Mon. 9/17 Tues.9/18 Wed. 9/19 Thurs. 9/20 Fri. 9/21	**2.4** Quadratic Air Resistance **3.1-.2** Momentum & Rockets **3.3-.4** Center of Mass and Angular Momentum for 1 particle	**HW2b (2.B-.F & code from Friday)** HW3a (3.A-.C)
Mon. 9/24 Tues. 9/25 Wed. 9/26	**3.5** Angular Momentum for multiple particles **4.1-.3, 4.9** Work & Energy, Force as a Gradient, 2 Particle Interaction ***Science Poster Session**: Hedco7~9pm*	**HW3b, Project Topic**

Post and save corrected EulerCromer-vector.py

Materials

Laptops (need to warn them in advance)
EulerCromer-vector traj shell.py code

Reminders:

Project topic – Next Tues (consult project sheet & possible topics & talk with me)

Poster Session for summer researchers (dept can print poster for you, but will want do that *this* week)

What I did last summer –the following week

Physics Theories; 'first principles' or 'good enough'

In today's discussion prep, someone asked about the nature of a "theory", which is particularly relevant to our current discussion of projectile motion.

Theory = explanation, or explanatory model

Now, Physics as a discipline has something of a split personality – on the one hand, we like to come up with the most fundamental theories we can, to understand things on the most basic of levels – no approximations, no handwaving – just accurate.

One the other hand, there are lots of cases in which a 'first principles' model is unnecessarily unwieldy, and certain mathematical and conceptual simplifications make a problem more tractable while still leading to 'good enough' answers. The classic example is interatomic bonds – to answer some situations, you really need quantum mechanics (heaven help us), but for others, it's sufficient to assume the potential energy can be expressed in a Taylor series and we can get away with ignoring higher order terms – viola, the 'Hook's Law / spring' approximation.

For projectile motion, a *truly* first principles approach, considering each particle of air colliding with the object, may be necessary for answering some questions, but for others, like 'to 99.9% accuracy, what's the trajectory of a smooth ball that's not spinning?' we certainly don't need it. What do we need? As this chapter points out, that depends on how big and fast the ball is.

General Drag review

Newton's second law for a projectile with drag is

$$m\dot{\vec{v}} = m\vec{g} - f(v)\hat{v}$$

For simple projectiles at reasonable speeds, for both mathematical and physical reasons, we expect to be able to represent the drag force with the first two terms in a Taylor series, so

$$m\dot{\vec{v}} \approx m\vec{g} - b\vec{v} - cv^2\hat{v}$$

Or

$$\dot{\vec{v}} \approx \vec{g} - \frac{b}{m}\vec{v} - \frac{c}{m}v^2\hat{v}$$

Zeroth-order approx: slow and massive. Clearly, if the ball's massive and slow enough, the v-dependent terms are always smaller than g and we can accurately enough predict the ball's motion by simply ignoring them – so our 'good enough' model is just

$$\dot{\vec{v}} \approx \vec{g}$$

First or second-order approx: faster and lighter. But if it's not that slow / not that massive, we do need to include one or both of these terms in our model.

Which do we need? As we can rephrase the constants in terms of properties of the air and diameter of the projectile, we can tell when one or the other term is negligible. For spheres in air at STP,

$$\frac{f_q}{f_l} = \frac{cv}{b} = \frac{\gamma D^2 v}{\beta D} = \left(1.6\times10^3\, s/m^2\right)Dv$$

Clearly, for a very small object and/or very small speed, this ratio is small and the quadratic term is negligible. But even if it starts out negligible (small launch speed), will it *remain* negligible throughout the flight? Of course, it something's in flight long enough it will reach terminal velocity. So, what does this ratio look like by the time terminal velocity is reached?

$$\frac{f_q}{f_l} = \frac{cv_{ter}}{b}$$

Recall that last week you'd found a general expression for the terminal speed of an object that's simply moving up/down: $v_{ter} = \dfrac{\sqrt{\left(\frac{b}{c}\right)^2 + 4\frac{mg}{c}} - \left(\frac{b}{c}\right)}{2}$

Exercise: What's the threshold mass for needing to consider quadratic vs. linear drag?

What would be the diameter of a raindrop (density $1 g/cm^3$) with this mass?

