Wed. 10/20 Thurs 10/21 Fri. 10/22	**5.7 -.8** Fourier Series *Summer Research Presentations* **6.1-.2** Calculus of Variations – Euler-Lagrange	**HW5, Project Bibliography**

White boards and pens

Ladder

O'scope & Fourier Synthesizer & speaker

Examples and Exercises:

Before plowing ahead, I want to take a moment to pause and work with what we've met so far. Here are the key relations:

Euler's Relations

$$\frac{e^{ix}+e^{-ix}}{2}=\cos x \quad \frac{e^{ix}-e^{-ix}}{2i}=\sin x$$

Taylor Series

$$f(x)=f(x_o)+\left.\frac{df(x)}{dx}\right|_{x_o}(x-x_o)+\tfrac{1}{2}\left.\frac{d^2f(x)}{dx^2}\right|_{x_o}(x-x_o)^2+\tfrac{1}{3!}\left.\frac{d^3f(x)}{dx^3}\right|_{x_o}(x-x_o)^3+\ldots$$

Hook's law

$$F_x(x)=m\ddot{x}=-k(x-x_{eq}) \quad U(x)=\tfrac{1}{2}k(x-x_{eq})^2 \quad x(t)=A\cos(\omega t-\delta) \quad \omega_o=\sqrt{\frac{k}{m}}=\frac{2\pi}{T}$$

(Linear) Damped Oscillator

$$\ddot{x}+2\beta\dot{x}+\omega_o^2x=0 \qquad x_h(t)=\begin{cases} e^{-\beta t}A\cos(\omega_1 t-\delta) & \text{under damped} \\ e^{-\beta t}\left(C_1e^{\sqrt{\beta^2-\omega_o^2}\cdot t}+C_2e^{-\sqrt{\beta^2-\omega_o^2}\cdot t}\right) & \text{over damped} \\ e^{-\beta t}(A+Bt) & \text{critically damped}\end{cases} \quad \text{where } \omega_1=\sqrt{\beta^2-\omega_o^2}$$

(Linearly) Damped & Driven Oscillator

$$\ddot{x}+2\beta\dot{x}+\omega_o^2x=f(t) \text{ with } f(t)=f_o\sin(\omega_D t) \quad x(t)=A\sin(\omega_D t-\delta)+x_h(t)$$

$$A=\frac{f_o}{\sqrt{(\omega_o^2-\omega_D{}^2)^2+(2\beta\omega_D)^2}} \qquad \delta=\arctan\left(\frac{2\beta\omega_D}{\omega_o^2-\omega_D{}^2}\right)$$

Exercises

I want you to get a little experience using these.

Pr. 5.21

Pr. 5.29

Example: (similar to Ex. 5.3) Suppose $\omega_o = 10\pi$ rad/s, $\beta = \omega_o/20 = \pi/2$ rad/s, $f_0 = 1000$ m/s^2, and $\omega = 4\pi$ rad/s (only difference from Ex. 5.3). If the oscillator starts at rest at the orign, find and plot the function for position as a function of time. Compare with the results for Ex. 5.3.

The frequency for the undriven oscillator (and the homogeneous solution) is:

$$\omega_1 = \sqrt{\omega_o^2 - \beta^2} = \sqrt{(10\pi)^2 - (\pi/2)^2} = 9.987\pi .$$

The amplitude of the particular solution is:

$$A = \frac{f_o}{\sqrt{(\omega_o^2 - \omega^2)^2 + 4\beta^2\omega^2}} = \frac{1000 \text{ m/s}^2}{\pi^2 \text{ rad/s}^2 \sqrt{(10^2 - 4^2)^2 + 4(1/2)^2(4)^2}} = 1.177 \text{ m},$$

and the phase angle is:

$$\delta = \tan^{-1}\left(\frac{2\beta\omega}{\omega_o^2 - \omega^2}\right) = \tan^{-1}\left(\frac{2(\pi/2)(4\pi)}{(10\pi)^2 - (4\pi)^2}\right) = 0.0465 \text{ radians}.$$

The general solution for an underdamped, driven oscillator can be written as:

$$x(t) = A\cos(\omega t - \delta) + e^{-\beta t}\left[B_1\cos(\omega_1 t) + B_2\sin(\omega_1 t)\right],$$

where the coefficients B_1 and B_2 must be determined from the initial conditions $x_o = v_o = 0$. From the equation above:

$$x_o = A\cos(-\delta) + B_1,$$

$$B_1 = x_o - A\cos\delta = 0 - (1.177 \text{ m})\cos(0.0465 \text{ rad}) = -1.176 \text{ m}.$$

