Thurs 10/11 Fri., 10/12	 **5.4-.5** Damped & Driven Oscillations	**HW5a** (5.2, 5.10)
Mon. 10/15 Tues. 10/16 Wed. 10/17 Thurs 10/18 Fri. 10/19	**5.5-.6** Resonance **5.7 -.8** Fourier Series **6.1-.2** Calculus of Variations – Euler-Lagrange	 **HW5b** (5.26-.43) **HW5c** (5.46-.52), **Project Bibliography**

Mass on spring with force probe

Why need two independent solutions?

What is δ?

Graph in 186

<u>Damped Oscillations</u>:

So far, we have just considered a restoring force. If we include linear resistive force (the simplest case), the net force on a particle moving in 1-D is:

$$F_x(x) = -kx - b\dot{x}\,.$$

Newton's second law gives:

$$m\ddot{x} = -kx - b\dot{x}\,,$$

or defining the *natural frequency* $\omega_o = \sqrt{k/m}$ (frequency without damping) and the *damping constant* $\beta = b/2m$ (the "two" makes later result neater) we get:

$$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = 0\,.$$

Note: As the book points out (and anyone who's taken our Electronics course would recognize) The form of the equation for the charge on a capacitor $q(t)$ in an RLC circuit is the same, replace x with q (for charge), replace m with L (for inductance), replace β with R (for resistance), and replace k with 1/C (for capacitance).

Second-order differential equation.

Think about when you solve the very simple second order differential equation $d^2y/dt^2 = 0$. You integrate once and get $v = v_0$, then you integrate again and get $y = y_o + v_o t$. Notice that, along the way, you introduced *two* constants that can only be set by knowing some boundary conditions; in this case, the initial position and initial velocity of the object. That's generally true of a 2^{nd}-order differential equation – your general solution must have two free parameters with which to fit specific boundary conditions.

We will once again guess that the solutions are of the form $x(t) = e^{rt}$, so $\dot{x} = re^{rt}$ and $\ddot{x} = r^2 e^{rt}$. Substituting this into our differential equation (and canceling out e^{rt} from each term) yields the *auxiliary equation*:

$$r^2 + 2\beta r + \omega_o^2 = 0,$$

Applying the quadratic equation solves this with:

$$r = \frac{-2\beta \pm \sqrt{(2\beta)^2 - 4\omega_o^2}}{2}.$$

Cancel out a "2" (the reason one is in the definition of β) and define:

$$r_1 = -\beta + \sqrt{\beta^2 - \omega_o^2},$$

$$r_2 = -\beta - \sqrt{\beta^2 - \omega_o^2}.$$

As long as $r_1 \neq r_2$, $e^{r_1 t}$ and $e^{r_2 t}$ are independent solutions (they are not constant multiples of each other). So, most generally, a solution can be linear combination of them. In the case that $r_1 \neq r_2$, this gives:

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} = e^{-\beta t}\left(C_1 e^{\sqrt{\beta^2 - \omega_o^2} \cdot t} + C_2 e^{-\sqrt{\beta^2 - \omega_o^2} \cdot t} \right).$$

Notice that this gives us our two free parameters, C_1 and C_2. This should be sufficient for fitting any specific scenario; any specific initial position and initial velocity for example.

The damping constant β is real and positive, so one effect of the resistive force is to make the size of $x(t)$ decrease over time.

Qualitatively different Cases.

There are 4 qualitatively different possibilities for β. $\beta = 0$, $\beta = \omega_o$, $\beta < \omega_o$, and $\beta > \omega_o$. These will give qualitatively different mathematical results / physical behaviors.

Let's look at the four possible cases:

(1) $\beta = 0$ **Undamped Oscillations**: If $\beta = 0$, there is no damping and:

$$x(t) = C_1 e^{i\omega_o t} + C_2 e^{-i\omega_o t},$$

Which can be rewritten as

$$x(t) = A\,Cos(\omega_o t - \delta)$$

as in the first section of this chapter. This describes an oscillation with an angular frequency ω_o and a constant amplitude. We already discussed the many ways of writing this solution.

