Wed. 9/12 Thurs. 9/13 Fri. 9/14	**2.1-.2** Air Resistance - Linear **2.3** Trajectory and Range with Linear Resistance	**HW2a** (2.A)

Equipment

- Paper handout on the Range approximations (reading for next time)
- Paper handout on computer exercise
- Whiteboards (pens and erasers)
- Email them EulerCromer-Vector.py

Next time:

They'll do simulation of ball moving in linear medium; bring laptops (with python and sample code) if you have them.

Discussion Prep for next time: Begin working on the handout, starting with the code "EulerCrommer-vector.py" that I've emailed you. You don't need to send me anything in advance, just begin working on it so you know what your questions are.

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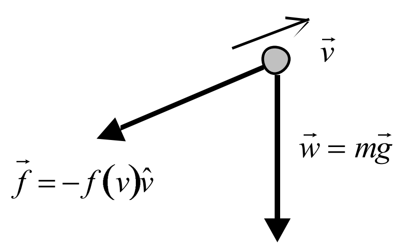
For this time, need a good example like homework to work

<u>Introduction</u>:

The drag force due to "air resistance" is in the opposite direction as the velocity, as shown below. For a projectile, the only other force that we will consider is an objects weight. No effects of spin (e.g. lift) will be considered.



The drag force is:

$$\vec{f} = -f(v)\hat{v},$$

where $\hat{v} = \vec{v}/|\vec{v}|$ is a unit vector in the direction of the velocity $\vec{v}$ and $f(v)$ is the size of the force. The function $f(v)$ is a complicated function, especially near the speed of sound in a medium. A good approximation at low speeds is:

$$f(v) \approx bv + cv^2.$$

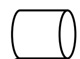
As the book says, this is simply the first two (non-zero) terms of a Taylor expansion in *v*. So, even if we have no good handle on the true functional dependence of air resistance on velocity, this would be a reasonable first guess. Better than that, we *do* have some simply reasons for expecting a linear term and a quadratic term.

Why Linear term: As you may learn in Statistical Mechanics / Thermal Dynamics (ch 1), if you imagine trying to drag one 'layer' of, say, water, across another layer, the force which opposes that sliding is proportional to *v*. It's called 'viscosity.' So, when an object flies into the air, it's trying to make air 'slide' past other air as it slides by. Of course, more obviously, the object is itself slamming through air! All those microscopic collisions means a force opposing the motion.

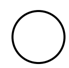
Why Quadratic term: As we argued in Phys 231, *that* drag force is proportional to the velocity squared (one factor has to do with the momentum transfer involved in the collisions and the other has to do with the rate of the collisions).

The coefficients *b* and *c* depend on the size and the shape of an object. For a spherical projectile, they are:

$b = \beta D$ (linear in D since depends on the area of the cylinder described by the sphere's motion A= π DΔx)



$c = \gamma D^2$ (quadratic in D since depends on the cross-sectional area of the sphere pushing through air: A = π r^2)



where D is the diameter of the sphere and β and γ depend on the medium. For a sphere moving in air at standard temperature and pressure (STP):

$$\beta = 1.6 \times 10^{-4}\ \mathrm{N \cdot s/m^2}$$

$$\gamma = 0.25\ \mathrm{N \cdot s^2/m^4}$$

Comparing Terms,

$$\frac{f_{\text{quad}}}{f_{\text{lin}}} = \frac{cv^2}{bv} = \frac{\gamma D v}{\beta} = \left(1.6 \times 10^3\ \mathrm{s/m^2}\right) D v$$

For sufficiently low speeds, the linear term dominates and the quadratic term can be neglected. For sufficiently high speeds, the quadratic term dominates. In between, the terms are comparable. What qualifies as a low or high speed depends on the size of an object.

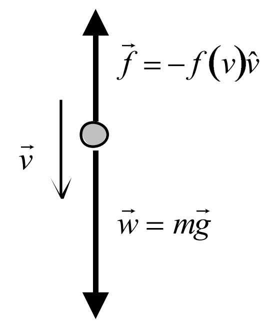
For a spherical object in air at STP:

$$\frac{f_{\text{quad}}}{f_{\text{lin}}} = \frac{\gamma D v}{\beta} = \left(1.6 \times 10^3\ \mathrm{s/m^2}\right) D v$$

If $\left(1.6 \times 10^3\ \mathrm{s/m^2}\right) D v = 1$ or $v = \left(6.25 \times 10^{-4}\ \mathrm{m^2/s}\right) / D$, the two terms are comparable. If $v << \left(6.25 \times 10^{-4}\ \mathrm{m^2/s}\right) / D$, then the linear term dominates. If $v >> \left(6.25 \times 10^{-4}\ \mathrm{m^2/s}\right) / D$, then the quadratic term dominates.

Terminal Speed (for strictly vertical motion):

Suppose a projectile is moving directly downward. That means that the weight and drag force are in opposite directions as shown below.