Taking the derivative of $x(t)$ gives:

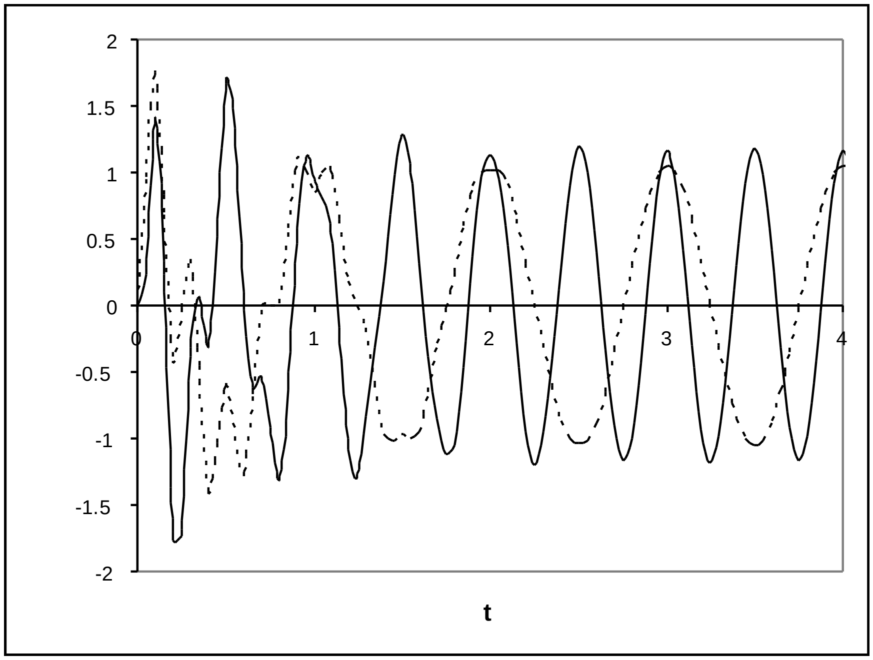
$$v(t) = -\omega A\sin(\omega t - \delta) - \beta e^{-\beta t}\left[B_1\cos(\omega_1 t) + B_2\sin(\omega_1 t)\right] + \omega_1 e^{-\beta t}\left[-B_1\sin(\omega_1 t) + B_2\cos(\omega_1 t)\right],$$

$$v_o = -\omega A\sin(-\delta) - \beta B_1 + \omega B_2,$$

$$B_2 = \frac{1}{\omega_1}(v_o - \omega A\sin\delta + \beta B_1) = \frac{1}{9.987}\left[0 - 4(1.177 \text{ m})\sin(0.0465 \text{ rad}) + (1/2)(-1.176 \text{ m})\right],$$

$$B_2 = -0.807 \text{ m}.$$

The graph of the solution is shown below (solid line) along with the solution of Ex. 5.3 (dashed line) where the driving frequency is $\omega = 2\pi$ rad/s. The steady state solution for this example has a slightly larger amplitude because the driving frequency is closer to the natural frequency. It also lags a little farther behind the driving force.



Fourier Series

This is one of those *very* powerful ideas in physics: any periodic function can be resolved into a (potentially infinite) discrete sum of sines and cosines of frequencies in the harmonic series with a fundamental frequency of the periodic function.

The book doesn't mention it, but there's the extension that any *non* periodic function can be resolved into a (potentially infinite) continuous sum of sines and cosines with a continuous range of periods. For example, a delta function (an isolated spike) can be built of sines and cosines. But we'll just concern ourselves with the periodic functions.

You actually use this fact every day, all the time. Every time you hear a sound, your ear mechanically decomposes it into the "pure tones" that make it up and send signals to the brain of the appropriate strength for each pure tone in the sound you're hearing. It can do this because different locations in your inner ear (which are connected to different nerve fibers) resonate at slightly different frequencies.