(2) $\beta < \omega_o$ **Weak Damping / Underdamped Oscillations**: The obvious thing to compare the damping constant β with is the natural frequency ω_o since they appear together under the square

root. We will say that the damping constant is small if $\beta < \omega_o$. In that case, the quantity under the square root is negative, so:

$$\sqrt{\beta^2 - \omega_o^2} = i\sqrt{\omega_o^2 - \beta^2} = i\omega_1,$$

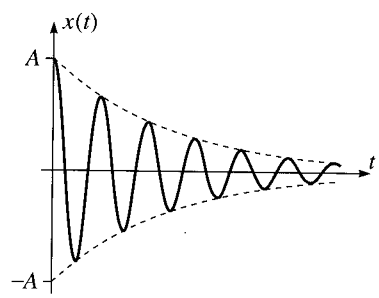
where $\omega_1 \equiv \sqrt{\omega_o^2 - \beta^2}$. This frequency ω_1 is less than the natural frequency ω_o, but if $\beta << \omega_o$ then $\omega_1 \approx \omega_o$. The general solution is:

$$x(t) = e^{-\beta t}\left(C_1 e^{i\omega_1 t} + C_2 e^{-i\omega_1 t}\right),$$

which can also be written as (the part in brackets is the same form as the undamped solution):

$$x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta).$$

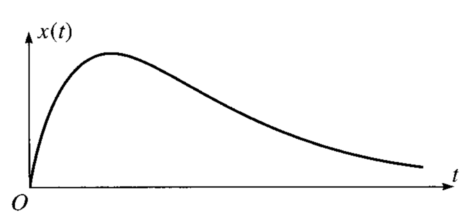
So this looks like sinusoidal oscillation with an exponentially-decaying amplitude. Sometimes β is called the *decay parameter* for this type of motion because it determines how quickly the amplitude decreases.



(3) $\beta > \omega_o$ **Strong Damping / Overdamped Oscillations**: We will say that the damping constant is large if $\beta > \omega_o$. In that case, the quantity under the square root is positive and the exponents are real, so the general solution is:

$$x(t) = C_1 e^{-\left(\beta - \sqrt{\beta^2 - \omega_o^2}\right)t} + C_2 e^{-\left(\beta + \sqrt{\beta^2 - \omega_o^2}\right)t}.$$

This solution does <u>not</u> oscillate because both terms are decaying exponentials. The factor in brackets is smaller for the first term, so it decays more slowly and determines the motion at large times. Therefore, the decay constant for overdamped motion is $\beta - \sqrt{\beta^2 - \omega_o^2}$. The greater the damping constant, the longer it takes for the system to return to equilibrium once it's been displaced (think of a mass on a spring hanging in oil.)



(4) $\beta = \omega_o$ **Critical Damping**: The last case to consider is the threshold between these two previous cases, when $\beta = \omega_o$. In this case $r_1 = r_2$ which means that our two previously-independent solutions aren't so independent anymore and we'll need to go hunting for a new

solution (there must be two because this is a second order linear differential equation), and we don't have our two constants with which to fit all possible initial conditions.

We know that there is one solution of the form $x_1(t) = e^{-\beta t}$.

The book proposes that $x_2(t) = te^{-\beta t}$ is also a solution. Of course, we can plug this back into the equation,

$$\ddot{x} + 2\beta\dot{x} + \beta^2 x = 0$$

(rephrased using that $\beta = \omega_o$ in this special case.)

Let's find those two derivatives and then plug their expressions in (use the product rule):

$$\dot{x}_2(t) = e^{-\beta t} - \beta te^{-\beta t},$$

$$\ddot{x}_2(t) = -\beta e^{-\beta t} - \beta\left(e^{-\beta t} - \beta te^{-\beta t}\right) = \left(\beta^2 t - 2\beta\right)e^{-\beta t}.$$

Okay, plugging these into the equation:

$$\ddot{x} + 2\beta\dot{x} + \beta^2 x = 0,$$

$$\left(\beta^2 t - 2\beta\right)e^{-\beta t} + 2\beta\left(e^{-\beta t} - \beta te^{-\beta t}\right) + \beta^2 te^{-\beta t} \overset{?}{=} 0,$$

which works.

