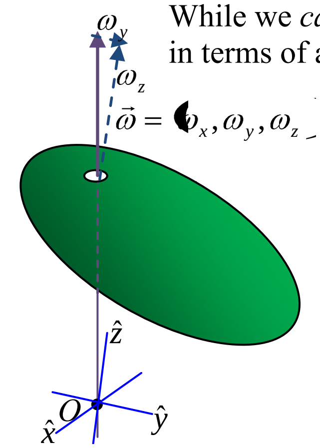
Phys 331: Finding Princpal Axes and Precession of a Top

Wed., 12/5 Thurs. 12/6 Fri., 12/7	**10.5-.6** Finding Principle Axes, Precession **10.7-.8** Euler's Equations	**HW10b** (10.36, 10.39)
Mon., 12/10	**Review for Final**	**Project**

Equipment

- Gyroscope
-

10.5 Finding Principal Axes:



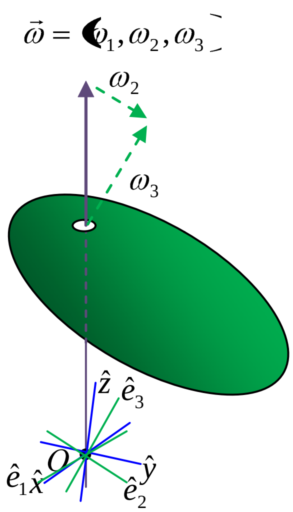
While we *can* generally describe the angular momentum of an object about some given point in terms of any conveniently or not-so-conveniently chosen coordinate axes, and it looks like

$$\vec{L} = \vec{I}\vec{\omega} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

the previous section asserts that, for any rigid object and any given point, O, there exist three perpendicular axes such that, if the object's rotated about any one of them, the angular momentum points in the direction of the angular velocity. These are the object's "principle axes" of rotation about that point.

A "principle axis" is an axis of symmetry; spinning around it means the angular momentum is:

$$\vec{L} = \vec{I}\vec{\omega} = I\vec{\omega} = \lambda\vec{\omega}\,.$$



(book uses λ, probably because I and 1 look a bit too much alike)

Call them the e_1, e_2, e_3 directions, then if we spun the object just around the e_1 axis, the angular momentum should be

About principle axis e_1: $\vec{L} = \vec{I}\vec{\omega} = I\vec{\omega} = I_{11}\omega_1\hat{e}_1$

(the stutter in my I subscript, 11 rather than just 1, will make sense soon)

Similarly, if we spun just around the e_2 axis,

About principle axis e_2: $\vec{L} = \vec{I}\vec{\omega} = I\vec{\omega} = I_{22}\omega_2\hat{e}_2$

Or, if we spun just around the e_3 axis,

About principle axis e_3: $\vec{L} = \vec{I}\vec{\omega} = I\vec{\omega} = I_3\omega_3\hat{e}_3$

More generally, then, if we used *these* three perpendicular axes as a *new* coordinate axes, then in terms of them, the angular velocity is

$$\vec{\omega} = (\omega_1, \omega_2, \omega_3)$$

And the angular momentum is simply

$$\vec{L} = (I_1\omega_1, I_2\omega_2, I_3\omega_3)$$

Now, in this case, matrix math seems a bit like using a hammer to drive in a tack, but we could express this as

$$\vec{L} = \overleftrightarrow{I}\vec{\omega} = \begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = (I_{11}\omega_1, I_{22}\omega_2, I_{33}\omega_3)$$

What I've demonstrated is that, in terms of the principle axes, e_1, e_2, e_3, the inertia tensor is *diagonal*.

We'd like to relate those principle axes back to the arbitrary x,y,z axes.

