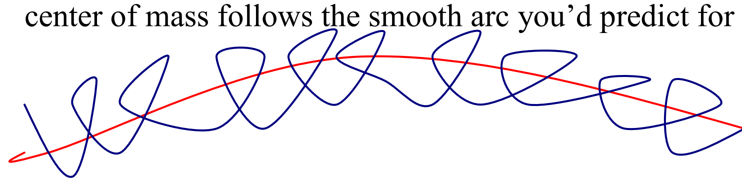
Mon. 9/24	**3.5** Angular Momentum for multiple particles	
Tues. 9/25		**HW3b** (3.D-.G), **Project Topic**
Wed. 9/26	**4.1-.3, 4.9** Work & Energy, Force as a Gradient, 2 Particle Interaction ***Science Poster Session***: *Hedco7~9pm*	
Thurs. 9/27		**HW4a** (4.A, 4.B)
Fri., 9/28	**4.4-.6** Curl of Conservative Force, Varying Potential, 1-D systems	
Mon. 10/1	**4.7-.8** Curvilinear 1-D, Central Force	
Tues. 10/2		**HW4b** (4.C-.F)
Wed. 10/3	**5.1-.3 (2.6)** Hooke's Law, Simple Harmonic (Complex Sol'ns) ***What (research) I Did Last Summer***: *AHoN 116 @ 4pm*	
Fri. 10/5	Review for ***Exam 1***	

- Equipment
  - Binary w & wo cm.py
  - Cross-product.py
  - Batton
  - Cone
  - Orbits noncircular.py
  - orbit with L.py form lab 8.
  - Integrating over a half-sphere handout
-
- **Q:** in terms of a system's dynamics, how its motion responds to external forces, what's special about its center of mass?
  - **A:** This is the point that responds to forces like a particle with all the system's mass.
    - $\vec{R}_{CM} = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + m_4\vec{r}_4 + ...}{m_1 + m_2 + m_3 + m_4 + ...} = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + m_4\vec{r}_4 + ...}{M_{total}}$
    - So
    - $\vec{V}_{CM} = \dfrac{m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + m_4\vec{v}_4 + \ldots}{m_1 + m_2 + m_3 + m_4 + \ldots} \approx \dfrac{\vec{P}_{system}}{M_{system}}$
    - So
    - $\vec{A}_{CM} = \dfrac{m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + m_4\vec{a}_4 + \ldots}{m_1 + m_2 + m_3 + m_4 + \ldots} \approx \dfrac{\dfrac{\Delta\vec{P}_{system}}{\Delta t}}{M_{system}} = \dfrac{\vec{F}_{net.ext}}{M_{system}}$
- Consider a system of many particles, perhaps it is a dust cloud in interstellar space and each speck of dust is one of our "particles." On the one hand, if we focus in, we see that each speck is naturally at a different location, has a different mass, and is moving with a different velocity. Yet, if we zoom out, we see a single cloud that behaves somewhat cohesively. If we watch the cloud for a while, as it moves through space, it makes sense to speak of the whole as having some velocity and going from some position to another – in short, we can think of it as a "particle" of its own. For that matter, if we zoom way in, each of our dust "particles" are themselves made of several, much tinier particles. How do we reasonably do this – treat a composite of several particles, each more-or-less doing their own thing, as a single object and yet properly account for the internal workings? That's what this chapter is about.
- **Demo: Lab_3 binary wo cm.py** You may remember writing code to simulate two stars gravitationally interacting. They went looping through space weaving around each other.

Yet, if we call the two stars together our "system", since there is not net external force on the system, the average velocity of its members must be unchanging – just a constant. What representative point is moving like that? The Center of Mass

- **Demo: Lab_3 binary w cm.py**
  - Note that the center of mass is just a mathematically defined point – it's not a fixed part of either object.
  - **Example: Tossed Baton.** Toss a baton – if you trace the trajectory of any point, *other than* the center of mass, it follows a complicated path through space, but the center of mass follows the smooth arc you'd predict for a tossed point mass.



Red = center of mass trajectory, Blue = baton end trajectory

  - **Example: Balancing a meter stick.** Gravity pulls equally on each morsel of an object. If you push up on the object's center of mass, or in line with it, then there's an equal amount of mass on the left and an equal amount on the right, so the object falls neither way, but balances. For this reason the center of mass is also known as the "center of gravity."

If we consider the mass to be distributed smoothly, instead of discretely, the sum in the calculation of the CM becomes an integral. The textbook gives equation (3.13):

$$\vec{R} = \lim_{\Delta m \to 0} \frac{\sum \vec{r}_i \, \Delta m_i}{M} = \frac{\int \vec{r} dm}{M} = \frac{1}{M} \int \vec{r} \, \rho(\vec{r}) dV \, ,$$

This nicely separates into

$$X = \lim_{\Delta m \to 0} \frac{\sum x_i \, \Delta m_i}{M} = \frac{\int x dm}{M} = \frac{1}{M} \int x \, \rho(\vec{r}) dV$$

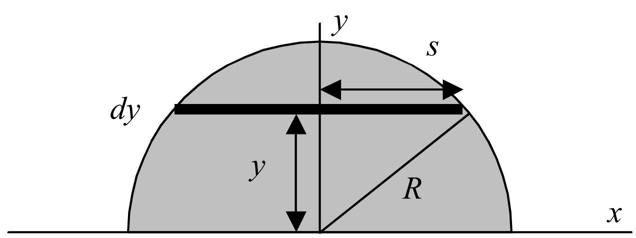
(ditto for the other components)

where ρ is the density and dV is an element of volume. The greatest challenge then is parameterizing the density as a function of position and making sure you set your integral limits appropriately. I suggest starting from the summation, not the integral form of the definition. This makes you to think about how to divide up an object.

Example 1: Find the CM of thin sheet of metal cut into a semicircle of radius R (this is Problem3.21 from the textbook).

