

Mon. 1/30	Ch 3 – the rest	
Wed. 2/1	Quiz Ch 3, Lab 3: AC Circuits	HW3: Ch3 Pr 2,6,7*, 8*, 11,12,16
Thurs. 2/2	more of the same	
Fri. 2/3	Ch 4.2-.3, Ch 5.1-.2: Transformers, Power Supplies, & Semiconductors	Lab 3 Notebook

Equipment

- Before class – write at top of board the basics for L, C, and R inductance & reactance
- O'Scope
- Fourier Transform box
- Speaker
- Necessary cables and connectors
- <http://www.falstad.com/circuit/>

Topics:

AC terminology for sinusoidal signals:

1. Sizes of voltages and currents: ANY of these can be used in Ohm's Law
 - a. Peak value (amplitude)
 - b. Peak-to-peak value (twice the amplitude)
 - c. Root mean square (rms) – related to average power dissipated
2. Period, frequency, & angular frequency
3. Phase – leading and lagging

Fourier decomposition – only need to figure out response of circuits to sine waves

Reactance & Inductance – using Ohm's Law & Kirchoff's Rules for AC circuits

Phases of components - ELI the ICE man (emf is voltage)

Phasors – similar to vectors, represent size and phase of inductances

Calculations for various filters

Study List for Quiz #3:

1. Sine waves – amplitude, frequency, angular frequency, period, and phase.
2. Reactances and impedances of R, L, and C - complex number (j-operator) representation.
3. Using the phasor representation to calculate the magnitudes and phase angles of voltages and currents.

Equation List:

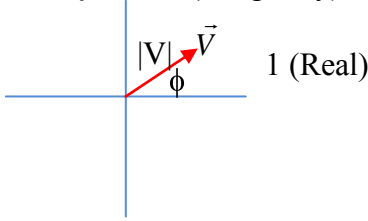
$$\begin{array}{lll}
 |X_C| = 1/\omega C = 1/2\pi f C & Z_C = -j/\omega C = -j/2\pi f C \quad [1/(\text{Hz} \cdot \text{F}) = 1\Omega] & j = \sqrt{-1} \\
 |X_L| = \omega L = 2\pi f L & Z_L = j\omega L = j2\pi f L \quad [1\text{Hz} \cdot \text{H} = 1\Omega] & \omega = 2\pi f \\
 V = \sqrt{(V_{\text{real}})^2 + (V_{\text{imaginary}})^2} & \phi = \tan^{-1}(V_{\text{imaginary}}/V_{\text{real}}) & \tilde{V} = V \angle \phi \\
 \tilde{A} \times \tilde{B} = A \times B \angle (\phi_A + \phi_B) & \tilde{A} / \tilde{B} = A / B \angle (\phi_A - \phi_B) & \tilde{v} = \tilde{i} \tilde{Z}
 \end{array}$$

From Last Time

Ch 3 AC Circuits: Impedances

Phasors

$j = \sqrt{-1}$ (Imaginary)



$$\vec{V} = V_R + V_I j = |V| \cos \phi + |V| j \sin \phi = |V| (\cos \phi + j \sin \phi) \equiv |V| e^{j\phi}$$

$$|V| = \sqrt{V_R^2 + V_I^2} \quad \phi = \tan^{-1} \left(\frac{V_I}{V_R} \right)$$

With this definition of a phasor, the algebraic rules easily relate to rules you're familiar with.

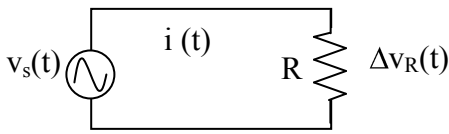
Say, $\vec{A} = A_R + A_I j$, $\vec{B} = B_R + B_I j$

Then, $\vec{A} + \vec{B} = (A_R + A_I j) + (B_R + B_I j) = (A_R + B_R) + (A_I + B_I) j$

Alternatively, for multiplication and division, it's easiest to use the other representation:

$$\vec{A} \cdot \vec{B} = |A| e^{j\phi_A} |B| e^{j\phi_B} = |A||B| e^{j\phi_A} e^{j\phi_B} = |A||B| e^{j(\phi_A + \phi_B)}$$