Plugging this expression for terminal speed into $\frac{f_q}{f_l} = \frac{cv_{ter}}{b}$ gives

$$\frac{f_q}{f_l} = \frac{c}{b}\frac{\sqrt{\left(\frac{b}{c}\right)^2 + 4\frac{mg}{c}} - \left(\frac{b}{c}\right)}{2} = \left(\sqrt{\tfrac{1}{4} + m\left(\tfrac{gc}{b^2}\right)} - \tfrac{1}{2}\right) = \left(\sqrt{\tfrac{1}{4} + m\left(\tfrac{g\gamma D^2}{\beta^2 D^2}\right)} - \tfrac{1}{2}\right) = \left(\sqrt{\tfrac{1}{4} + m\left(\tfrac{g\gamma}{\beta^2}\right)} - \tfrac{1}{2}\right)$$

$$\frac{f_q}{f_l} = \left(\sqrt{\tfrac{1}{4} + m\left(3.4\times10^7/kg\right)} - \tfrac{1}{2}\right)$$

To get the exact break-even mass,

$$\sqrt{\tfrac{1}{4} + m(3.4\times10^{7}/kg)} - \tfrac{1}{2} = 1 \Rightarrow \frac{2}{3.4\times10^{7}}kg = 5.9\times10^{-8}kg = 5.9\times10^{-5}g = 59\mu g$$

What would be the diameter of a raindrop (density 1g/cm^3=1000kg/m^3) with this mass?

D = 0.5 mm.

Point being, for most things, if they fly long enough, the quadratic term is most important and we can ignore the linear term.

Computational Exercise: Trajectory Comparisons

Let's see how this plays out in our simulation. It's already set up to model the *zero* drag case with

$$\vec{a}_o = \vec{g}$$

You'll add in code to trace the trajectory if we only consider linear, quadratic, or both drag terms. Once you have that, you'll run it for 'waterballs' of vastly different diameters: 10m, 1m,0.1m, 0.01m, 0.001m, and finally 0.0001m to see for which we need the different quality approximations.

Add in a line to calculate what the acceleration would be if there were linear drag:

$$\vec{a}_1 = \vec{g} - \tfrac{b}{m}\vec{v}_1$$

Since a different acceleration means velocity and position evolve differently, you'll also want to write lines to initialize and update v1 and r1 and create and update a funct1 to trace the trajectory.

If it's quadratic drag, we have

$$\vec{a}_2 = \vec{g} - \tfrac{c}{m}{v_2}^2\hat{v}_2 = \vec{g} - \tfrac{c}{m}|v_2|\vec{v}_2$$

Again, you'll want to add lines to initialize and update v2, r2, and funct2.

And if both are significant, we have

$$\vec{a}_{12} = \vec{g} - \tfrac{b}{m}\vec{v}_{12} - \tfrac{c}{m}|v_{12}|\vec{v}_{12}$$

Quadratic Drag

Newton's second law for a projectile with just quadratic drag is:

$$m\dot{\vec{v}} \approx m\vec{g} - cv^2\hat{v} = m\vec{g} - cv\vec{v} = m\vec{g} - c\sqrt{v_x^2 + v_y^2}\ \vec{v}\ .$$

Terminal Speed

Remember, it's really easy to get the limit – for what speed is there *no* acceleration, what's the terminal speed?

$$m\dot{\vec{v}} = m\vec{g} - cv^2\hat{v}$$
$$0 = m\vec{g} - c{v_{t.q}}^2\hat{v}$$
$$v_{t.q} = \sqrt{\frac{mg}{c}}$$

If the object's only moving vertically,

$$0 = mg(-\hat{y}) - c{v_{t.q}}^2(-\hat{y})$$
$$v_{t.q} = \sqrt{\frac{mg}{c}}$$

Note: if there's also a horizontal component to the velocity, then there are two contradictory relations

$$0 = mg(-\hat{y}) - c{v_{t.q}}^2(-\hat{y}),\ 0 = -c{v_{t.q}}^2(-\hat{x})$$

So there isn't truly a "terminal velocity" (at which there is no force) for a purely quadratic drag force, but then again, there isn't truly a "purely quadratic drag force." In practice, the horizontal motion would decay enough that the quadratic approximation would no longer be good – we'd need to consider the linear term. Eventually, we're essentially back to purely vertical motion with the same terminal speed just found.

V(t)

Pulling back from the limiting case, generally, if a projectile is moving in both the *x* and *y* directions, the differential equation for v_x will involve v_y in a complicated way, and vice versa. In this case, there is no analytical solution so we are forced to solve the problem numerically. If the motion is constrained to be horizontal ($v_y = 0$) or is strictly vertical ($v_x = 0$), then there are analytical solutions. We'll look at those two cases first.