First, qualitatively: On this drawing, roughly where should we expect the center of mass to be? (we'll compare our quantitative result with this to make sure we didn't make any math errors.)

By symmetry, the CM must be on the y axis, so $X = 0$. To find Y, we can consider slices of the object that have the same y coordinate (see the diagram).



The area of the slice is $dA = 2s\,dy$, where:

$$s = \sqrt{R^2 - y^2}\,.$$

The mass of the slice is:

$$\sigma = \frac{M}{A} = \frac{dm}{dA} \Rightarrow dm = \left(\frac{M}{A}\right)dA$$

$$\text{mass of slice} = \text{dm} = \left(\frac{M}{\pi R^2/2}\right)\left(2\sqrt{R^2 - y^2}\,dy\right)$$

Put the mass and y coordinate into the definition of Y:

$$Y = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha y_\alpha = \frac{1}{M}\sum M\left(\frac{2\sqrt{R^2 - y^2}\,dy}{\pi R^2/2}\right)y\,.$$

Summing over all of the masses means converting this to an integral:

$$Y = \frac{4}{\pi R^2}\int_0^R \sqrt{R^2 - y^2}\;y\,dy\,.$$

Student Exercise: massage this into the form of something times $\int_1^{w=0} \sqrt{w}\,dw$

$$Y = \frac{4}{\pi R^2}\int_0^R \sqrt{R^2 - y^2}\;y\,dy = \frac{4}{\pi R^2}\int_0^R \sqrt{R^2 - y^2}\;\tfrac{1}{2}\,dy^2 = \frac{4R}{\pi}\int_0^R \sqrt{1 - \left(\frac{y}{R}\right)^2}\;\tfrac{1}{2}\,d\left(\frac{y}{R}\right)^2$$

$$\frac{2R}{\pi}\int_0^{u=1} \sqrt{1-u}\;du = -\frac{2R}{\pi}\int_1^{w=0} \sqrt{w}\;dw$$

Make the change of variables $q = y^2$ and $dq = 2y\,dy$:

$$Y = \frac{2}{\pi R^2}\int_0^{R^2}\sqrt{R^2 - q}\,dq = -\frac{2}{\pi R^2}\left[\left(\frac{2}{3}\right)w^{3/2}\right]_1^0 = \frac{4R}{3\pi} = 0.424R\,.$$

Q: How does this compare with our initial guess?

As you should expect, $Y < R/2$.

Q: What if we set another such half disc, but with half the total mass, right on top of this one; where would the combined system's center of mass be?

The center of mass of the new half-disc would simply be $Y_2 = \frac{4R}{3\pi} + R$

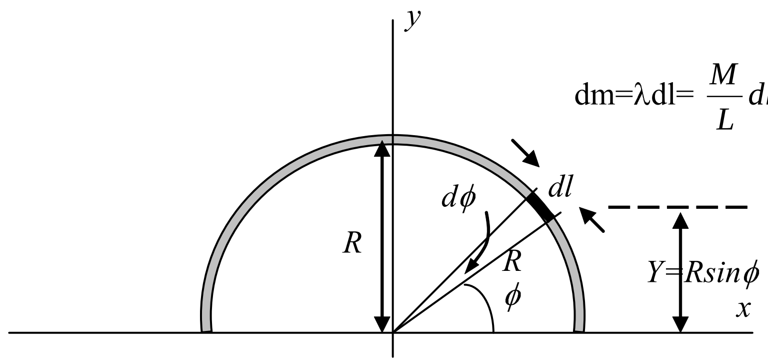
Then the center of mass of the combined system would be

$$Y_{cm} = \frac{Y_1 m + Y_2 (m/2)}{m + m/2} = \frac{\frac{4R}{3\pi} + \left(\frac{4R}{3\pi} + R\right)\frac{1}{2}}{1 + \frac{1}{2}} = \frac{\frac{3}{2}\left(\frac{4R}{3\pi}\right) + \frac{1}{2}R}{\frac{3}{2}} = R\frac{1}{3}\left(\frac{4}{\pi} + 1\right)$$

Example 2: Find the CM of a of thin wire bent into semicircle of radius R.

First, you must choose an origin. Let's use the "center" of the semicircle (see the diagram).

$$dm = \lambda dl = \frac{M}{L}dl = \frac{M}{\pi R}dl = \frac{M}{\pi R}R\,d\phi$$



By symmetry, the CM must be on the y axis, so $X = 0$. It is convenient to use polar coordinates to find Y. Consider a small segment of the wire between angles ϕ and $\phi + d\phi$. The length of the segment is $dl = R\,d\phi$. If M is the total mass of the wire, the mass of the segment is: $(dl/\pi R)M$. The y coordinate of the segments position (its small, so assume its all at one point) is $y = R\sin\phi$. Put the mass and y coordinate into the definition of Y:

$$Y = \frac{1}{\sum_{\alpha=1}^{N} m_\alpha}\sum_{\alpha=1}^{N} m_\alpha y_\alpha = \frac{1}{M}\int y\,dm = \frac{1}{M}\int y\lambda\,dl = \frac{1}{M}\int_0^{\pi} y\left(\frac{M}{\pi R}\right)(R\,d\phi) = \frac{1}{M}\int_0^{\pi}(R\sin\phi)\left(\frac{M}{\pi R}\right)(R\,d\phi).$$

Summing over all of the masses means converting this to an integral:

$$Y = \int_0^{\pi} \frac{R\sin\phi}{\pi}\, d\phi = \frac{R}{\pi}\left(-\cos\phi\right)_0^{\pi} = \frac{2R}{\pi}$$

Q: What if the wire's mass density varied, say $\lambda(y) = \lambda_o\left(1 - \frac{y}{R}\right)$**.** Trace through to the integral (no need to actually *do* the integral, though I've included the result here).