So, the general solution in this case is:

$$x(t) = C_1 e^{-\beta t} + C_2 te^{-\beta t} = \left(C_1 + C_2 t\right)e^{-\beta t}.$$

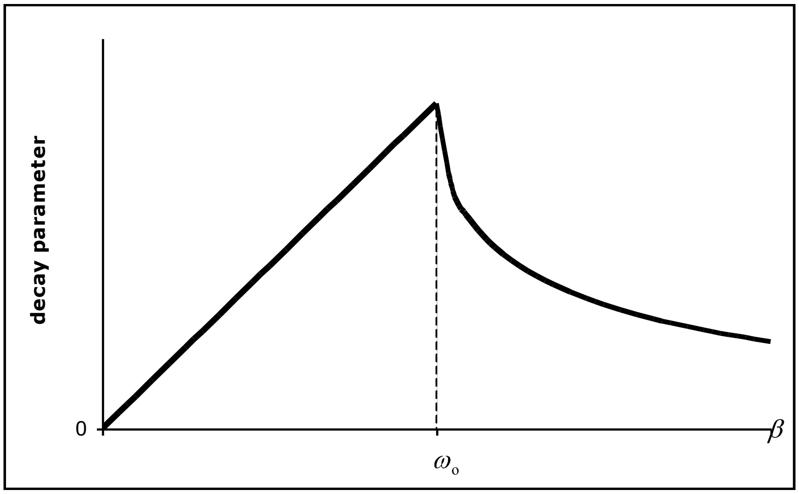
Both terms decay with the same decay parameter $\beta = \omega_o$. Graphically, these solutions are similar to the overdamped solutions. There are no oscillations. This is actually the case that decays the quickest – it's decay rate depends on just, β, as for the underdamped case, however, we've cranked up the value of β to be as large as it can be without flipping over to the overdamped case which then decays according to $\beta - \sqrt{\beta^2 - \omega_o^2}$.

Summary:

There are two coefficients in each solution to be determined by the initial conditions (usually $x(0)$ and $\dot{x}(0)$) of a particular problem.

For cases (1), (2), and (4) where $0 \le \beta \le \omega_o$, the decay parameter is β, which is zero for case (1) and ω_o for case (4).

For case (3) where $\beta > \omega_o$, the decay parameter is $\beta - \sqrt{\beta^2 - \omega_o^2}$ which <u>decreases</u> as β increases!



Too much damping (overdamping) does <u>not</u> cause the motion to decay more quickly! If you want an object subjected to a restoring force to come to rest as quickly as possible, you should make sure $\beta \approx \omega_o$. Applications include the needle on a meter and the motion of a car body relative to its wheels (springs and shocks in between).

<u>Example #1</u>: For an underdamped case, show that the ratio of the amplitudes at two successive maxima in the displacement is constant. (The maxima do <u>not</u> occur where $x(t)$ contacts the curve $Ae^{-\beta t}$.)

Find the times when:

$$x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta),$$

is maximum by setting the derivative equal to zero:

$$\dot{x}(t_m) = -\beta A e^{-\beta t_m}\cos(\omega_1 t_m - \delta) - \omega_1 A e^{-\beta t_m}\sin(\omega_1 t_m - \delta) = 0,$$

$$\omega_1 \sin(\omega_1 t_m - \delta) = -\beta\cos(\omega_1 t_m - \delta),$$

$$\tan(\omega_1 t_m - \delta) = -\beta/\omega_1,$$

$$t_m = \frac{1}{\omega_1}\left[\tan^{-1}(-\beta/\omega_1) + \delta\right].$$

Actually, we've found the times for maxima and minima, so every other solution is for a maximum. The tangent repeats every π radians, $\tan(\theta) = \tan(\theta + \ell\pi)$ where $\ell = 1,2,\cdots$. If t_1 is the time for the first maximum, the subsequent maxima are at:

$$t_n = t_1 + (n-1)(2\pi/\omega_1),$$

or:

$$t_{n+1} = t_n + (2\pi/\omega_1).$$

The ratio of the amplitudes at two successive maxima is:

$$\frac{x(t_{n+1})}{x(t_n)} = \frac{Ae^{-\beta[t_n + (2\pi/\omega_1)]}\cos(\omega_1(t_n + (2\pi/\omega_1)) - \delta)}{Ae^{-\beta t_n}\cos(\omega_1 t_n - \delta)} = e^{-2\pi\beta/\omega_1},$$

because $\cos(\theta)=\cos(\theta+2\pi)$.

Example #2: Suppose the angular frequency of a underdamped oscillator is 628 Hz (frequency of 100 Hz) and the ratio of the amplitudes at two successive maxima is one half. What is the natural frequency ω_o?