Resistor



$$i_s(t) = i_o \sin(\omega t)$$

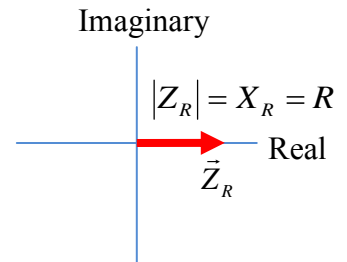
$$\Delta v_R(t) = -\Delta V_{R,p} \sin(\omega t)$$

$$\frac{\Delta V_{R,p}}{I_p} = R$$

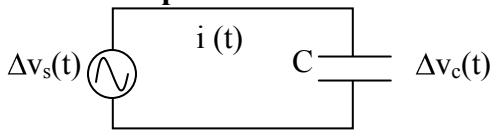
$$\vec{i}_s = i_o$$

$$\Delta \vec{v}_R = -\Delta V_{R,p}$$

$$\vec{Z}_R = R$$



Capacitive Reactance



$$i(t) = i_o \sin(\omega t)$$

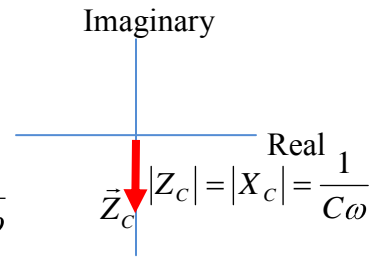
$$\Delta v_C(t) = -\Delta V_{C,p} \sin(\omega t - \frac{\pi}{2})$$

$$\frac{\Delta V_{C,p}}{I_p} = \frac{1}{C\omega} \equiv |X_C|$$

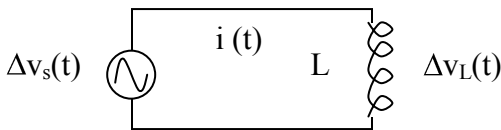
$$\vec{i}_s = i_o$$

$$\Delta \vec{v}_R = -\Delta V_{R,p} e^{-j\frac{\pi}{2}}$$

$$\vec{Z}_c = X_c e^{-j\frac{\pi}{2}} = -\frac{j}{C\omega}$$



Inductive Reactance



$$i(t) = i_o \sin(\omega t)$$

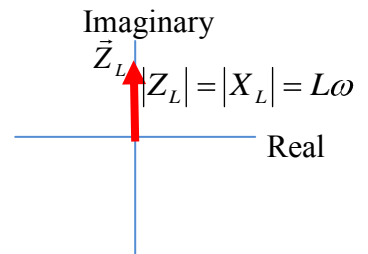
$$\Delta v_L(t) = -\Delta V_{L,p} \sin(\omega t + \frac{\pi}{2})$$

$$\frac{\Delta V_{L,p}}{I_p} = L\omega \equiv |X_L|$$

$$\vec{i}_s = i_o$$

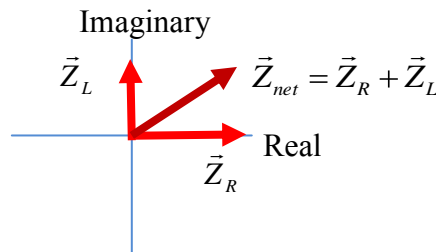
$$\Delta \vec{v}_R = -\Delta V_{R,p} e^{j\frac{\pi}{2}}$$

$$\vec{Z}_L = X_L e^{j\frac{\pi}{2}} = jL\omega$$



Combinations

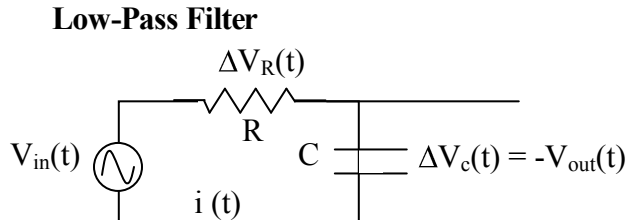
If you have a circuit with, say an inductor *and* a resistor in series, then the net impedance would simply be the vector sum of the two:



This Time

Putting it all to use:

I'll do two examples. The first one I'll fully narrate, so you can follow what's going on, but it'll look much longer than it needs to. The second one, I'll just crunch through, so you'll get a better sense of how much work is involved.



Say, $v_{in}(t) = v_{in} \sin(\omega t)$
What's $v_{out}(t) = v_{out} \sin(\omega t + \phi_{out})$?

So, we've got our physical / tangible / measurable properties that we're interested in.

