{L}}{\partial y_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}_1}, \quad \cdots, \quad \frac{\partial \mathcal{L}}{\partial z_N} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{z}_N}.$$

The Lagrangian can also be written in terms of $3N$ generalized coordinates $q_1, q_2, \cdots, q_{3N}$:

$$\mathcal{L} = \mathcal{L}\left(q_1, q_2, \cdots, q_{3N}, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_{3N}\right),$$

and there will be $3N$ equations of motion in terms of the generalized coordinates:

$$\frac{\partial \mathcal{L}}{\partial q_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1}, \quad \frac{\partial \mathcal{L}}{\partial q_2} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_2}, \quad \cdots, \quad \frac{\partial \mathcal{L}}{\partial q_{3N}} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_{3N}}.$$

Example: 7.8

Time for HW #6 questions

$$\mathcal{L} = \tfrac{1}{2}m\dot{x}_1^{\,2} + \tfrac{1}{2}m\dot{x}_2^{\,2} - \tfrac{1}{2}k\left(x_1 - x_2 - l\right)^2$$

$$x \equiv x_1 - x_2 \quad X \equiv \tfrac{1}{2}\left(x_1 + x_2\right)$$

Re-phrase L

Next two classes:

- Friday – Lagrangian Approach for Constrained Motion
- Monday – Examples of Lagrange's Equaitons