Now dm=λdl= $\lambda_o\left(1-\frac{y}{R}\right)dl = \lambda_o\left(1-\frac{y}{R}\right)dl = \lambda_o\left(1-\frac{y}{R}\right)Rd\phi = \lambda_o R d\phi - \lambda_o y d\phi$,

Both sums probably need to be turned into integrals

$$Y = \frac{1}{\sum_{\alpha=1}^{N} m_\alpha}\sum_{\alpha=1}^{N} m_\alpha y_\alpha = \frac{\int y dm}{\int dm} = \frac{\int y\lambda dl}{\int \lambda dl} = \frac{\int y\lambda_o\left(1-\frac{y}{R}\right)dl}{\int \lambda_o\left(1-\frac{y}{R}\right)dl} = \frac{\int y\lambda_o dl - \int y\lambda_o \frac{y}{R}dl}{\int \lambda_o dl - \int \lambda_o \frac{y}{R}dl} = \frac{\int y dl - \int y\frac{y}{R}dl}{\int dl - \int \frac{y}{R}dl}$$

$$\frac{\int y\left(R\,d\phi\right) - \int y\frac{y}{R}\left(R\,d\phi\right)}{\int\left(R\,d\phi\right) - \int\frac{y}{R}\left(R\,d\phi\right)} = \frac{\int\left(R\sin\phi\right)\left(R\,d\phi\right) - \int\left(R\sin\phi\right)^2\frac{1}{R}\left(R\,d\phi\right)}{\int\left(R\,d\phi\right) - \int\left(R\sin\phi\right)\frac{1}{R}\left(R\,d\phi\right)} = R\frac{\int\left(\sin\phi\right)\left(d\phi\right) - \int_0^{\pi}\left(\sin\phi\right)^2\left(d\phi\right)}{\int\left(d\phi\right) - \int\left(\sin\phi\right)\left(d\phi\right)}$$

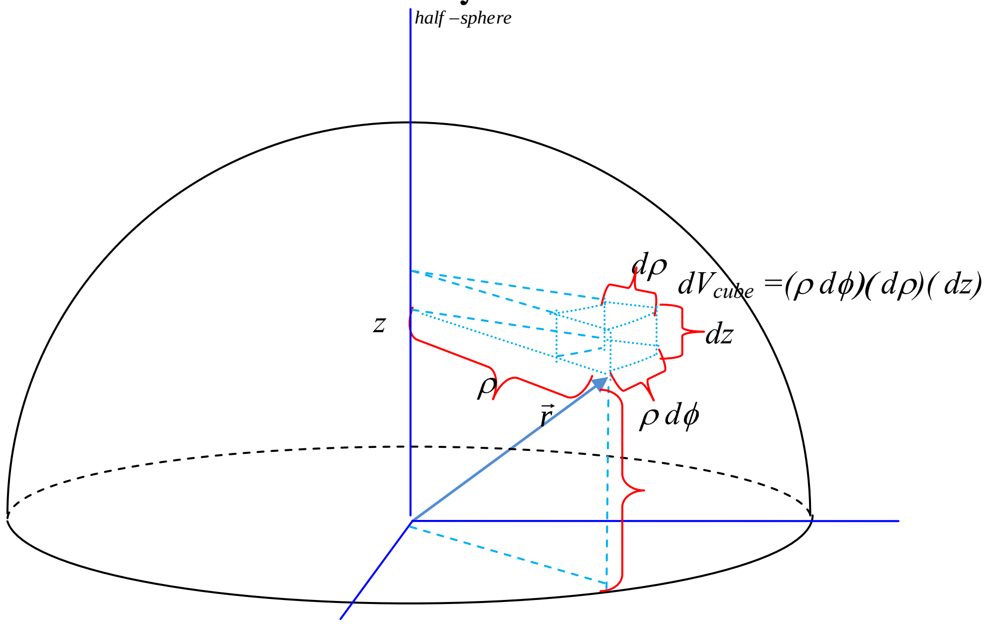
$$R\frac{2 - 2\int_0^{\pi/2}\left(\sin\phi\right)^2\left(d\phi\right)}{\pi - 2} = R\frac{2 - 2\int_0^{\pi/2}\left(\sin\phi\right)\left(d\cos\phi\right)}{\pi - 2} = R\frac{2 - 2\int_0^{\pi/2}\sqrt{1-\left(\cos\phi\right)^2}\left(d\cos\phi\right)}{\pi - 2}$$

$$R\frac{2 - 2\int_1^{0}\sqrt{1-u}\left(du\right)}{\pi - 2} = R\frac{2 - \frac{4}{3}\left(1-u\right)^{3/2}\Big|_1^0}{\pi - 2} = R\frac{2-\frac{4}{3}}{\pi - 2} = R\frac{2}{3\pi - 6}$$

Integrating-over-half-circle

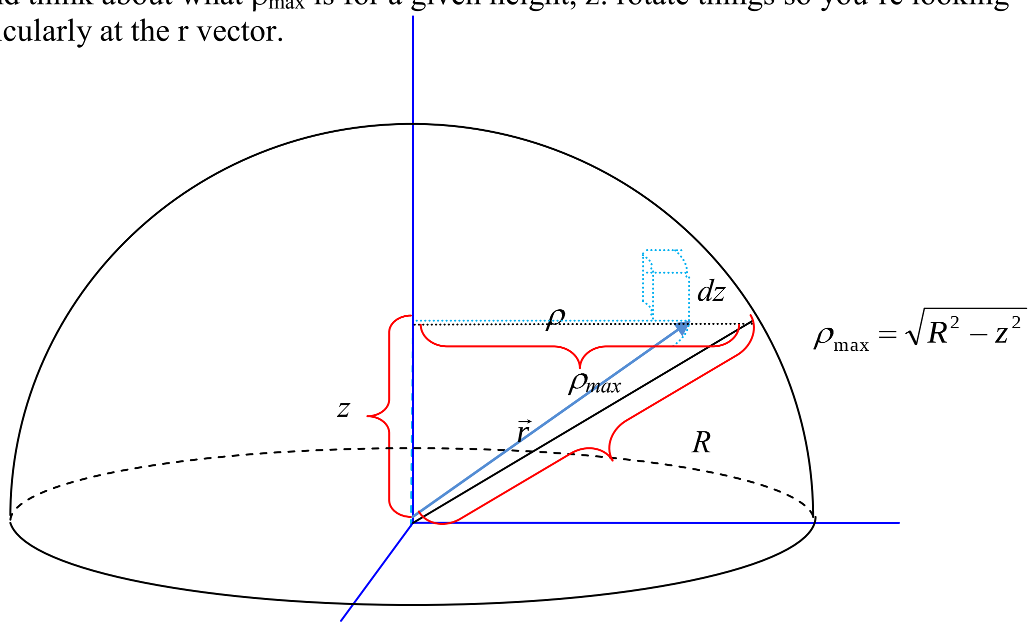
No doubt, you once knew how to do this, but perhaps it's been a while. Since you'll need to do something like this on the homework, here's a reminder.