- **Demo.** Before analyzing it theoretically, let's look at a simulation to get a sense of the behavior our math should model: <http://www.falstad.com/circuit/> select **Circuits** –

Passive Filters – Low-Pass RC

- Notice that as the input *frequency* increases the output *amplitude* decreases. That makes sense, for times much less than $\tau = RC$, the capacitor hardly charges up – hardly any voltage across it, current passes right across it. For *high* frequency, the period, $T \ll \tau$.
- The relative phase of the two signals is a bit harder to see; for low frequencies, it looks pretty much the same; for high frequencies it's hard to tell.

Now for the math. The easiest way to go about determining the amplitude and phase of the output is to go off to mathland and use the powerful tool of phasors. The phasors corresponding to the properties we're interested in are

Relevant Phasors

$$\vec{v}_{in} = V_{in} e^{j0} = v_{in}$$

$$\vec{v}_{out} = V_{out} e^{j\phi_{out}}$$

$$\vec{Z}_R = \text{Re}^{j0} = R$$

$$\vec{Z}_C = \frac{1}{C\omega} e^{-j\frac{\pi}{2}} = -j \frac{1}{C\omega}$$

$$\vec{i} = i e^{j\phi_i}$$

Solution

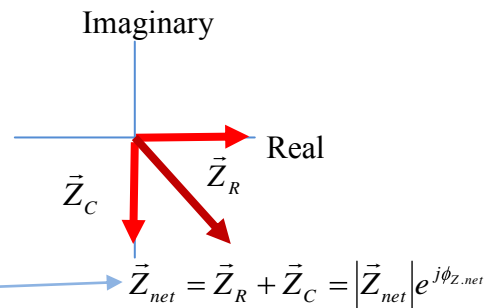
Kirchoff's Loop rule gives us

$$\vec{v}_{out} = -\Delta\vec{v}_C = \Delta\vec{v}_R + \vec{v}_{in}$$

In terms of currents and reactances that is

$$\vec{v}_{out} = -\left(\vec{i}\vec{Z}_C\right) = -\vec{i}\vec{Z}_R + \vec{v}_{in} \Rightarrow \vec{i} = \frac{\vec{v}_{in}}{\vec{Z}_R + \vec{Z}_C} \Rightarrow \vec{Z}_{net} = \vec{Z}_R + \vec{Z}_C = |\vec{Z}_{net}| e^{j\phi_{Z.net}}$$

Plugging this relation for the current's phasor back into $\vec{v}_{out} = \vec{i}\vec{Z}_C$ gives



$$\vec{v}_{out} = \frac{\vec{v}_{in}}{\vec{Z}_R + \vec{Z}_C} \vec{Z}_C = \frac{\vec{v}_{in}}{\vec{Z}_R / \vec{Z}_C + 1}$$

AC Voltage Divider

Notice that this is the same reasoning we'd employ and result that we'd get for simple 2-resistor voltage divider. In general, the logic of impedances is the same as that of resistances, so analyzing a circuit of resistors, capacitors, and inductors is as simple as analyzing the circuit where all elements are resistors.

Well, it was pretty simple solving for the output's phasor; now we've got to try to manipulate it into the form $Ve^{j\phi}$ so we can read off the magnitude and phase angle (the two things we're interested in.)

Plugging in $\vec{v}_{in} = V_{in}$, $\vec{Z}_R = R$, and $\vec{Z}_C = -j\frac{1}{C\omega}$ we have

$$\vec{v}_{out} = \frac{\vec{v}_{in}}{\vec{Z}_R / \vec{Z}_C + 1} = \frac{V_{in}}{-jRC\omega + 1}$$

The denominator conveniently has the real and imaginary terms nicely separated. It's of the form $\vec{D} = D_{real} + jD_{imaginary}$. So, I can use pathagorean's theorem to find *its* magnitude

$\phi = \sqrt{D_{real}^2 + D_{imaginary}^2}$ and use arctan to find its phase angle $\phi_D = \tan^{-1} \frac{D_{imaginary}}{D_{real}}$. Here goes:

$$\vec{v}_{out} = \frac{V_{in}}{-jRC\omega + 1} = \frac{V_{in}}{\left(\sqrt{RC\omega^2 + 1}\right) e^{j \tan^{-1}(-RC\omega)}} = \frac{V_{in}}{\left(\sqrt{RC\omega^2 + 1}\right)} e^{-j \tan^{-1}(-RC\omega)}$$

In the last step, I made use of the fact that $\frac{1}{e^x} = e^{-x}$.