The projectile will continue to accelerate downward until the air resistance is the same size as its weight. At that point, there will be no net force on the object, so it will continue at the same velocity. The speed at which this occurs is called the *terminal speed*.

For linear air resistance, the terminal speed is reached when:

$$\begin{aligned} &\vec{w} + \vec{f} = 0 \Rightarrow \vec{w} = -\vec{f} \\ &mg = bv_y \end{aligned},$$

so:

$$v_{\text{ter},l} = \frac{mg}{b} \qquad \text{(linear case).}$$

For quadratic air resistance, the condition is:

$$\begin{aligned} &\vec{w} = -\vec{f} \\ &mg = cv_y^2 \end{aligned},$$

so:

$$v_{\text{ter},q} = \sqrt{\frac{mg}{c}} \qquad \text{(quadratic case).}$$

Be careful to use the right equation for the terminal speed! In either case, if two objects of the same size, the one with a larger mass (the denser one) will have a larger terminal speed. You will do a problem (2.36) related to Galileo's experiments with objects of the same density, but different sizes.

<u>Exercise for Students</u>: What is the terminal speed with both linear and quadratic terms?

<u>Solution</u>: The condition for reaching terminal speed is:

$$\vec{w} = -\vec{f}$$
$$mg = bv_y + cv_y^2 ,$$
$$cv_y^2 + bv_y - mg = 0 .$$

The solution (which reduces to the two result above for $b = 0$ or $c = 0$) to this quadratic equation is (speeds are always positive by definition):

$$v_{\text{ter}} = \frac{-b + \sqrt{b^2 + 4cmg}}{2c} .$$

<u>Motion with Linear Air Resistance</u>:

This is the easier case, but unfortunately it doesn't apply to as many situations. If the quadratic term can be ignored, the drag force is:

$$\vec{f} = -bv\hat{v} = -b\vec{v} .$$

The second law gives:

$$m\ddot{\vec{r}} = m\vec{g} - b\vec{v}$$

or a differential equation for $\vec{v}$, since $\ddot{\vec{r}} = \dot{\vec{v}}$:

$$m\dot{\vec{v}} = m\vec{g} - b\vec{v} .$$

This equation can be used to find the velocity as a function of time, then the equation $\vec{v}(t) = d\vec{r}/dt$ can be integrated to find the position as a function of time. Choosing a coordinate system with *x* to the right and *y* <u>downward</u>, the equation above separates into:

$$m\dot{v}_x = -bv_x ,$$
$$m\dot{v}_y = mg - bv_y .$$

These differential equations are easy to solve because they each only involve one component of the velocity (and its derivative). We'll use the initial conditions $\vec{r}_{\text{o}} = (0,0)$ and $\vec{v}_{\text{o}} = (v_{x\text{o}}, v_{y\text{o}})$.

(1) <u>Horizontal Component</u>: The horizontal equation can be rewritten as:

$$dv_x/dt = -(b/m)v_x .$$

Separate v_x & *t* and integrate the equation:

$$\int_{v_{xo}}^{v_x(t)} \frac{dv'_x}{v'_x} = -(b/m)\int_0^t dt'$$

$$\ln(v'_x)_{v_{xo}}^{v_x(t)} = \ln\left(\frac{v_x(t)}{v_{xo}}\right) = -(b/m)(t')_0^t = -(b/m)t$$

$$v_x(t) = v_{xo}e^{-(b/m)t} = v_{xo}e^{-t/\tau_l},$$

where define the parameter $\tau_l = m/b$ which has units of time.

Then again, we could just make an 'inspired" guess:

$$dv_x/dt = -(b/m)v_x$$

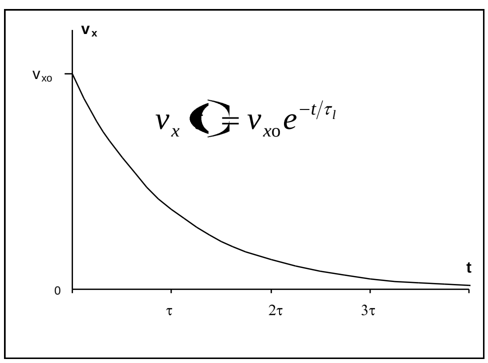
Has the functional form of

$$\frac{d}{dt}f(t) = -kf(t)$$

And that's pretty uniquely a property of the exponential function

Qualitatively:

The ball's initially moving very quickly so it experiences a lot of drag, that means that it slows down rapidly (steep slope in a v vs. t plot). At a later time, it's now moving more slowly, so it's experiencing less drag and thus slowing down a little less quickly (shallower slope).