$$\int_{half-sphere} f(\vec{r})dVol = ?$$



$$\int_{half-sphere} f(\vec{r})dVol = \int_{z=0}^{z=R}\int_{\rho=0}^{\rho=\rho_{max}}\int_{\phi=0}^{\phi=2\pi} f(\rho,z,\phi)(\rho d\phi)(d\rho)(dz) = \int_{z=0}^{z=R}\int_{\rho=0}^{\rho=\rho_{max}}\int_{\phi=0}^{\phi=2\pi} f(\rho,z,\phi)\rho d\phi d\rho dz$$

Pause and think about what ρ_{max} is for a given height, z: rotate things so you're looking perpendicularly at the r vector.



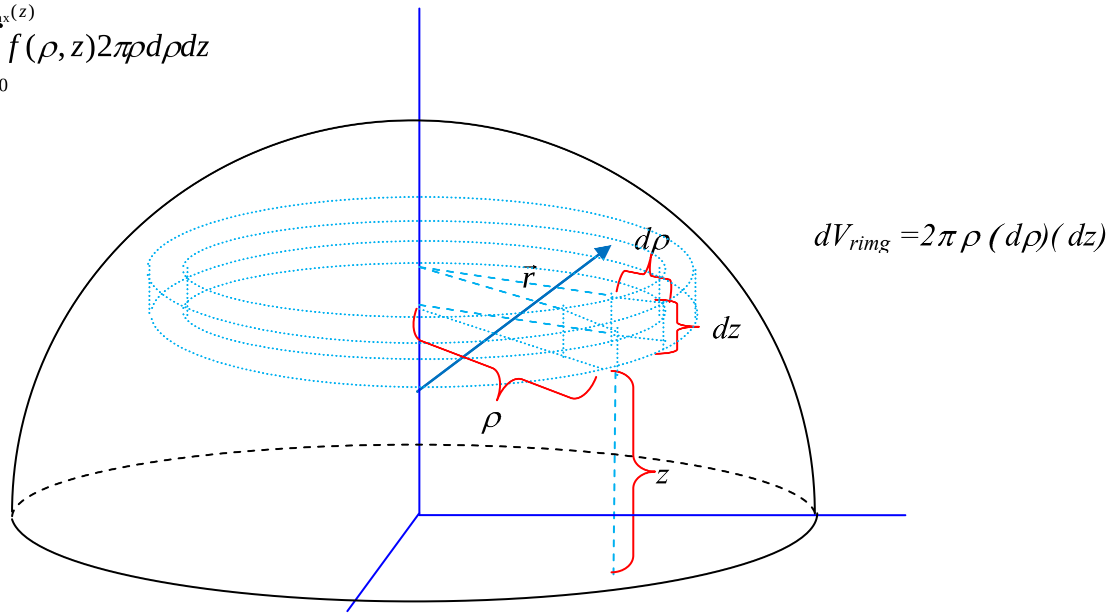
Just as a reminder that it depends upon the height, z, we'll write $\rho_{max}(z)$

$$\int_{half-sphere} f(\vec{r})dVol = \int_{z=0}^{z=R}\int_{\rho=0}^{\rho=\rho_{max}(z)}\int_{\phi=0}^{\phi=2\pi} f(\rho,z,\phi)\rho d\phi d\rho dz$$

Now, If f doesn't really depend upon ϕ, then you can pull it out of the ϕ integral and build up a differentially thin ring of volume.

$$\int_{half-sphere} f(\vec{r})dVol = \int_{z=0}^{z=R}\int_{\rho=0}^{\rho=\rho_{\max}(z)}\int_{\phi=0}^{\phi=2\pi} f(\rho,z)\rho d\phi d\rho dz = \int_{z=0}^{z=R}\int_{\rho=0}^{\rho=\rho_{\max}(z)} f(\rho,z)\left(\int_{\phi=0}^{\phi=2\pi} d\phi\right)\rho d\rho dz$$