Here's what the theorem looks like mathematically.

$$F(\omega t) = \sum_{n=0}^{\infty} a_n \cos(n\omega t) + b_n \sin(n\omega t)$$

That's a fine idea and all, but not very useful if you can't figure out what the coefficients / amplitudes of each term are. Fortunately, that's pretty easy to do if you know the function you're expressing:

$$\int_{-T/2}^{T/2} F(\omega t)\cos(m\omega t)\,dt = \int_{-T/2}^{T/2}\left[\sum_{n=0}^{\infty} a_n\cos(n\omega t)+b_n\sin(n\omega t)\right]\cos(m\omega t)\,dt$$

$$\sum_{n=0}^{\infty}\left[a_n\int_{-T/2}^{T/2}\cos(n\omega t)\cos(m\omega t)\,dt+b_n\int_{-T/2}^{T/2}\sin(n\omega t)\cos(m\omega t)\,dt\right]$$

I do just cosine terms, they'll then do Sine terms.

Now, we could, merely by observation, note that the second integral is 0 since it's the product of an even and an odd function, thus itself an odd function, integrated between symmetric points about 0. Or we can show it's fate.

$$\sum_{n=0}^{\infty}\left[a_n\int_{-T/2}^{T/2}\tfrac{1}{2}\left[\cos(n\omega t+m\omega t)+\cos(n\omega t-m\omega t)\right]dt+b_n\int_{-T/2}^{T/2}\tfrac{1}{2}\left[\sin(n\omega t+m\omega t)+\sin(n\omega t-m\omega t)\right]dt\right]$$

$$\sum_{n=0}^{\infty}\left[a_n\int_{-T/2}^{T/2}\tfrac{1}{2}\left[\cos((n+m)\omega t)+\cos((n-m)\omega t)\right]dt+b_n\int_{-T/2}^{T/2}\tfrac{1}{2}\left[\sin((n+m)\omega t)+\sin((n-m)\omega t)\right]dt\right]$$

$$\sum_{n=0}^{\infty}\left[a_n\tfrac{1}{2}\left(\frac{\sin((n+m)\omega t)}{(n+m)\omega}+\frac{\sin((n-m)\omega t)}{(n-m)\omega}\right)\Bigg|_{-T/2}^{T/2}+b_n\left(-\frac{\cos((n+m)\omega t)}{(n+m)\omega}-\frac{\cos((n-m)\omega t)}{(n-m)\omega}\right)\Bigg|_{-T/2}^{T/2}\right]$$

Clearly something special happens when n = m since we've got these terms with numerators and denominators that go to 0; we'll look at the other case first.

$$\sum_{n=0}^{\infty}\left[a_n\tfrac{1}{2}\left(\frac{\sin\left((n+m)\omega\frac{T}{2}\right)}{(n+m)\omega}+\frac{\sin\left((n-m)\omega\frac{T}{2}\right)}{(n-m)\omega}\right)\Bigg|_{-T/2}^{T/2}+b_n\left(-\frac{\cos\left((n+m)\omega\frac{T}{2}\right)}{(n+m)\omega}-\frac{\cos\left((n-m)\omega\frac{T}{2}\right)}{(n-m)\omega}\right)\Bigg|_{-T/2}^{T/2}\right]$$

Now, sine $\sin\left((n\pm m)\omega\frac{T}{2}\right)=\sin((n\pm m)\pi)=0$ since n and m are integers. So the first term dies.

As for the second term, $\cos\left((n\pm m)\omega\frac{T}{2}\right)=\cos\left((n\pm m)\omega\frac{-T}{2}\right)$ so evaluating it at its two limits kills it off. So all terms in the sum vanish except, possibly, the n =m, so we'll look at that specifically.

$$\int_{-T/2}^{T/2} F(\omega t)\cos(n\omega t)\,dt = \int_{-T/2}^{T/2} \left[a_m \cos(m\omega t) + b_m \sin(m\omega t)\right]\cos(m\omega t)\,dt$$

$$a_m \int_{-T/2}^{T/2} \cos(m\omega t)\cos(m\omega t)\,dt + b_m \int_{-T/2}^{T/2} \sin(m\omega t)\cos(m\omega t)\,dt$$

$$a_m \int_{-T/2}^{T/2} \tfrac{1}{2}\left(\cos(2m\omega t) + 1\right)dt + b_m \int_{-T/2}^{T/2} \tfrac{1}{2}\left(\sin(2m\omega t)\right)dt$$