In language that may be familiar from another physics class many of you are taking, the inertia tensor/matrix is an operator, $\vec{\omega}$ is an *eignvector* , and I_{nn} (what the book denotes λ) is an *eigenvalue* . In this *eigenvalue equation*, the right hand side of the equation can be rewritten using the *unit matrix*:

$$\overleftrightarrow{1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Obviously,

$$\vec{\omega} = \overleftrightarrow{1}\vec{\omega}.$$

So, the eigenvalue equation becomes:

$$\overleftrightarrow{I}\vec{\omega}_n = I_{nn}\overleftrightarrow{1}\vec{\omega}_n,$$

(the subscript denotes that we're considering the special case of rotating about the n^{th} principle axis, and so $\vec{\omega}_n = \omega\hat{e}_n$ and the moment of inertia is essentially the corresponding element of the diagonalized inertia tensor).

$$(\overleftrightarrow{I} - I_{nn}\overleftrightarrow{1})\vec{\omega}_n = 0.$$

This equation has a nonzero solution if and only the *characteristic equation*:

$$\det\left|\overleftrightarrow{I} - I_{nn}\overleftrightarrow{1}\right| = 0,$$

is satisfied.

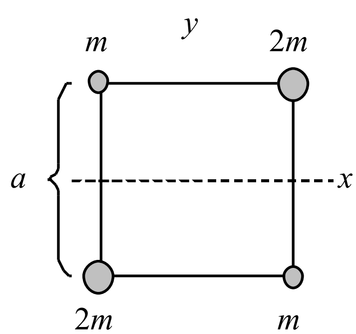
The procedure goes something like this:

- First use this equation to find the *eignevalues*, $\det\left|\overleftrightarrow{I} - I_{nn}\overleftrightarrow{1}\right| = 0$
    - Essentially finding the roots of a 3rd-order polynomial – there will be three of them, I_{11}, I_{22}, and I_{33}.

- then find the eigenvectors using the first equation. $\left(\overleftrightarrow{I} - I_{nn}\overleftrightarrow{1}\right)\vec{\omega}_n = \vec{0}$.
    - Plug in one of the moments I_{nn}'s found above and this will generate three equations which place constraints on how the x,y, and z components $\vec{\omega}_n$ can be related to each other. Since $\vec{\omega}_n$ points in the direction of a principle axis, $\hat{e}_n$, we're equivalently placing constraints on how its projection on the x, y, and z axes are related.
    - Do this for each of the three I_{nn}*'s* in turn. If this wasn't enough to uniquely determine $\hat{e}_1, \hat{e}_2, \hat{e}_3$, in itself, the fact that $\hat{e}_1 \perp \hat{e}_2 \perp \hat{e}_3$ places yet another constraint on them.

The directions of the eigenvectors are the directions of the principal axes. Since we only care about the directions, we will add the condition that the eigenvectors are unit vectors. This is easiest to explain by example.

<u>Example</u>: Find the principal moments and principal axes for the object shown below.



From yesterday, the inertia tensor using the axes shown is:

$$\overleftrightarrow{I} = \begin{bmatrix} 3/2 & -1/2 & 0 \\ -1/2 & 3/2 & 0 \\ 0 & 0 & 3 \end{bmatrix} ma^2 .$$

The characteristic equation is:

$$\det\left(\overleftrightarrow{I} - I_{nn}\overleftrightarrow{1}\right) = \det\begin{bmatrix} \frac{3}{2}ma^2 - I_{nn} & -\frac{1}{2}ma^2 & 0 \\ -\frac{1}{2}ma^2 & \frac{3}{2}ma^2 - I_{nn} & 0 \\ 0 & 0 & 3ma^2 - I_{nn} \end{bmatrix} = 0 ,$$

Since all non-zero terms have a factor of ma^2, it's tempting to make the math look a little cleaner by factoring that out.

$$\det\left(\overleftrightarrow{I} - I_{nn}\overleftrightarrow{1}\right) = ma^2 \det\begin{bmatrix} \frac{3}{2}-\eta_{nn} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{3}{2}-\eta_{nn} & 0 \\ 0 & 0 & 3-\eta_{nn} \end{bmatrix} = 0$$

where $\eta_{nn} \equiv \dfrac{I_{nn}}{ma^2}$

$$\left(\tfrac{3}{2}-\eta_{nn}\right)\left(\tfrac{3}{2}-\eta_{nn}\right)\left(3-\eta_{nn}\right) - \left(-\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right)\left(3-\eta_{nn}\right) = 0$$