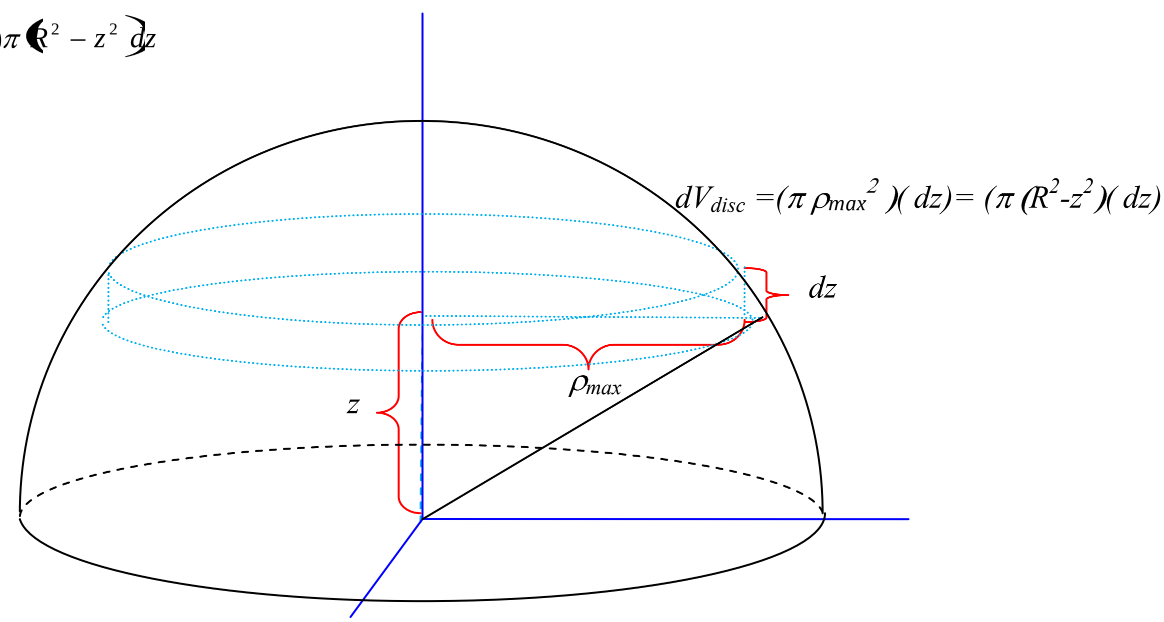
$$= \int_{z=0}^{z=R}\int_{\rho=0}^{\rho=\rho_{\max}(z)} f(\rho,z)2\pi\rho d\rho dz$$



If f doesn't really depend upon ρ, then you can pull it out of the ρ integral and build up a differentially thin disc of volume.

$$\int_{half-sphere} f(\vec{r})dVol = \int_{z=0}^{z=R}\int_{\rho=0}^{\rho=\rho_{\max}(z)} f(\rho,z)2\pi\rho d\rho dz = \int_{z=0}^{z=R}\int_{\rho=0}^{\rho=\rho_{\max}(z)} f(z)2\pi\rho d\rho dz = \int_{z=0}^{z=R} f(z)2\pi \int_{\rho=0}^{\rho=\rho_{\max}(z)} \rho d\rho dz = \int_{z=0}^{z=R} f(z)2\pi \tfrac{1}{2}\rho_{\max}^2 dz$$

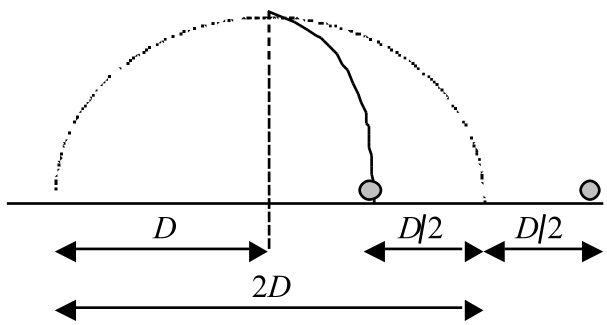
$$= \int_{z=0}^{z=R} f(z)\pi\left(R^2 - z^2\right)dz$$



Conceptual Question:

Example 3: A projectile of mass M is launched and explodes into two equal size pieces of $M/2$ at the top of its trajectory which is a horizontal distance D away. Suppose the two pieces fly apart horizontally and one piece lands a distance $1.5D$ from the launch site. Where does the second piece land? Ignore air resistance.

Regardless of the fact that the projectile exploded, it's *center of mass* will have followed the same parabolic trajectory. Being familiar with that kind of trajectory, you know that if the peak was a distance D from the launch point, then the landing was a distance 2D. So that's where the center of mass will end up. The projectile was going horizontally at the top of the trajectory. Since the two pieces fly apart horizontally, they will land at the same time. While they are in the air, the CM follows the same path as if it hadn't exploded because the net external force is the total weight.



The first piece ends up a distance $D/2$ from the path of the CM, so the other also does. The means the second piece ends up a distance $2.5D$ from the launch site.

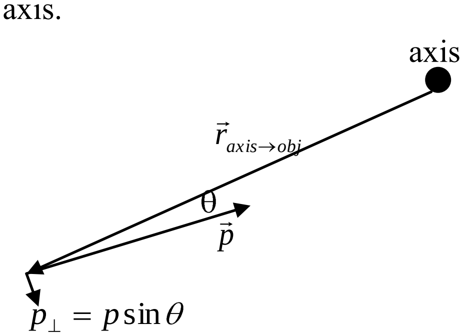
This is only solvable because the pieces land at the same time! The CM doesn't follow the parabolic path once either piece hits the ground because of the extra force.

Angular Momentum

Much of what follows is a review of what you learned in your introductory physics class (but may not have thought about in a long time.) So this is here more as an opportunity for you to skim over than something we're going to fully cover in class.

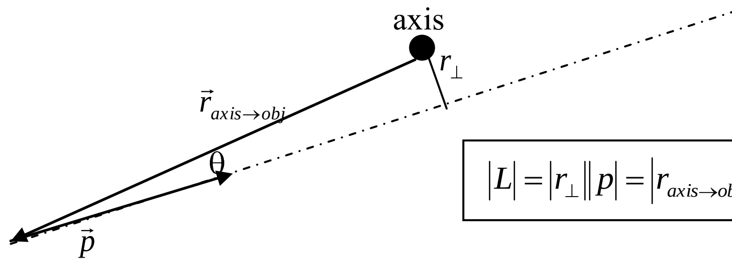
Introduction.

- "Angular Momentum" is a measure of how much *angular* or *rotational* motion something has (relative to a given axis.) So, of the full momentum vector, it's interested in just the component that points *around* the axis rather than *toward* or *away from* the axis.



$$|L| = |r_{axis\to obj}||p_\perp| = |r_{axis\to obj}||p|\sin\theta$$

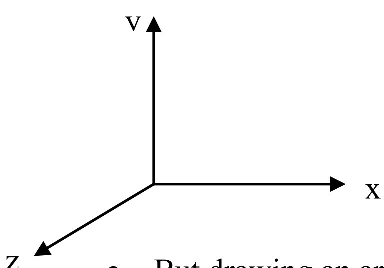
- Alternatively, you might have seen it described as the full momentum times the distance of closest approach (if it were to stay on its present course), or the "impact parameter" for collisions.