Using the result of Example 1 to find β:

$$\frac{x(t_{n+1})}{x(t_n)}=\frac{1}{2}=e^{-2\pi\beta/\omega_1},$$

$$e^{+2\pi\beta/\omega_1}=2,$$

$$2\pi\beta/\omega_1=\ln 2,$$

$$\beta=(\omega_1/2\pi)\ln 2=(628\text{ Hz}/2\pi)\ln 2=69.3\text{ Hz}.$$

The damped frequency is related to the natural frequency by:

$$\omega_1\equiv\sqrt{\omega_o^2-\beta^2},$$

so:

$$\omega_o=\sqrt{\omega_1^2+\beta^2}=\sqrt{(628\text{ Hz})^2+(69.3\text{ Hz})^2}=632\text{ Hz}.$$

Linear Differential Operators:

The book sets up the discussion of the driven oscillator by first pointing out some general properties of linear differential operators. I'm not sure that that discussion isn't more work than it's worth.

The left side of the differential equation we solved today:

$$\ddot{x}+2\beta\dot{x}+\omega_o^2x=0,$$

can be thought of as the result of an operator acting on the function $x(t)$. Define the *differential operator D* by:

$$D=\frac{d^2}{dt^2}+2\beta\frac{d}{dt}+\omega_o^2.$$

When this acts on x, it gives:

$$Dx=\ddot{x}+2\beta\dot{x}+\omega_o^2x,$$

so the differential equation above can be written as $Dx=0$. The operator is *linear* if:

$$D(ax_1+bx_2)=aDx_1+bDx_2.$$

We use the fact that the operator we've been dealing with is linear when we form general solutions.

Driven Damped Oscillations:

Before we embark on this, let's pause and think about complex numbers.

If you write out the Taylor series for $e^{i\theta}$, all even terms will be real and all odd terms will be imaginary, gathering all those even/real terms together, you'll recognize the Taylor expansion for $\cos(\theta)$; similarly, if you gather all the imaginary/odd terms together, you'll recognize the Taylor expansion for $\sin(\theta)$. So, strange but true,

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

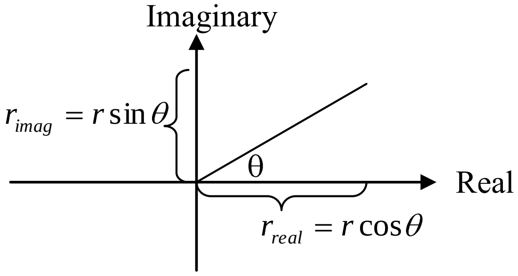
Rather trivially, if we multiply this by some constant, cal it *r,* then we have

$$re^{i\theta} = (r\cos(\theta)) + i(r\sin(\theta)) = r_{real} + ir_{imaginary}$$

I did this because the expression on the right looks kind of familiar; it looks a lot like the two components of a 2-D vector:

$$\vec{r} = (r\cos(\theta))\hat{x} + (r\sin(\theta))\hat{y}$$

We can push this analogy further by representing it in the complex plain: that's



That might help you to remember how to translate between the two representations

$$\vec{r} = |r|e^{i\theta)} = r_r + ir_I$$

where $|r| = \sqrt{(r_r)^2 + (r_I)^2} = \sqrt{r^* r}$

and

$\theta = \tan^{-1}(r_I / r_r)$

Damed-Driven Oscillator

Okay, this will come in handy shortly. Now on to the task at hand. Suppose a damped oscillator is "driven" by an additional time-dependent force $F_d(t)$, so the net force on the oscillator is $-kx - bv + F(t)$. Newton's second law gives:

$$-kx - bv + F_d(t) = ma\ ,$$

or:

$$m\ddot{x} + b\dot{x} + kx = F_d(t).$$

If we define the force per unit mass $f_d(t) = F_d(t)/m$ and recall the definition of the damping constant $\beta = b/2m$, then we can write:

$$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = f_d(t).$$

Fourier's theorem can be rephrased as saying that any complicated function can be phrased as a linear combination of complex exponential terms. Got to love exponentials – it's *so* easy to take their derivatives!