$$\left[\left(\tfrac{3}{2}-\eta_{nn}\right)\left(\tfrac{3}{2}-\eta_{nn}\right) - \left(-\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right)\right]\left(3-\eta_{nn}\right) = 0 \qquad ,$$

$$\left[\left(\tfrac{3}{2}-\eta_{nn}\right)^2 - \left(-\tfrac{1}{2}\right)^2\right]\left(3-\eta_{nn}\right) = 0$$

$$\left[2-3\eta_{nn}+\eta_{nn}^{\;2}\right]\left(3-\eta_{nn}\right) = 0\,,$$

$$\left(1-\eta_{nn}\right)\left(2-\eta_{nn}\right)\left(3-\eta_{nn}\right) = 0\,.$$

The solutions are:

$$\eta_{nn} = 1,2,3$$
$$I_{nn} = ma^2, 2ma^2, 3ma^2\,.$$

Find eigen axis 1: If we set $I_{11} = ma^2$, the equation for the eigenvector is:

$$\left(\overleftrightarrow{I} - I_{11}\overleftrightarrow{1}\right)\vec{\omega}_1 = ma^2\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} \omega_{1.x} \\ \omega_{1.y} \\ \omega_{1.z} \end{bmatrix} = 0\,.$$

Or

$$\begin{bmatrix} \frac{1}{2}\omega_{1.x} - \frac{1}{2}\omega_{1.y} \\ -\frac{1}{2}\omega_{1.x} + \frac{1}{2}\omega_{1.y} \\ \frac{1}{2}\omega_{1.z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This gives two independent equations:

$$\tfrac{1}{2}\left(\omega_{1.x} - \omega_{1.y}\right) = 0 \quad \text{and} \quad \tfrac{1}{2}\omega_{1.z} = 0\,,$$

Or

$$\omega_{1.x} = \omega_{1.y} \quad \text{and} \quad \omega_{1.z} = 0\,.$$

So, the angular velocity that corresponds to $I_{11} = ma^2$ has the form $\vec{\omega}_1 = c\,(1,1,0)$, so it points in the direction

$$\hat{e}_1 = \tfrac{1}{\sqrt{2}}(1,1,0).$$

The multiplying factor ensures that $|\hat{e}_1| = \hat{e}_1 \cdot \hat{e}_1 = 1$.

Find eigen axis 2: If we set $I_{22} = 2ma^2$, the equation for the eigenvector is:

$$\left(- I_{22}\overleftrightarrow{1}\right)\vec{\omega} = ma^2 \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & ma^2 \end{bmatrix} \begin{bmatrix} \omega_{2.x} \\ \omega_{2.y} \\ \omega_{2.z} \end{bmatrix} = 0$$

This gives two independent equations:

$$\omega_{1.x} + \omega_{1.y} = 0 \quad \text{and} \quad \omega_{1.z} = 0 .$$

Rotating the object about the other diagonal axis, $\hat{e}_2 = \frac{1}{\sqrt{2}}(1,-1,0)$, would ensure that.

Find eigen axis 3: If we set $I_{33} = 3ma^2$, the equation for the eigenvector is:

$$\left(- I_{33}\overleftrightarrow{1}\right)\vec{\omega} = \begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \omega_{3.x} \\ \omega_{3.y} \\ \omega_{3.z} \end{bmatrix} = 0$$

This gives two independent equations:

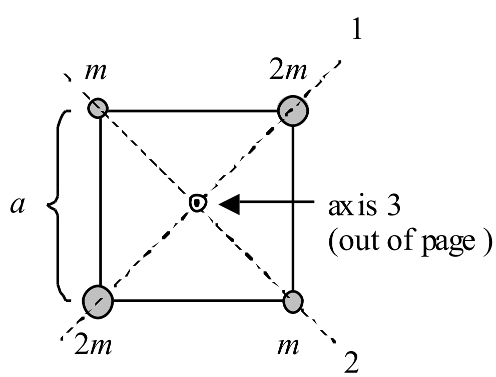
$$3\omega_{3.x} + \omega_{3.y} = 0 ,$$

$$\omega_{3.x} + 3\omega_{3.y} = 0 .$$

And no constraint is placed on $\omega_{3.z}$.