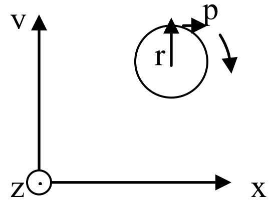
$$|L| = |r_\perp||p| = |r_{axis\rightarrow obj}||p|\sin\theta$$

- Angular Momentum is going to prove a very useful tool for discussing angular motion like things orbiting or spinning.

- **Coordinate System & in & out vector representation.** Just as with regular *linear* momentum, we'd like this tool to describe not just the *magnitude* of motion, but also the *direction* of motion. Since we're talking vectors and directions, we need a coordinate system. Let's define a Cartesian coordinate system with x & y in the plane parallel to the board and z coming out of the board at you. I could draw the coordinate axes at a slight kilter like so:



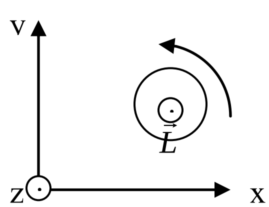
- But drawing an arrow like this for the z-axis doesn't quite look like the z-axis is pointing straight at you, it's pointing down & to your left a little. So I'll represent the z-axis with just the tip of an arrow, so it really looks like it's coming straight at you.



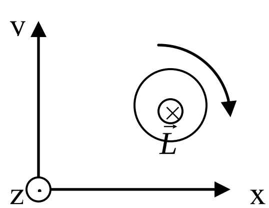
-
- This is a common convention for representing a vector pointing "out-of-the-page."
- **What's the Cartesian direction of rotation?** Now, consider a spot on a wheel laying in the x-y plane and spinning clockwise, relative to you (looking back along the z-axis). At any given instant, the point is moving in the +x direction, the –y direction, the – x direction, the + y direction, and every direction in between. Graphically it's easy enough to represent the direction of rotation with a *bent* arrow. If all you're interested in is being able to draw a clear picture, that's good enough. But if you want to use that picture to help translate the physical world into the language of mathematics, as we do in physics,

we'll have to get creative, for a bent arrow doesn't correspond to a mathematical Cartesian vector, they *don't bend.* How to mathematically designate the direction of spin in terms of these Cartesian coordinates?

- **Where are the individual points going?** As we've noted, through the course of one full revolution, the direction of the point's motion sweeps through all directions in the x-y plane. In fact, the only direction that it *never* points is along the z-axis!
- **Where it *isn't* going – axis of rotation.** So it is mathematically most concise to describe the rotational motion, not in terms of the directions that the particle *does* move, but the one direction that it does *not* – the direction of the axis of rotation, in this case, the z-axis.
- **Clockwise vs. Counterclockwise: + and - .** So, rather than talking about the direction of motion, we talk about the direction of the axis of rotation. For example, a wheel in the z-y plane has its (axis of) rotation along the x-axis, and a wheel in the z-x plane has its (axis of) rotation along the y-axis. So we can concisely describe the direction of rotation in terms of the direction of the axis. However, say I've got two wheels, both in the x-y plane, one rotating clockwise and one rotating counter clockwise. These clearly have opposite rotations, so it would be nice to say that one's direction is + while the other's is -, but which is which?
  - **Right-hand rule.** Truth be told, it's arbitrary, just like choosing whether to call up or down the + direction; you just need to remain consistent throughout your work. But for the sake of clarity, there is a handy, or rather, handed convention: the right hand rule. Take your right hand, start with palm open and fingers pointed in the direction of *r,* radially out from the axis toward the rotating object. Next, contort your hand as need be so you can then curl your fingers in the direction that the object is moving, in the direction of *p*. Now, your thumb points along the axis in the associated direction.

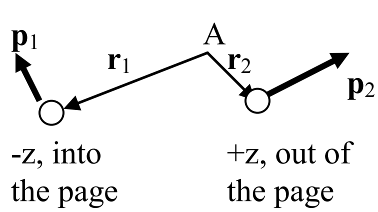


    - **Counterclockwise:** Say the wheel is spinning counterclockwise. At any instant, curling your fingers from the radial direction to the motion's direction points your thumb along the ***positive*** z-axis. $\vec{L} = \langle 0,0,+|L|\rangle \Rightarrow \vec{L} = \langle 0,0,|pr|\rangle$



    - **Clockwise:** Say the wheel is spinning clockwise in the x-y plane, then curling your fingers leaves your thumb pointing along the ***negative*** z-axis. $\vec{L} = \langle 0,0,-|L|\rangle \Rightarrow \vec{L} = \langle 0,0,-|pr|\rangle$
      - To illustrate this into-the-board vector, I draw the tail end of an arrow – the crossed tail-feathers.

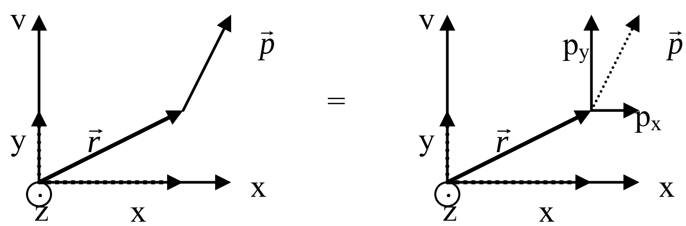
  - **CW 2:** More angular momentum directions. In this case, we, rather logically, referenced our motion to the center of the wheel. Now consider, more abstractly, the direction of angular momentum for particle 1 and for particle 2 relative to point A.



- The direction defined to tell us which way the object is circling the axis by the "right-hand rule" convention: imagine grabbing the axis so that your fingers curl around it the way the object is moving, then your thumb points in the direction assigned to the angular momentum.