Of course, the driving force can have any functional dependence on time; however, we'll focus on a specific driving force. Fourier's theorem tells us that any complicated function can be built of a combination of sines and cosines. So a natural place to start is solving this problem for a cosine driving force:

$$\ddot{x}_c + 2\beta\dot{x}_c + \omega_o^2 x_c = f_o \cos(\omega_d t)$$

Or maybe for a sine driving force

$$\ddot{x}_s + 2\beta\dot{x}_s + \omega_o^2 x_s = f_o \sin(\omega_d t)$$

Or here's a crazy thought, since this is a *linear* differential equation, add these two equations and simultaneously find a solution for a cosine *and* sine driving forces, or, heck if the cosine and sine have different amplitudes, say by a factor of *i*, we could multiply the second equation by *i* and then add it:

$$(\ddot{x}_c + a\ddot{x}_s) + 2\beta(\dot{x}_c + a\dot{x}_s) + \omega_o^2(x_c + a x_s) = f_o(\cos(\omega_d t) + a\sin(\omega_d t))$$

Okay, so we *could* do that, but why? Well, only if it makes life easier, and there is one choice if *a* that *does* make life easier: i.

The actual problem we'll tackle is

$$\ddot{x}(t) + 2\beta\dot{x}(t) + \omega_o^2 x(t) = f_o e^{i\omega_d t}$$

So, rather perversely, it's actually going to be easier for us to solve this equation for a 2-D vector than for a scalar. The solutions we'll get will also be complex, so they could be written in either of two forms

$$z(t) = r(t)e^{i\theta(t)} = x(t) + iy(t)$$

where $r(t) = \sqrt{(x(t))^2 + (y(t))^2} = |z(t)| = \sqrt{z(t)^* z(t)}$

and

$$\theta(t) = \tan^{-1}(y(t)/x(t))$$

Formally then, say we find a complex solution, that is

$$\ddot{z}(t) + 2\beta\dot{z}(t) + \omega_o^2 z(t) = f_o e^{i\omega_d t}$$

Then that's equivalent to saying

$$\left(\ddot{x}(t)+i\ddot{y}(t)\right)+2\beta\left(\dot{x}(t)+i\dot{y}(t)\right)+\omega_o^2\left(x(t)+y(t)\right)=f_o\cos(\omega_d t)+if_o\sin(\omega_d t)$$

or we can break this one 'vector' equation into two component equations: one relating just the real terms and one relating just the imaginary terms.

$$\ddot{x}(t)+2\beta\dot{x}(t)+\omega_o^2x(t)=f_o\cos(\omega_d t)$$

$$\ddot{y}(t)+2\beta\dot{y}(t)+\omega_o^2y(t)=f_o\sin(\omega_d t)$$

So, we'll actually solve the problem of a complex driving force, but from that we'll be able to extract the solution for a cosine driving force or the solution for the sine driving force, and of course, we can build any complicated driving force (and *its* solutions) from linear combinations of these.

Okay, here we go; our equation has the form

$$\ddot{z}(t)+2\beta\dot{z}(t)+\omega_o^2z(t)=f_oe^{i\omega_d t}$$

It's hard to imagine being able to generate a "characteristic equation" unless our solution had the same time dependence as does the driving force, that is

$$z(t)=Ce^{i\omega_d t},$$

where C is an undetermined complex constant. Substitute this into the differential equation to get:

$$\left(\frac{d^2}{dt^2}Ce^{i\omega_d t}+2\beta\frac{d}{dt}Ce^{i\omega_d t}+\omega_o^2Ce^{i\omega_d t}\right)=\left(-\omega_d{}^2+2i\beta\omega_d+\omega_o^2\right)Ce^{i\omega_d t}=f_oe^{i\omega_d t}.$$

This is a good solution if:

$$C=\frac{f_o}{\omega_o^2-\omega_d{}^2+2i\beta\omega_d}.$$

Looking at the denominator, that has one of two classic forms of a complex number, and we can rewrite it in the other:

$$z(t)=r(t)e^{i\theta(t)}=x(t)+iy(t)$$

where $r(t)=\sqrt{\left(x(t)\right)^2+\left(y(t)\right)^2}=|z(t)|=\sqrt{z(t)^*z(t)}$

and

$\theta(t)=\tan^{-1}\left(y(t)/x(t)\right)$

$$C=\frac{f_o}{\sqrt{\left(\omega_o^2-\omega_d{}^2\right)^2+\left(2\beta\omega_d\right)^2}\,e^{i\delta}}=\frac{f_o}{\sqrt{\left(\omega_o^2-\omega_d{}^2\right)^2+\left(2\beta\omega_d\right)^2}}e^{-i\delta}$$

where

$$\delta = \tan^{-1}\left(\frac{2\beta\omega_d}{\omega_o^2 - \omega_d^{\,2}}\right)$$