The only way the first two equations can be true is if $\omega_{3.x} = \omega_{3.y} = 0$, so the third angular velocity points purely along the z-axis, that is, eigenvector is $\hat{e}_3 = (0,0,1)$.

This can also be found using that the principal axes are perpendicular, so $\hat{e}_3 = \hat{e}_1 \times \hat{e}_2$. The principal axes are shown below.



We could also arrive at the results above in a different way. Choose the three perpendicular axes (1,2, and 3) because the object has reflection symmetry for the planes defined by each pair of axes. For example, there is reflection symmetry for the plane defined by the 1 & 3 axes. Because of the symmetry with respect to the planes defined by the principal axes, the off-diagonal elements of the inertia tensor are zero. All of the masses are a distance $a/\sqrt{2}$ from the origin. The diagonal elements are the principal moments:

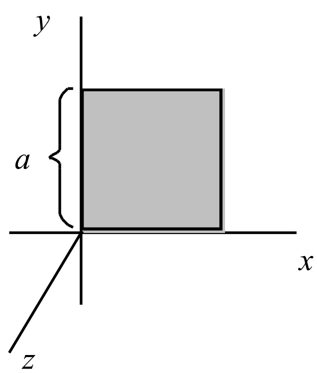
$$\lambda_1 = \sum m_\alpha \left(d_2^2 + d_3^2\right) = m\left(a/\sqrt{2}\right)^2 + m\left(a/\sqrt{2}\right)^2 = ma^2,$$

$$\lambda_2 = \sum m_\alpha \left(d_1^2 + d_3^2\right) = 2m\left(a/\sqrt{2}\right)^2 + 2m\left(a/\sqrt{2}\right)^2 = 2ma^2,$$

$$\lambda_3 = \sum m_\alpha \left(d_1^2 + d_2^2\right) = 2m\left(a/\sqrt{2}\right)^2 + 2m\left(a/\sqrt{2}\right)^2 + m\left(a/\sqrt{2}\right)^2 + m\left(a/\sqrt{2}\right)^2 = 3ma^2.$$

Since $d_3 = 0$ for all of the masses, it is easy to see from the definitions why $\lambda_3 = \lambda_1 + \lambda_2$. This result holds for any object that lies in the 1-2 plane and is sometimes called the *Perpendicular Axis Theorem*.

Exercise: (F&C Ex. 9.6) Suppose uniform square "lamina" (thin plate) of side *a* and mass *m* is spun about a corner.



Using the axes shown above, the inertia tensor is:

$$\overleftrightarrow{I} = ma^2 \begin{bmatrix} \frac{1}{3} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} \end{bmatrix}$$

Find the principal moments and principal axes.

The principal moments are found by solving the characteristic equation:

$$\det\left(\overleftrightarrow{I} - \lambda\overleftrightarrow{1}\right) = \det\begin{bmatrix} \frac{1}{3}ma^2 - \lambda & -\frac{1}{4}ma^2 & 0 \\ -\frac{1}{4}ma^2 & \frac{1}{3}ma^2 - \lambda & 0 \\ 0 & 0 & \frac{2}{3}ma^2 - \lambda \end{bmatrix} = 0$$

$$\det\left(\overleftrightarrow{I} - \lambda\overleftrightarrow{1}\right) = ma^2 \det\begin{bmatrix} \frac{1}{3} - \eta_{nn} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & \frac{1}{3} - \eta_{nn} & 0 \\ 0 & 0 & \frac{2}{3} - \eta_{nn} \end{bmatrix} = 0$$

$$\left(\tfrac{1}{3}-\eta_{nn}\right)^2\left(\tfrac{2}{3}-\eta_{nn}\right)-\left(-\tfrac{1}{4}\right)^2\left(\tfrac{2}{3}-\eta_{nn}\right)=0$$

$$\left(\tfrac{2}{3}-\eta_{nn}\right)\left[\left(\tfrac{1}{3}-\eta_{nn}\right)^2-\left(\tfrac{1}{4}\right)^2\right]=0$$