The vector cross product & angular momentum

- Now, we know how to determine angular momentum's direction and its magnitude. However, the techniques we have are rather rudimentary, and not generally versatile. For example, instead of having a nice simple case, say with the momentum in the x direction and the radial vector in the y direction, what if they pointed in some arbitrary directions, how then would we calculate the angular momentum and designate its direction?
- Let's see if we can reason out one step in this generalization and extrapolate from there.

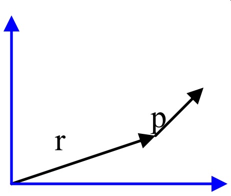


- **Motion in the X-Y plane.** For simplicity, consider **p** and **r** vectors confined to the x-y plane. Now, the momentum vector we have is the same as p_y in the y direction plus p_x in the x direction. So, what's the angular momentum associated with each of these? We'll then add them together to get the angular momentum associated with the total motion.
  - **X-component of momentum:**
    - **Magnitude:** Recall that $L = pr_{\perp}$. Just considering the x-component of the momentum, then the perpendicular vector from the axis to the point would be y, so $|L_x| = yp_x$.
    - **Direction:** and L would be in the negative z direction by our right-hand rule convention:
    - $\langle 0,0,-yp_x \rangle$.
  - Similarly, if we just consider the y-component of the momentum, then the perpendicular vector from the axis to the point would by x and L would be in the positive z direction: $\langle 0,0,xp_y \rangle$.
- Then the total angular momentum, associated with the total vector **p** who lies in the x-y plane, is $\vec{L} = \langle 0,0,xp_y - yp_x \rangle$.
- **Generalize to 3-D motion.** We could run through this same argument for **p** and **r** vectors confined to the y-z plane or to the x-z plane, and we'd get similar terms.
  - Say our momentum and position vectors were in the z-y plane instead, that would give rise to an angular momentum vector pointing in the x direction which we could reason out to be: $\langle yp_z - zp_y ,0,0 \rangle$.

- Similarly, say the momentum and position vectors were in the z-x plane, then we'd get an angular momentum of $\langle 0, zp_x - xp_z, 0\rangle$.
- Now, in the most general case, momentum and radial vectors with x, y, and z components: $\vec{L} = \langle (yp_z - zp_y), (zp_x - xp_z), (xp_y - yp_x)\rangle = \vec{r} \times \vec{p}$.

- This kind of vector multiplication is known as the vector cross-product.
- This is the most general representation of angular momentum for a point object and, in lab, this is what you used to determine the orbital angular momentum of the Earth around the Sun.

- **Demo: 09_Cross-product.py** This program computes the cross-product of the red and green vectors, and represents it as the yellow vector. You can see that it is always perpendicular to the two and that the more perpendicular they are, the large the cross-product is.
- **Example.** Giving this a whirl, say you have a mass that, at some instant, has linear momentum $\vec{p} = \langle 4,2,0\rangle kg \cdot m/s$ and, relative to a reference point, A, it is at $\vec{r}_A = \langle 5,3,0\rangle m$, then what is its angular momentum about A?



$\vec{L}_A = \langle 0,0,(xp_y - yp_x)\rangle$

$= \langle 0,0,(5m \cdot 2kg \cdot m/s - 3m \cdot 4kg \cdot m/s)\rangle = \langle 0,0,-2\rangle kg \cdot m^2/s$

Now, the cross-product gets magnitude and direction both correct for us. Depending on how information is given to you – components or direction and magnitude, either using the cross product or $|L| = |r_\perp||p| = |r_{axis\to obj}||p|\sin\theta$ with the right-hand rule will be simplest.

Conservation of Angular Momentum

Next time, we'll see how you keep track of the angular momentum of a multi-particle system and we're going to get a little more rigorous about showing what kind of interactions change a system's angular momentum. It's pretty clear that the angular momentum of an isolated object remains constant – for example, consider the child running toward the merry-go-round; constant linear momentum meant constant angular momentum. Beyond that though, I will simply *assert* for now that angular momentum can be conserved even in *non*-isolated object as long as the force is radial, that is, it's along the line between the axis you've chosen and the object. Put another way, it takes an *angularly* applied force to change *angular* momentum. That claimed, we'll just get a little practice thinking about angular momentum in the simplest of systems.

Circular Motion

Just from the definition of angular momentum, it's pretty darn clear that an object executing uniform circular motion has constant angular momentum relative to the axis of rotation – neither speed nor radial distance change, so

$\vec{L}_i = \vec{L}_f$

$\vec{r}_i \times \vec{p}_i = \vec{r}_f \times \vec{p}_f$

Of course, as we found when first discussing uniform circular motion, it takes a constant *radial* force to drive this kind of motion. For example, when a toy plane flies in circles, there's a net force always pointing toward the center of its orbit.

Planetary Orbits

How about elliptical orbits like the Earth's? That too has a force that's always directed toward an approximately fixed point – the sun, so the Earth's angular momentum about the Sun should be constant.

Demo: Orbit noncircular.py

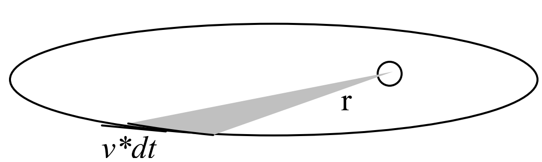
- Let's see if we can reason that out. Let's imagine that the angular momentum *is* changing over some small instant: $d\vec{L}_{E-S} = d\left(\vec{r}_{E-S} \times \vec{p}_E\right) = d\vec{r}_{E-S} \times \vec{p}_E + \vec{r}_{E-S} \times d\vec{p}_E$
- The common approach in calculus is to look at how much it changes due to a small change in the first variable, and then look at the effect of a small change in the second variable. We ignore the effect of both variables changing at once since that's a doubly small change.
- Now dp is in the direction of the force and the force is directed along r, back to the sun, so dp is parallel to r, thus their cross product is 0. On the other hand, dr, the displacement is of course in the direction of that the planet is going, i.e., the direction of its momentum, so their cross product is 0 too!
- $d\vec{L}_{E-S} = 0 + 0 = 0!$
- **Demo:** orbit with L.py form lab 8.