<u>Strictly Horizontal Motion</u>:

If the *only* force on an object moving horizontally is a quadratic drag force (the weight must be balanced by a normal force), then the second law is:

$$m\frac{dv}{dt} = -cv^2.$$

A subscript on *v* is not needed because this is a one-dimensional problem. Separate the variables *v* & *t* and integrate the equation:

$$\frac{dv}{v^2} = -(c/m)dt$$
$$\int_{v_o}^{v}\frac{dv'}{v'^2} = -(c/m)\int_0^t dt'$$

$$\left[\frac{-1}{v'}\right]_{v_o}^{v} = \frac{1}{v_o} - \frac{1}{v} = -(c/m)t.$$

Solve for *v*:

$$\frac{1}{v} = \frac{1}{v_o} + (c/m)t = \frac{1 + (cv_o/m)t}{v_o}$$

$$v(t) = \frac{v_o}{1 + (cv_o/m)t} = \frac{v_o}{1 + t/\tau_q},$$

where we define the parameter $\tau_q = m/cv_o$ which has units of time.

Since $v = dx/dt$, this equation can be separated and integrated again:

$$dx = \frac{v_o dt}{1 + t/\tau_q}$$

$$\int_{x_o}^{x} dx' = \int_0^t \frac{v_o dt'}{1 + t'/\tau_q}$$

Math idea of the day: change of variables (it's an important tool to have)

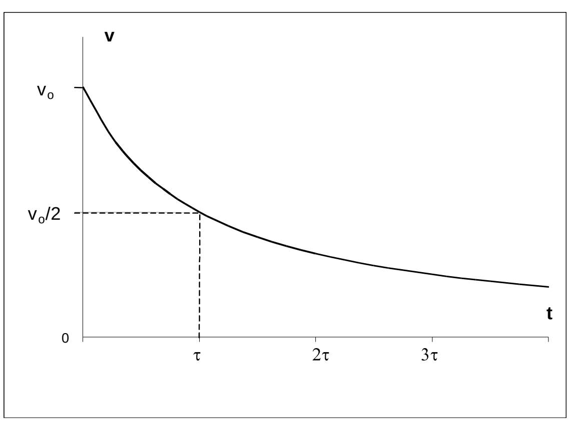
The integration can be done by making the change of variables $q = 1 + t'/\tau_q$ and $\tau_q dq = dt'$:

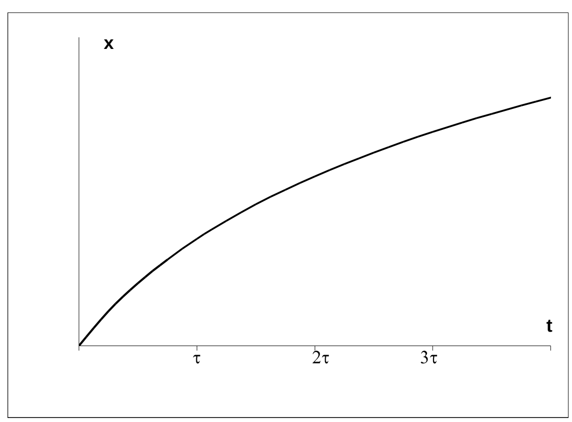
$$\int_{x_o}^{x} dx' = \int_1^{(1+t/\tau_q)} \frac{v_o \tau_q dq}{q}$$

$$[x']_{x_o}^{x} = x(t) - x_o = v_o \tau_q \ln(q)\Big|_1^{1+t/\tau_q} = v_o \tau_q \ln\left(1 + t/\tau_q\right)$$

$$x(t) = x_o + v_o \tau_q \ln\left(1 + t/\tau_q\right).$$

Graphs of the velocity and position are shown below. After a time $t = \tau_q$, the velocity is half of its original value. The velocity drops more slowly than in the linear medium, so the position continues to increase without limit.

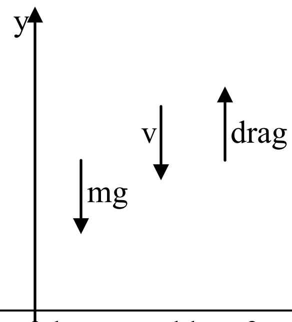




Strictly Vertical Motion:

Consider an object that is dropped from rest. Choose the y axis to be pointing downward. If an object is *moving* vertically downward with a quadratic drag force, then the second law is (A subscript on v is not needed because this is a one-dimensional problem):

$$m\frac{dv}{dt} = mg - cv^2 \qquad \left(m\frac{dv}{dt} = -mg + cv^2 \quad \text{if the } y \text{ axis is } \underline{\text{upward}} \right)$$



Question: What is the y-component of the second law for an upward moving particle?

Answer: It depends on the coordinate system and direction of motion!