So,

$$\eta_{33}=\tfrac{2}{3}$$
$$I_{33}=\tfrac{2}{3}ma^2$$

The square brackets give:

$$\tfrac{1}{3}-\eta=\pm\tfrac{1}{4}\Rightarrow\eta=\tfrac{1}{3}\pm\tfrac{1}{4}=\tfrac{1}{12},\tfrac{7}{12}$$

so:

$$I_{22}=\tfrac{1}{12}ma^2 \quad \text{and} \quad I_{33}=\tfrac{7}{12}ma^2$$

Now to find how the angular velocities must be oriented for these to be the moments of inertia:

$$\left(\overleftrightarrow{I}-I_{22}\overleftrightarrow{1}\right)\vec{\omega}=\begin{bmatrix}\tfrac{1}{3}ma^2-\tfrac{1}{12}ma^2 & -\tfrac{1}{4}ma^2 & 0\\ -\tfrac{1}{4}ma^2 & \tfrac{1}{3}ma^2-\tfrac{1}{12}ma^2 & 0\\ 0 & 0 & \tfrac{2}{3}ma^2-\tfrac{1}{12}ma^2\end{bmatrix}\begin{bmatrix}\omega_{1.x}\\ \omega_{1.y}\\ \omega_{1.z}\end{bmatrix}=0,$$

$$\begin{bmatrix}\tfrac{1}{4}ma^2 & -\tfrac{1}{4}ma^2 & 0\\ -\tfrac{1}{4}ma^2 & \tfrac{1}{4}ma^2 & 0\\ 0 & 0 & \tfrac{7}{12}ma^2\end{bmatrix}\begin{bmatrix}\omega_{2.x}\\ \omega_{2.y}\\ \omega_{2.z}\end{bmatrix}=0.$$

This gives two independent equations:

$$\omega_{2.x}-\omega_{2.y}=0 \quad \text{and} \quad \omega_{2.z}=0.$$

We can choose the eigenvector $\hat{e}_2=\frac{1}{\sqrt{2}}(1,1,0)$.

For $I_{33}=\frac{7}{12}ma^2$, the equation for the eigenvector is:

$$\left(\overleftrightarrow{I}-I_{33}\overleftrightarrow{1}\right)\vec{\omega}=\begin{bmatrix}\tfrac{1}{3}ma^2-\tfrac{7}{12}ma^2 & -\tfrac{1}{4}ma^2 & 0\\ -\tfrac{1}{4}ma^2 & \tfrac{1}{3}ma^2-\tfrac{7}{12}ma^2 & 0\\ 0 & 0 & \tfrac{2}{3}ma^2-\tfrac{7}{12}ma^2\end{bmatrix}\begin{bmatrix}\omega_{3.x}\\ \omega_{3.y}\\ \omega_{3.z}\end{bmatrix}=0,$$

$$\begin{bmatrix}-\tfrac{1}{4}ma^2 & -\tfrac{1}{4}ma^2 & 0\\ -\tfrac{1}{4}ma^2 & -\tfrac{1}{4}ma^2 & 0\\ 0 & 0 & \tfrac{1}{12}ma^2\end{bmatrix}\begin{bmatrix}\omega_{3.x}\\ \omega_{3.y}\\ \omega_{3.z}\end{bmatrix}=0.$$

This gives two independent equations:

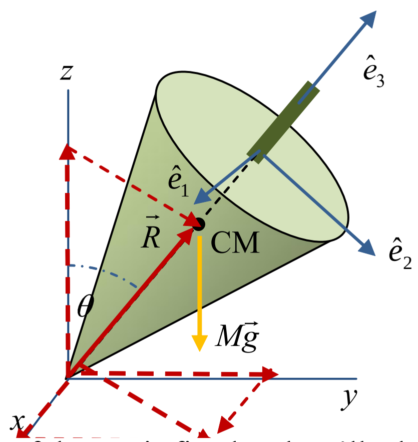
$$\omega_{3.x} + \omega_{3.y} = 0 \quad \text{and} \quad \omega_{3.z} = 0\,.$$

We can choose the eigenvector $\hat{e}_3 = \frac{1}{\sqrt{2}}(1,-1,0)$. Since the principal axes are perpendicular, the third one must be

$\hat{e}_3 = (0,0,1)$.

These are the same as in the previous example, since it has the same symmetry.