$$a_m \tfrac{1}{2}\left(\frac{\sin(2m\omega t)}{2m\omega} + t\right)\Bigg|_{-T/2}^{T/2} + b_m\left(-\frac{\cos(2m\omega t)}{4m\omega}\right)\Bigg|_{-T/2}^{T/2}$$

$$a_m \tfrac{1}{2}\left(\frac{0}{2m\omega} + T/2 - \frac{0}{2m\omega} - T/2\right) + b_m\left(-\frac{\cos(2m\omega T/2) - \cos(2m\omega(-T/2))}{4m\omega}\right)$$

$$\int_{-T/2}^{T/2} F(\omega t)\cos(m\omega t)\,dt = a_m \tfrac{T}{2}$$

$$\frac{2}{T}\int_{-T/2}^{T/2} F(\omega t)\cos(m\omega t)\,dt = a_m$$

Similarly,

$$\frac{2}{T}\int_{-T/2}^{T/2} F(\omega t)\sin(m\omega t)\,dt = b_m$$

Now let’s put this to use. The book did the simplest periodic function – the square wave. Let’s try the triangle wave.

Fourier Synthesis machine – build some

Sawtooth

$$F(\omega t) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n}\sin(n\omega t)$$

Square

$$F(\omega t) = \frac{4}{\pi}\sum_{\substack{n=1\\ odd}}^{\infty} \frac{1}{n}\sin(n\omega t)$$

Triangle

$$F(\omega t) = \frac{8}{\pi^2}\sum_{\substack{n=1\\ odd}}^{\infty} \frac{1}{n^2}\sin(n\omega t)$$

Fourier Solutions to Damped-Driven Oscillator

The relevance to this chapter, on harmonic oscillators, is that, if the driving force for a harmonic oscillator can be expressed as a sum of cosines, then the solutions can be expressed as the sum of solutions for each individual cosine driving term.

More formally, if we have

$$\ddot{x}+\beta\dot{x}+\omega_o^2 x = f_{ocn}\cos(\omega_{Dn}t) \qquad \text{where } \omega_{Dn} = n\omega_{D1}$$

Just to be a little specific, let's say that we're in the under-damped situation, $\beta<\omega_o$. Then we know that the solution is, in its most general form

$$x(t) = x_h(t) + x_{p.n}(t) \text{ where } x_h(t) = e^{-\beta t}A\cos(\omega_1 t-\delta) \text{ for which } \omega_1 = \sqrt{\beta^2-\omega_o^2}$$

$$\text{And } x(t) = A_{cn}\cos(\omega_{Dn}t-\delta_n) \text{ for which } A_n = \frac{f_{\text{oc,n}}}{\sqrt{(\omega_o^2-\omega_{Dn}{}^2)^2+(2\beta\omega_{Dn})^2}}$$

$$A_n = \frac{f_{\text{oc,n}}}{\sqrt{(\omega_o^2-n^2\omega_D{}^2)^2+(2\beta n\omega_D)^2}}$$

$$\delta_n = \arctan\left(\frac{2\beta\omega_{Dn}}{\omega_o^2-\omega_{Dn}{}^2}\right) = \arctan\left(\frac{2\beta n\omega_D}{\omega_o^2-n^2\omega_D{}^2}\right)$$

Similarly, if the driving force had a different frequency, say $\omega_{D,n+1} = (n+1)\omega_{D1}$

We'd have the same set of solutions, but all n's would be replaced by n+1.

$$\ddot{x}+\beta\dot{x}+\omega_o^2 x = f_{on+1}\cos(\omega_{D,n+1}t)$$
$$x(t) = x_h(t) + x_{p.n+1}(t)$$

Now, if we had a complicated driving force, of $f_{ocn}\cos(\omega_{Dn}t) + f_{oc,n+1}\cos(\omega_{Dn+1}t)$

Then a solution would be

$$\ddot{x}+\beta\dot{x}+\omega_o^2 x = f_{ocn}\cos(\omega_{D,n}t) + f_{ocn}\cos(\omega_{D,n+1}t)$$
$$x(t) = x_h(t) + x_{p.n}(t) + x_{p.n+1}(t)$$