So then our solution is

$$z(t) = \frac{f_o}{\sqrt{(\omega_o^2 - \omega_d^{\,2})^2 + (2\beta\omega_d)^2}} e^{i(\omega_d t - \delta)}.$$

The amplitude gets large and the phase shift gets small when $\omega \approx \omega_o$, so the driving frequency is near the natural frequency. We'll talk more about that next time.

$$z(t) = Ce^{i\omega t} = Ae^{i(\omega t - \delta)},$$

the real part, corresponding to when a cosine driving force is applied is:

$$x_p(t) = A\cos(\omega t - \delta).$$

Similarly, if a sine driving force is applied:

$$y_p(t) = A\sin(\omega t - \delta)$$

So, this solves the differential equation.

But that's not the completely general solution yet. Then again, recall that the solutions for the *un* driven equation, if plugged into the left hand side will give 0. Which means that we can add them to this solution and get another solution!

That is

$$\ddot{x}_{un} + 2\beta\dot{x}_{un} + \omega_o^2 x_{un} = 0$$

Where we'd found that $x_{un}(t) = e^{-\beta t}\left(C_1 e^{\sqrt{\beta^2 - \omega_o^2}\cdot t} + C_2 e^{-\sqrt{\beta^2 - \omega_o^2}\cdot t}\right)$

And now we've found that

$$\ddot{x}_{driv} + 2\beta\dot{x}_{driv} + \omega_o^2 x_{driv} = f_o\cos(\omega_d t)$$

Where we've now found that $x_{driv}(t) = A\cos(\omega t - \delta)$

But adding these two equations together, we'd also get that

$$(\ddot{x}_{driv} + \ddot{x}_{un}) + 2\beta(\dot{x}_{driv} + \dot{x}_{un}) + \omega_o^2(x_{driv} + x_{un}) = f_o\cos(\omega_d t) + 0$$

So, apparently, $(x_{driv} + x_{un})$ is *also* a solution to this driven case. *That* is our most general solution.

$$x(t) = x_{driv} + x_{un} = A\cos(\omega t - \delta) + e^{-\beta t}\left(C_1 e^{\sqrt{\beta^2-\omega_o^2}\cdot t} + C_2 e^{-\sqrt{\beta^2-\omega_o^2}\cdot t} \right)$$

The book couches this result in more general differential-equations language; for many of you, I hope that helps to make connections, and understand this better. I've avoided using that language (homogeneous, inhomogeneous, particular,…) in case it was more useful just focusing on the specific case at hand.

The homogeneous solution determined by the initial conditions must be added to this to get the general solution. Note that the homogeneous solution decays, so it can also be called the *transient solution*. Only the early motion of the oscillator depends on how it starts out. The particular solution is an oscillatory solution with the same frequency as the driving frequency, which can also be called the *steady-state solution*. The motion for large times only depend on the parameters of the system (including the driving force), not the initial conditions.

Example: (similar to Ex. 5.3) Suppose $\omega_o = 10\pi$ rad/s, $\beta = \omega_o/20 = \pi/2$ rad/s, $f_0 = 1000$ m/s^2, and $\omega = 4\pi$ rad/s (only difference from Ex. 5.3). If the oscillator starts at rest at the orign, find and plot the function for position as a function of time. Compare with the results for Ex. 5.3.

The frequency for the undriven oscillator (and the homogeneous solution) is:

$$\omega_1 = \sqrt{\omega_o^2 - \beta^2} = \sqrt{(10\pi)^2 - (\pi/2)^2} = 9.987\pi .$$

The amplitude of the particular solution is:

$$A = \frac{f_o}{\sqrt{(\omega_o^2-\omega^2)^2 + 4\beta^2\omega^2}} = \frac{1000 \text{ m/s}^2}{\pi^2 \text{ rad/s}^2 \sqrt{(10^2-4^2)^2 + 4(1/2)^2(4)^2}} = 1.177 \text{ m},$$

and the phase angle is:

$$\delta = \tan^{-1}\left(\frac{2\beta\omega}{\omega_o^2-\omega^2}\right) = \tan^{-1}\left(\frac{2(\pi/2)(4\pi)}{(10\pi)^2-(4\pi)^2}\right) = 0.0465 \text{ radians.}$$