We're essentially done. To make it obvious, I'll write the left hand side in exponential form.

$$\vec{v}_{out} = V_{out} e^{j\phi_{out}} = \frac{V_{in}}{\left(\sqrt{RC\omega^2 + 1}\right)} e^{-j \tan^{-1}(-RC\omega)}$$

Clearly,

$$V_{out} = \frac{V_{in}}{\left(\sqrt{RC\omega^2 + 1}\right)} \text{ and } \phi_{out} = -\tan^{-1}(-RC\omega)$$

Though we went off to the Complex Plane to calculate these, they are the very real and measurable amplitude and phase that appear in $v_{out}(t) = V_{out} \sin(\omega t + \phi_{out})$.

Note: the book gets a different expression for the angle; I believe that you can convince yourself they are equivalent. Here's how one can use my approach to get the book's expression for the angle.

$$\vec{v}_{out} = \frac{\vec{v}_{in}}{\vec{Z}_R + \vec{Z}_C} \vec{Z}_C = \frac{v_{in}}{R - j\frac{1}{C\omega}} \frac{1}{C\omega} e^{-j\frac{\pi}{2}} = \frac{v_{in}}{\sqrt{R^2 + \left(\frac{1}{C\omega}\right)^2}} \frac{1}{C\omega} e^{-j\frac{\pi}{2}} = \frac{v_{in}}{\sqrt{RC\omega^2 + 1}} e^{j\left(-\tan^{-1}\left(\frac{1}{C\omega R}\right) - \frac{\pi}{2}\right)}$$

Analysis

Now that we've got a result in hand, let's get familiar with it. In particular, note the frequency dependence of the amplitude (and the phase angle, for that matter). So, input signals of different frequencies will generate output signals of different amplitudes / strengths.

Qualitatively, this makes perfect sense – we're dealing with a capacitor after all, and if a capacitor is charged and discharged with very low frequency, it has plenty of time to accumulate charge and thus develop a large potential difference across it; on the other hand, if it's very quickly charged and discharged (high frequency), it doesn't have time to accumulate much charge and little charge separation across the capacitor means little voltage across it.

Quantitatively, that's exactly what we see.

$V_{out} = \frac{V_{in}}{\sqrt{RC\omega^2 + 1}}$ is the largest when the frequency is the lowest / it's the smallest when

frequency is the highest. So, this block of circuitry “passes” low frequency signals and “filters out” high frequency signals. An application would be in a stereo speaker. You only want to send the low frequency signals to the woofer. So the signal sent from the stereo's amp to the speaker gets run through a “low pass filter” on its way to the woofer. They're also commonly tacked onto the outputs of electronic devices – if you know that the signal your device will output shouldn't exceed a certain frequency, then you can safely run it out through a “low-pass” filter that dampens out any higher frequency noise.

A handy way of looking at how v_{out} and v_{in} are related is looking at their ratio, a.k.a., the “voltage gain.”

$$Gain_v \equiv \frac{V_{out}}{V_{in}} = \frac{1}{\sqrt{RC\omega^2 + 1}} \text{ . (this is what the book calls “A”)}$$

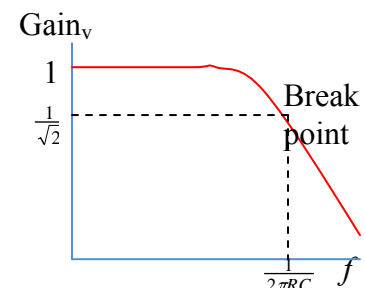
Break-Point Frequency, f_B

The expression for gain has two limiting approximations for low and high frequencies:

For **low frequencies**, $RC\omega \ll 1$, $Gain_v \approx \frac{1}{\sqrt{0+1}} = 1$

For **high frequencies**, $RC\omega \gg 1$, $Gain_v \approx \frac{1}{\sqrt{RC\omega^2}} = \frac{1}{RC\omega}$ which

goes to 0 for high frequency.



So, frequency that defines the threshold between the two, i.e., between gain of 1 and of 0 for which gain ‘breaks’ away from 1, satisfies

$$RC\omega_B = 1 \Rightarrow \omega_B = \frac{1}{RC}, \text{ or } f_B = \frac{1}{2\pi RC}.$$

Plugging this back into the gain expression gives $Gain_{v,B} = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}$.

So, on a plot of $Gain_v$ vs. f , the “Break-Point” is $\langle Gain_v, f \rangle = \langle \frac{1}{\sqrt{2}}, \frac{1}{2\pi RC} \rangle$.

In lab, you’ll make a plot like this and identify the “Break-Point” frequency experimentally by observing the frequency for which the gain drops to $\frac{1}{\sqrt{2}}$.