- **Equal Area in Equal Time.**

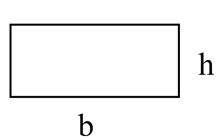
  - Historically, before folks knew what made the Earth orbit the Sun, before we even had the vocabulary of forces, a related observation was made. Kepler noted that as a planet moved through the sky, the area swept out by the radial line between it and the sun swept out equal area in equal times. When the planet was far away, it moved slowly, but the radius was long, so it swept out a great deal of area. When the planet was near the sun, though the radius was short, the planet moved quickly, so it swept out large are again.

Kepler's Law ppt.

- Let's look at that.

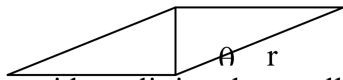


Now, what is this area? This can be seen in two steps. First consider the area of a rectangle:



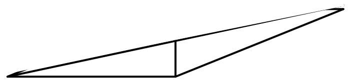
A=bh

Now consider the parallelogram made by slicing the rectangle down a hypotenuse and joining the two triangles:



A=bh=b rsinθ

Finally, consider splitting the parallelogram in ½, down the long diagonal. This is the shape whose area we're interested in. Having cut it in ½, we've clearly cut the area in ½ too.



A=1/2 bh=1/2 b rsinθ

In our case b = v dt. So A = ½*v***dt r*sinθ_{v-r}

- Then again, L = mv r sinθ_{v-r} and we've just reasoned that this is a constant, thus (as long as the planet's mass doesn't change and Kepler watches for equal time intervals) the areas are constant.
- Thus, Kepler's observation of equal areas implied equal angular momenta. Furthermore, equal angular momenta implies a force pulling the planet to the sun. Kepler's observation helped test / prove Newton's theory of a gravitational force.

<u>Torque</u>

The time rate of change of $\vec{\ell}$ is:

$$\dot{\vec{\ell}} = \frac{d}{dt}(\vec{r}\times\vec{p}) = (\dot{\vec{r}}\times\vec{p}) + (\vec{r}\times\dot{\vec{p}}).$$

The first term is zero, since $\dot{\vec{r}} \parallel \vec{p}$. Use the second law to write what is sometimes called the "rotational version of Newton's second law":

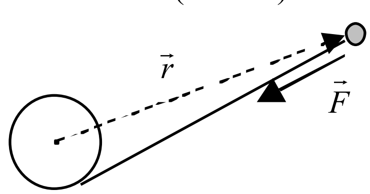
$$\dot{\vec{\ell}} = \vec{r}\times\vec{F} = \vec{\Gamma},$$

where we define the torque $\vec{\Gamma} = \vec{r}\times\vec{F}$. Numerous different symbols are used for torque! Of course, torque also depends on the choice of the origin. Any force directed toward or away from the origin produces no torque.

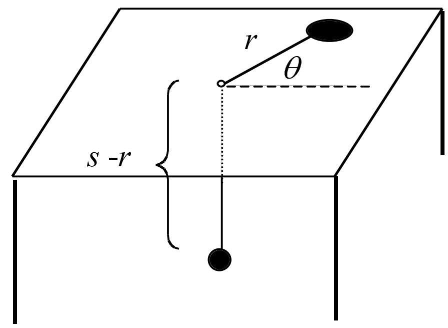
Conceptual Question

Suppose a small ball of mass *m* is at the end of a light string that wraps itself around a vertical cylinder of radius *R*. Using the center of the cylinder as the origin and ignore gravity. Is the angular momentum of the ball conserved?

No, the position is <u>not</u> parallel to the force (tension) so the torque $\vec{\Gamma} = \vec{r}\times\vec{F}$ is <u>not</u> zero.



Example 4: Suppose a string of length *s* connects a puck on a frictionless table and an object with the same mass through a hole. Find the equations of motion of the puck in terms of polar coordinates. Can the puck be spun so that it rotates with a constant radius?



Measuring the position of the mass below with downward positive. The second law is:

$$mg - T = m\frac{d^2}{dt^2}(s - r) = -m\ddot{r}$$

where *T* is the tension of the string. This makes sense because its weight tends to shorten *r*. The second law for the puck in polar coordinates gives:

$$\begin{aligned} \hat{r}\text{-direction:} \quad & -T = m\left(\ddot{r} - r\dot{\phi}^2\right) \\ \hat{\phi}\text{-direction:} \quad & 0 = m\left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right) \end{aligned}$$

The second equation can be rewritten as:

$$0 = \frac{1}{r}\frac{d}{dt}\left(mr^2\dot{\phi}\right) = \frac{1}{r}\frac{d}{dt}\left(mrv\right) = \frac{1}{r}\frac{d\ell}{dt},$$

which means that the angular momentum of the puck is conserved. We also know this because the tension exerts no torque on the puck relative to the hole. Eliminate *T* from the other two equations to get:

$$mg + m\left(\ddot{r} - r\dot{\phi}^2\right) = -m\ddot{r}$$

$$2m\ddot{r} = mr\dot{\phi}^2 - mg$$

$$2m\ddot{r} = \frac{\left(mr^2\dot{\phi}\right)^2}{mr^3} - mg$$

$$2m\ddot{r} = \frac{\ell^2}{mr^3} - mg\ .$$

If the initial conditions are $\dot{r}(0) = 0$ and $\ddot{r}(0) = 0$ then the puck will remain at a constant radius. The second condition means:

$$\ell^2 = m^2r^3g$$

$$\left(mr^2\dot{\phi}\right)^2 = m^2r^3g$$

$$\dot{\phi}^2 = g/r$$

$$\dot{\phi} = \sqrt{g/r}\ ,$$

No matter what, the puck will never reach the hole because its angular momentum can't be constant if it does.

Note: we'll have a much more streamlined approach to such problems when we develop the tools of Lagrangians.

Next two classes:

- Monday – Angular Momentum for Systems of Particles
- Wednesday – start Ch. 4