10.6 Precession of a Top for a Small Torque:



We will assume the contact point of the top is fixed and we'll take that to be the origin relative to which we'll measure angular momentum and torque.

$$\vec{\Gamma} = \dot{\vec{L}}$$

$$\vec{R} \times M\vec{g} = \dot{\vec{L}}$$

Now,

$$\vec{L} = \overleftrightarrow{I}\vec{\omega}$$

In general,

$$\overleftrightarrow{I} = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{12} & I_{22} & I_{23} \\ I_{23} & I_{32} & I_{33} \end{bmatrix}$$

However, it's pretty easy to correctly guess the principle axes of the top: one along the shaft and two perpendicular as shown. Resolving the Inertia tensor onto that basis set, we have just

$$\overleftrightarrow{I} = \begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix}.$$

Better yet, symmetry tells us that the moments for rotation about axes 1 and 2 are the same.

A top is symmetric about one axis, call it $\hat{e}_3$. We will choose the direction of the symmetry axis to be out of the handle of the top (away from the contact point).

Two of the principal moments are equal, $\lambda_1 = \lambda_2$, so the inertia tensor is:

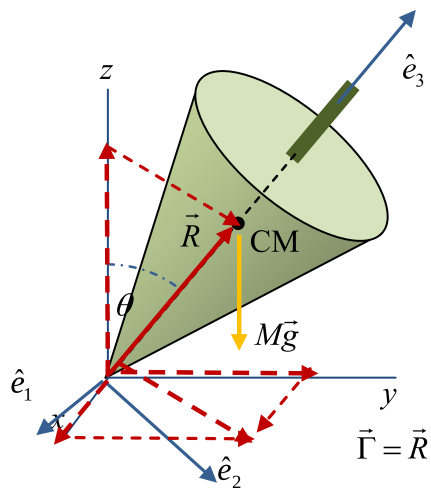
$$\vec{I} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

If the top is spinning about the symmetry axis, then the angular velocity is $\vec{\omega} = \omega \hat{e}_3$ and the angular momentum is

$$\vec{L} = \lambda_3 \vec{\omega} = \lambda_3 \omega \hat{e}_3$$

When symmetry axis is vertical, there is no net torque on the top so $\vec{L}$ is constant. If the top is tipped by an angle θ, there is a torque on it. We can find the torque by assuming the entire weight is applied at the center of mass (see the diagram below).

The torque is:



$$\vec{\Gamma} = \vec{R} \times M\vec{g},$$

so its magnitude is $\Gamma = RMg\sin\theta$.

Take the z axis upward, so the vectors can be written as $\vec{R} = R\,\hat{e}_3$ and $\vec{g} = -g\hat{z}$. The torque on the top leads to a change in the angular momentum:

$$\dot{\vec{L}} = \vec{\Gamma}.$$

The angular velocity must change. If the torque is small, the other components (ω_1 and ω_2) of the angular velocity will stay small (we will show this tomorrow). That means that we can continue to assume that:

$$\vec{L} = \lambda_3 \omega \hat{e}_3.$$

This gives:

$$\frac{d}{dt}\left(\lambda_3 \omega \hat{e}_3\right) = \vec{R} \times M\vec{g},$$

$$\lambda_3 \omega \, \frac{d\hat{e}_3}{dt} = \left(R\,\hat{e}_3\right) \times \left(-Mg\hat{z}\right) = +MgR\,\hat{z} \times \hat{e}_3,$$

$$\frac{d\hat{e}_3}{dt} = \frac{MgR}{\lambda_3 \omega}\,\hat{z} \times \hat{e}_3 = \vec{\Omega} \times \hat{e}_3,$$

where we define:

$$\vec{\Omega} = \frac{MgR}{\lambda_3 \omega}\,\hat{z}.$$

This means that $\hat{e}_3$ rotates about the *z* axis with an angular frequency $\Omega = MgR/\lambda_3\omega$. Note that the rate does <u>not</u> depend on how far the top is tilted (θ).

Demo: Show the motion of a gyroscope. Discuss the direction of the precession, which is in the opposite direction if the angular momentum is the opposite way. It must precess the opposite way to have the same change in angular momentum.