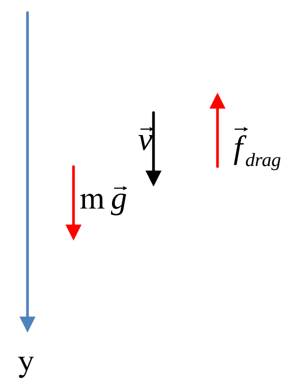
Say it's moving down, draw the pictures as above and decide the appropriate signs.

I'll do one possible combination of velocity and axis choices, you do the other three.

If you choose the *y axis* to point downward

and consider the *velocity* downward, then:

$$m\frac{dv}{dt} = mg - cv^2.$$



Your turn:

and the *velocity* upward, then:

$$m\frac{dv}{dt} = mg + cv^2$$

If you choose the *y axis* to point upward,

and consider the *velocity* downward, then:

$$m\frac{dv}{dt} = -mg + cv^2.$$

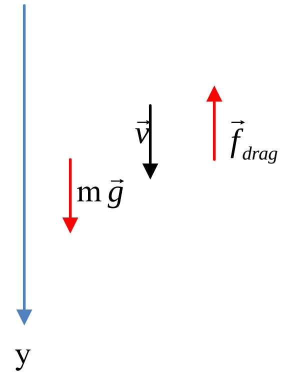
and consider the *velocity* upward, then:

$$m\frac{dv}{dt} = -mg - cv^2$$

Moral: you've got to be careful about both your choice of axis and whether the object is heading up or back down again.

To move forward, let's consider the case the book does (**object's falling, and down is positive**)

Separate the variables v & t and integrate the equation. Use the definition $v_{\text{ter},q} = \sqrt{mg/c}$.



$$\frac{dv}{dt} = g - \frac{c}{m}v^2 = g\left(1 - \frac{c}{gm}v^2\right) = g\left(1 - \frac{v^2}{v_{\text{ter},q}^2}\right)$$

$$\frac{dv}{1 - v^2 / v_{\text{ter},q}^2} = g\,dt$$

so (integral in front cover):

$$\int_{v_o}^{v} \frac{dv'}{1 - v'^2 / v_{\text{ter},q}^2} = \int_0^t g\,dt'$$

$$v_{\text{ter},q} \tanh^{-1}\left(\frac{v'}{v_{\text{ter},q}}\right)\Bigg|_{v_o}^{v} = v_{\text{ter},q}\left[\tanh^{-1}\left(\frac{v}{v_{\text{ter},q}}\right) - \tanh^{-1}\left(\frac{v_0}{v_{\text{ter},q}}\right)\right] = gt$$

Solve for v:

$$v(t) = v_{\text{ter},q} \tanh\left(\frac{gt}{v_{\text{ter},q}} + \tanh^{-1}\left(\frac{v_o}{v_{\text{ter},q}}\right)\right).$$

(note: the tanh is of the whole darn thing, you can't cleanly break it into parts.)

Separate and integrate again (integral in front cover):

$$dy = v_{\text{ter},q} \tanh\left(\frac{gt}{v_{\text{ter},q}} + \tanh^{-1}\left(\frac{v_o}{v_{\text{ter},q}}\right)\right)dt\,.$$

$$\int_{y'=0}^{y'=y} dy = v_{\text{ter},q} \int_{t'=0}^{t'=t} \tanh\left(\frac{gt'}{v_{\text{ter},q}} + \tanh^{-1}\left(\frac{v_o}{v_{\text{ter},q}}\right)\right)dt'$$

This may look pretty ugly, but doing a change of variables will help a good deal

In case you're rusty on how to do such things, here's a blow-by-blow

First, let's define

$$u \equiv \frac{gt'}{v_{\text{ter},q}} + \tanh^{-1}\left(\frac{v_o}{v_{\text{ter},q}}\right)$$

Now let's consider what the limits of integration will be in terms of this variable

as t' runs from t'_{min}=0 to t'_{max}= t,

u runs from $u_{min} = \tanh^{-1}\left(\frac{v_o}{v_{\text{ter},q}}\right)$ to $u_{max} = \frac{gt}{v_{\text{ter},q}} + \tanh^{-1}\left(\frac{v_o}{v_{\text{ter},q}}\right)$

Now let's think about what differential step in u is equivilant to *dt'*,

$$\frac{du}{dt'} = \frac{d}{dt'}\left(\frac{gt'}{v_{\text{ter},q}} + \tanh^{-1}\left(\frac{v_o}{v_{\text{ter},q}}\right)\right) = \frac{g}{v_{\text{ter},q}} \Rightarrow dt' = du\frac{v_{\text{ter},q}}{g}$$