The general solution for an underdamped, driven oscillator can be written as:

$$x(t) = A\cos(\omega t - \delta) + e^{-\beta t}\left[B_1\cos(\omega_1 t) + B_2\sin(\omega_1 t)\right],$$

where the coefficients B_1 and B_2 must be determined from the initial conditions $x_o = v_o = 0$. From the equation above:

$$x_o = A\cos(-\delta) + B_1,$$

$$B_1 = x_o - A\cos\delta = 0 - (1.177 \text{ m})\cos(0.0465 \text{ rad}) = -1.176 \text{ m}.$$

Taking the derivative of $x(t)$ gives:

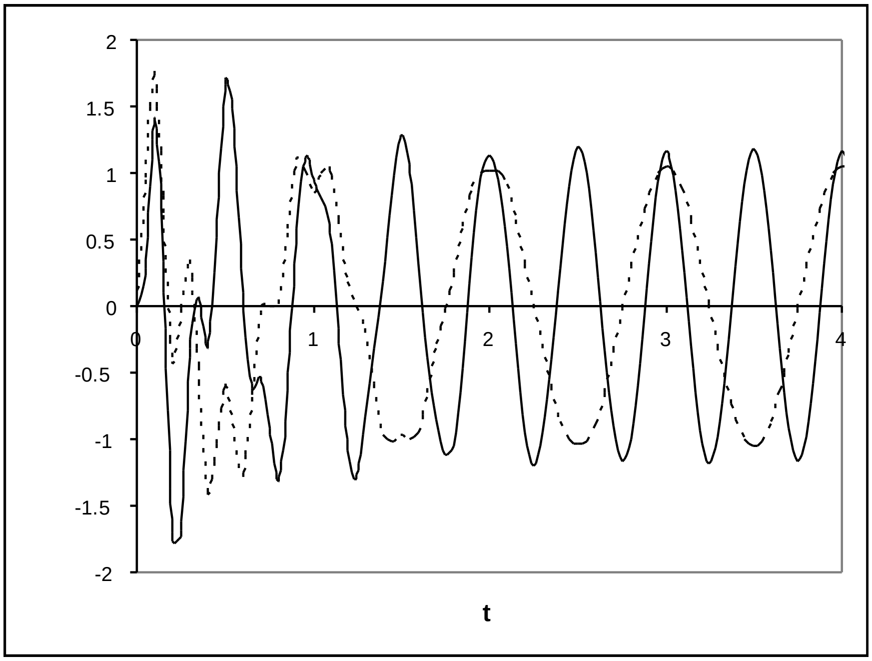
$$v(t) = -\omega A\sin(\omega t - \delta) - \beta e^{-\beta t}\left[B_1\cos(\omega_1 t) + B_2\sin(\omega_1 t)\right] + \omega_1 e^{-\beta t}\left[-B_1\sin(\omega_1 t) + B_2\cos(\omega_1 t)\right],$$

$$v_o = -\omega A\sin(-\delta) - \beta B_1 + \omega B_2,$$

$$B_2 = \frac{1}{\omega_1}\left(v_o - \omega A\sin\delta + \beta B_1\right) = \frac{1}{9.987}\left[0 - 4(1.177 \text{ m})\sin(0.0465 \text{ rad}) + (1/2)(-1.176 \text{ m})\right],$$

$$B_2 = -0.807 \text{ m.}$$

The graph of the solution is shown below (solid line) along with the solution of Ex. 5.3 (dashed line) where the driving frequency is $\omega = 2\pi\,\text{rad/s}$. The steady state solution for this example has a slightly larger amplitude because the driving frequency is closer to the natural frequency. It also lags a little farther behind the driving force.



In the formal language of Differential Equations, what I just argued is that if you have a particular, linear differential equation,

$\hat{D}x_p(t) = f(t)$ (using the Quantum notation of 'hat' for an operator)

Then the general solution is the linear combination of a *particular* solution to this equation, $x_p(t)$ and the solutions to this simpler "homogenious" equation

$$\hat{D}x_h(t) = 0$$

Since

$$\hat{D}(x_p + x_h) = \hat{D}(x_p) + \hat{D}(x_h) = f(t) + 0$$