This point is also sometimes rather loosely referred as the “3dB” point. The book’s explanation for this name is verges on incorrect: the *voltage gain* is not -3dB, the *power gain* is -3dB.

What’s a dB?

A “Bel” is a logarithmic unit, it’s essentially a measure of the ‘order of magnitude’ of a number.

$$YBel = \log_{10} X$$

So, the bel is literally the order of magnitude. For example $\log_{10} 100 = \log_{10} 10^2 = 2Bel$

Often, that’s too gross a measure for what one’s interested in, instead we use deci-bels which are $1/10^{\text{th}}$ of a Bel (just like deci-meters are $1/10^{\text{th}}$ of a meter). So, 2Bel would be 20 dB.

$$\log_{10} 100 = \log_{10} 10^2 = 2Bel = 20dB$$

Why is the break point called the “3dB” point?

Recall that $P = IV = V^2/R$. So the *power gain* is $Gain_p = \frac{P_{out}}{P_{in}} = \frac{V_{out}^2}{V_{in}^2} = \left(\frac{V_{out}}{V_{in}} \right)^2 = Gain_v^2$

In dB’s, the break-point *Power gain* is then $\log(Gain_{v,B}^2) = \log\left(\frac{1}{2}\right) \approx 0.3Bel = 3dB$.

Generalization for other maximum gains

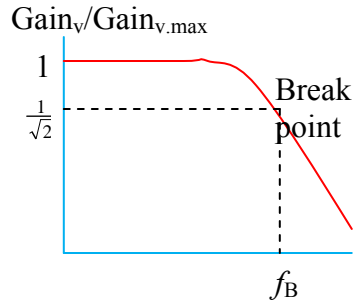
In this particular circuit, the maximum gain is 1. When we get to Operational Amplifier circuits, you can set the maximum gain; maybe it’s 2, maybe it’s 1000. Though it’s not generally considered a desirable feature, the gains of those circuits also have frequency dependence. For

them, one often plots the ratio of the gain to the maximal / desired gain, $\frac{Gain_v(\omega)}{Gain_{v,max}}$, against

frequency. The plot looks quite similar: flat over a range of frequencies, and then a drop off. In this context, the “break point”, a.k.a., “3dB” point, is again used as the representative threshold between the two (flat and dropping) segments of the plot. Now, the “break point” is where this

ratio of voltage gains is $\frac{Gain_v(\omega_B)}{Gain_{v,max}} = \frac{1}{\sqrt{2}}$, and the ratio of the power gains is $\frac{Gain_p(\omega_B)}{Gain_{p,max}} = \frac{1}{2}$,

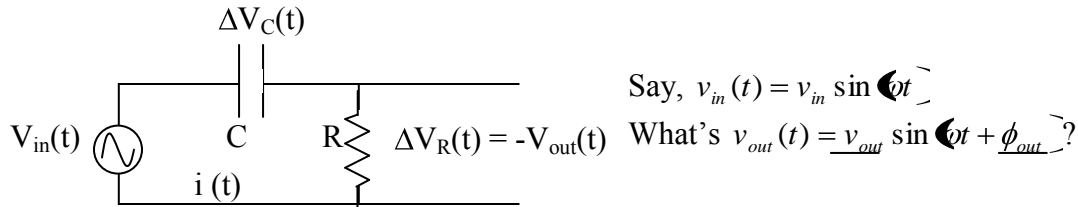
which, in terms of deci-bels is $\approx 3dB$.



The bottom line is that, the idea of a “break point” or “3dB point” is more generally the threshold between flat / frequency-*insensitive* response and frequency-*dependent* response.

High-Pass Filter

Now, let’s look back at the same circuit, but across the resistor rather than the capacitor. Since all the tools and tricks are essentially the same as for the Low-Pass filter, I’ll skip the narrative and just show the work.