Since

$$\ddot{x}_h+\beta\dot{x}_h+\omega_o^2 x_h = 0$$
$$\ddot{x}_{p,n}+\beta\dot{x}_{p,n}+\omega_o^2 x_{p,n} = f_{ocn}\cos(\omega_{D,n}t)$$
$$\ddot{x}_{p,n+1}+\beta\dot{x}_{p,n+1}+\omega_o^2 x_{p,n+1} = f_{ocn+1}\cos(\omega_{D,n+1}t)$$

Adding the left sides and the right sides gives

$$(\ddot{x}_h + \ddot{x}_{p,n} + \ddot{x}_{p,n+1}) + \beta(\dot{x}_h + \dot{x}_{p,n} + \dot{x}_{p,n+1}) + \omega_o^2 (x_h + x_{p,n} + x_{p,n+1}) = 0 + f_{ocn}\cos(\omega_{D,n}t) + f_{ocn+1}\cos(\omega_{D,n+1}t)$$

Or,

$$(\ddot{x}) + \beta(\dot{x}) + \omega_o^2 (x) = f_{ocn}\cos(\omega_{D,n}t) + f_{ocn+1}\cos(\omega_{D,n+1}t)$$

Well, if that's the case, then there's nothing to stop us from extending this logic to an infinite series. Say the driving force is some periodic function, F(ωt) which can, of course, be expressed as

$$F(\omega t) = \sum_{n=0}^{\infty} a_n \cos(n\omega t) + b_n \sin(n\omega t)$$

Then the solution, the position of the driven object, as a function of time is

$$\ddot{x} + \beta\dot{x} + \omega_o^2 x = f_{ocn}\cos(\omega_{D,n}t) + f_{osn}\cos(\omega_{D,n+1}t)$$

$$x(t) = x_h(t) + \sum_{n=0}^{\infty} (x_{p.c.n}(t) + x_{p.s.n}(t))$$

$$x(t) = x_h(t) + \sum_{n=0}^{\infty} (A_{cn}\cos(n\omega_D t - \delta_n) + A_{sn}\sin(n\omega_D t - \delta_n))$$

RMS Displacement: Parseval's Theorem

$$F_{rms} \equiv \sqrt{\langle F^2(\omega t)\rangle} = \sqrt{\frac{\int_{-T/2}^{T/2} F^2(\omega t)dt}{T}}$$

Rewrite the function in terms of its Fourier Series:

$$F(\omega t) = \sum_{n=0}^{\infty} a_n \cos(n\omega t) + b_n \sin(n\omega t)$$

Square it

$$F(\omega t) * F(\omega t) = \left(\sum_{n=0}^{\infty} a_n \cos(n\omega t) + b_n \sin(n\omega t)\right)\left(\sum_{m=0}^{\infty} a_m \cos(m\omega t) + b_m \sin(m\omega t)\right)$$

$$F(\omega t) * F(\omega t) = \left(\sum_{\substack{n=0 \\ m=0}}^{\infty} a_n a_m \cos(n\omega t)\cos(m\omega t) + b_n b_m \sin(n\omega t)\sin(m\omega t) + 2a_n b_m \cos(n\omega t)\sin(m\omega t)\right)$$

Average

$$\int_{-T/2}^{T/2} \frac{F(\omega t) * F(\omega t) dt}{T}$$

$$= \left(\sum_{\substack{n=0 \\ m=0}}^{\infty} a_n a_m \int_{-T/2}^{T/2} \frac{\cos(n\omega t)\cos(m\omega t)dt}{T} + b_n b_m \int_{-T/2}^{T/2} \frac{\sin(n\omega t)\sin(m\omega t)dt}{T} + 2a_n b_m \int_{-T/2}^{T/2} \frac{\cos(n\omega t)\sin(m\omega t)dt}{T} \right)$$

As we say when doing similar integrals, the only ones that survive are same trig function with same integer

$$\int_{-T/2}^{T/2} \frac{F(\omega t) * F(\omega t) dt}{T}$$

$$= \left(a_o^2 + \sum_{n=1}^{\infty} a_n{}^2 \frac{1}{2} + b_n{}^2 \frac{1}{2} \right)$$

The leading term comes from when n=m=0 and the cosine integrand is 1.

So,

$$F_{rms} \equiv \sqrt{\langle F^2(\omega t) \rangle} = \sqrt{\frac{\int_{-T/2}^{T/2} F^2(\omega t) dt}{T}} = \sqrt{a_o^2 + \frac{1}{2} \sum_{n=1}^{\infty} a_n{}^2 + b_n{}^2}$$