Position as a function of time: Separate *x* & *t* and integrate this equation:

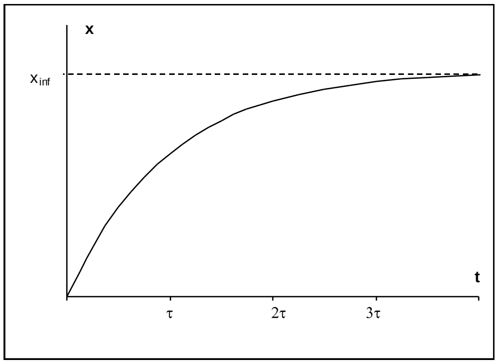
$$dx/dt = v_{xo}e^{-t/\tau_l}$$

$$\int_0^{x(t)} dx' = v_{xo}\int_0^t e^{-t'/\tau_l}dt'$$

$$(x')_0^{x(t)} = v_{xo}\left(-\tau_l e^{-t'/\tau_l}\right)_0^t$$

$$x(t) = v_{xo}\tau_l\left(1-e^{-t/\tau_l}\right) = x_\infty\left(1-e^{-t/\tau_l}\right),$$

where we define $x_\infty = v_{xo}\tau_l$.



The x component of the velocity drops exponential and τ_l is the time for it to decrease by a factor of 1/e. The limit of the x component of the position as $t \to \infty$ is x_∞.

(2) Vertical Motion: We have already determined that the terminal speed is:

$$v_{\text{ter},l} = \frac{mg}{b}.$$

The differential equation for the vertical direction can be rewritten as:

$$m\dot{v}_y = -b\left(-mg/b + v_y\right) = -b\left(v_y - v_{\text{ter},l}\right).$$

Consider a change of variables $u \equiv v_y - v_{ter,l}$

$$m\dot{u} = -bu$$

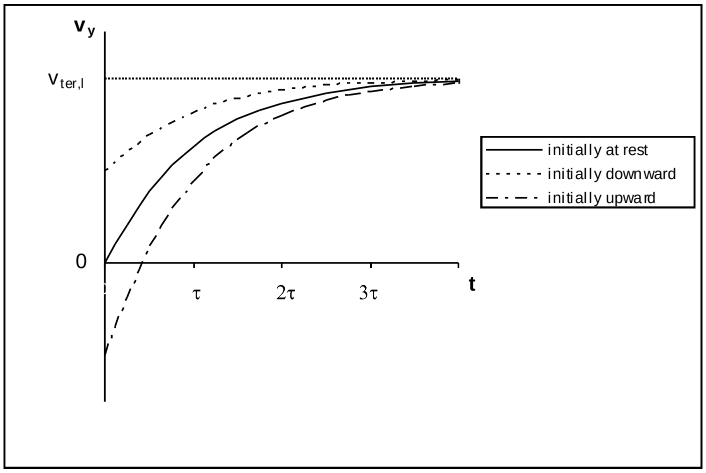
So we can again say that $u = u_o e^{-\frac{b}{m}t} = v_y - v_{ter,l}$

Of course, at t=0, v_y=v_o, so

$$v_y(t) - v_{\text{ter},l} = \left(v_{yo} - v_{\text{ter},l}\right)e^{-t/\tau_l}$$
$$v_y(t) = v_{\text{ter},l} + \left(v_{yo} - v_{\text{ter},l}\right)e^{-t/\tau_l}$$

Qualitatively: the difference between the actual speed and the terminal speed exponentially decays away.

The first term (associated with the initial speed) decays with time and the second term (associated with the terminal speed) grows with time. The following graph shows the three possibilities: $v_{yo} = 0$, $v_{yo} > 0$ (downward), and $v_{yo} < 0$ (upward) (recall that the y axis points downward!). In all cases, $v_y \to v_{\text{ter},l}$ as $t \to \infty$.



Question: What would the graph look like without air resistance? *Answer*: All have slope +g.

The last equation can be separated and integrated:

$$dy/dt = v_{\text{ter},l} + \left(v_{yo} - v_{\text{ter},l}\right)e^{-t/\tau_l}$$

$$\int_0^{y(t)} dy' = \int_0^t \left[v_{\text{ter},l} + \left(v_{yo} - v_{\text{ter},l}\right)e^{-t'/\tau_l}\right]dt'$$

$$\left(y'\right)_0^{y(t)} = \left[v_{\text{ter},l}t' + \left(v_{yo} - v_{\text{ter},l}\right)\tau_l e^{-t'/\tau_l}\right]_0^t$$

$$y(t) = v_{\text{ter},l}t + \left(v_{yo} - v_{\text{ter},l}\right)\tau_l\left(1 - e^{-t/\tau_l}\right)$$

This is <u>with the *y* axis pointing downward</u>!

Group Work: problem 2.11, but don't take b to the limit.

Next two classes:

- Friday – Linear Air Resistance (contd. And simulation)
- Monday – Quadratic Air Resistance

Next two classes:

- Wednesday – Air Resistance
- Friday – Linear Air Resistance