So we have

$$\int_0^y dy' = y = v_{\text{ter},q}\int_{u=u_{\min}}^{u=u_{\max}} \tanh(u)\frac{v_{\text{ter},q}}{g}du = \frac{v_{\text{ter},q}^2}{g}\int_{u=u_{\min}}^{u=u_{\max}} \tanh(u)\,du$$

$$= \frac{v_{\text{ter},q}^2}{g}\ln\left[\cosh(u)\right]_{u_{\min}}^{u_{\max}} = \frac{v_{\text{ter},q}^2}{g}\left(\ln\left[\cosh(u_{\max})\right] - \ln\left[\cosh(u_{\min})\right]\right).$$

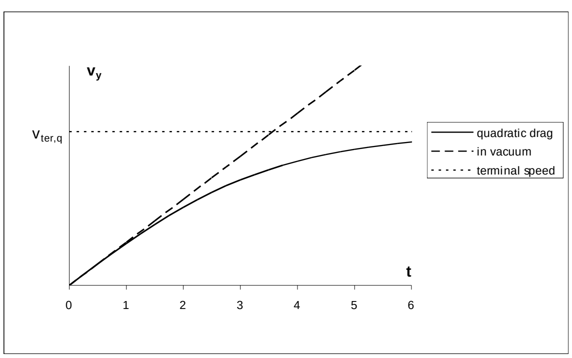
$$\frac{v_{\text{ter},q}^2}{g}\ln\left[\cosh\left(\frac{gt}{v_{\text{ter},q}} + \tanh^{-1}\left(\frac{v_o}{v_{\text{ter},q}}\right)\right)\right] - \ln\left[\cosh\left(\tanh^{-1}\left(\frac{v_o}{v_{\text{ter},q}}\right)\right)\right]$$

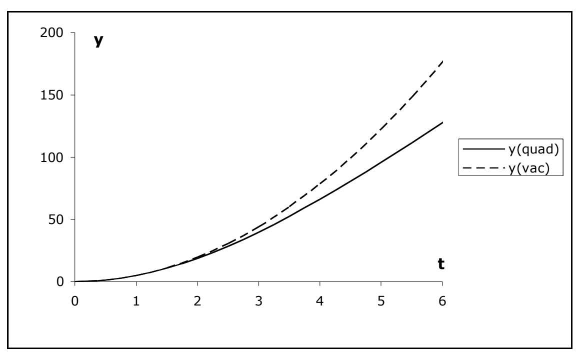
Special case, v_o=0.

In the case that the book considers, dropping from rest (v_o=0) this reduces to the slightly simpler expression

$$y(t) = \frac{(v_{\text{ter},q})^2}{g}\ln\left[\cosh\left(\frac{gt}{v_{\text{ter},q}}\right)\right], \text{ similarly } \; v(t) = v_{\text{ter},q}\tanh\left(\frac{gt}{v_{\text{ter},q}}\right)$$

This functions are plotted below with $v_{\text{ter},q} = 35$ m/s. Notice that the slope of y vs. t approaches a constant.





Combined Horizontal and Vertical Motion:

The second law is:

$$m\ddot{\vec{r}} = m\dot{\vec{v}} = m\vec{g} - cv^2\hat{v} = m\vec{g} - c|v|\vec{v}\,.$$

> *Question*: If we take the *y* axis to be vertically upward, then what are the component equations?
>
> *Answer*: Draw arrows representing the different terms' directions and figure out signs.
>
> $$m\dot{v}_x = -c\sqrt{v_x^2 + v_y^2}\; v_x\,,$$
>
> $$m\dot{v}_y = -mg - c\sqrt{v_x^2 + v_y^2}\; v_y\,,$$
>
> or:
>
> $$\ddot{x} = -(c/m)\sqrt{\dot{x}^2 + \dot{y}^2}\; \dot{x}\,,$$
>
> $$\ddot{y} = -g - (c/m)\sqrt{\dot{x}^2 + \dot{y}^2}\; \dot{y}\,.$$

These equations are not separable! There is not an analytical solution to these equations, so numerical solutions such as your simulation are necessary.

Computer Exercise: Use the Euler-Cromer method in VPython to solve Problem (2.43).

HW note: for the computational problem in the HW, you can modify today's program, using just the 12 trajectory, to do it.

Next two classes:

- Wednesday – Conservation of Momentum & Rockets (skipping 2.5 since you get it in our E&M courses; we'll return to 2.6 when we need it in this class.)
- Friday – Center of Mass & Angular Momentum of a Particle