To find the amplitude and phase angle, I’ll work with the phasors that correspond to the voltages, reactances, and current.

$$\text{Now } \vec{v}_{out} = -\Delta\vec{v}_R = \vec{v}_{in} + \Delta\vec{v}_C$$

$$-\vec{i}\vec{Z}_R = \vec{v}_{in} - \vec{i}\vec{Z}_C \Rightarrow \vec{i} = \frac{\vec{v}_{in}}{\vec{Z}_R + \vec{Z}_C} \text{ just as before (it is the same circuit after all)}$$

$$\vec{v}_{out} = -\Delta\vec{v}_R = \vec{i}\vec{Z}_R = \frac{\vec{v}_{in}}{\vec{Z}_R + \vec{Z}_C} \vec{Z}_R \quad \text{Again, this is the same form as for a voltage divider.}$$

$$v_{out} e^{j\phi_{out}} = \frac{\vec{v}_{in}}{1 + \vec{Z}_C / \vec{Z}_R} = \frac{v_{in}}{1 - \frac{j}{Rc\omega}} = \frac{v_{in}}{\sqrt{1 + \left(\frac{1}{Rc\omega}\right)^2}} e^{j \tan^{-1}\left(\frac{1}{Rc\omega}\right)}$$

$$v_{out} e^{j\phi_{out}} = \frac{v_{in}}{\sqrt{1 + \left(\frac{1}{Rc\omega}\right)^2}} e^{j \left(\tan^{-1}\left(\frac{1}{Rc\omega}\right)\right)}$$

Comparing the left and right hand sides, apparently $v_{out} = \frac{v_{in}}{\sqrt{1 + \left(\frac{1}{Rc\omega}\right)^2}}$ and $\phi_{out} = -\tan^{-1}\left(\frac{1}{Rc\omega}\right)$.

Coming back from phasor-land, these are the amplitude and phase angles that appear in the expression for the real voltage: $v_{out}(t) = v_{out} \sin(\omega t + \phi_{out})$.

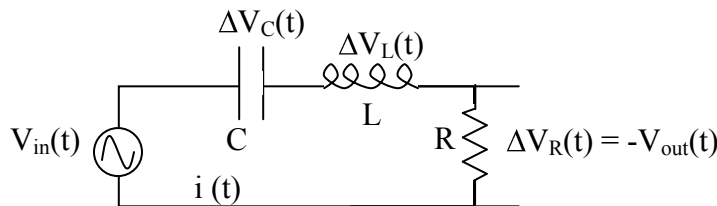
Looking at the limits, voltage gain starts at 0 for low frequencies and climbs to 1 at high frequencies. This is a *high-pass* filter. You could use this in a stereo speaker box to select only the high frequency signals to go to the tweeter. Another application – when the o’scope works in AC mode, it uses a high-pass filter to filter out any DC offset.

- **Demo.** Now that we've analyzed it theoretically, let's look at a simulation to get a sense of the behavior our math is modeling: <http://www.falstad.com/circuit/> select **Circuits – Passive Filters – High-Pass RC**
 - Notice that as the input *frequency* increases the output *amplitude* does indeed increase. As for phase, it's easy to see that at low frequencies the phase is about 90° off, while for higher frequencies it's hard to see.

3.1 RLC Resonant Circuit

Now here's a fun one. Say we have all three circuit elements in series, an R, a C, *and* an L.

Qualitative Introduction: You're already qualitatively familiar with this circuit. We've noted that the differential equation that describes the current through this circuit is of the same form as the differential equation that describes the displacement of a mass on a spring when it's subject to drag. In lab you saw that if you gave the system a good “slap” (hit it with a square wave), then it oscillated at its natural frequency and the oscillations slowly died away. This is exactly what happens if you slap a mass on a spring. Sticking with the mass-on-a-spring analogy, think about hitting it repeatedly. If you slap it at some random frequency, not much will happen, but if you slap it at its natural frequency, the oscillation grows, and grows, ... you've got resonance! We'll see the same kind of behavior is predicted for an RLC circuit – “drive” it with an oscillating input voltage at the natural frequency of the circuit, and the output voltage gets big.



As usual, say that $v_{in}(t) = v_{in} \sin(\omega t)$, What's $v_{out}(t) = v_{out} \sin(\omega t + \phi_{out})$?

We'll go off to phasor-land to figure out the amplitude and phase shift for the output voltage.

I'll make use of the following specific relations:

$$\vec{v}_{in} = v_{in} e^{j\phi_{in}} = v_{in} e^{j0} = v_{in}$$

$$\vec{v}_{out} = v_{out} e^{j\phi_{out}}$$

$$\vec{Z}_R = R$$

$$\vec{Z}_C = j \frac{-1}{C\omega}$$

$$\vec{Z}_L = jL\omega$$

As well as the general relation for any vector in the complex plane:

$$\vec{A} = A_{\text{Real}} + jA_{\text{Imaginary}} = |A|e^{j\phi_A} \text{ where } |A| = \sqrt{A_{\text{real}}^2 + A_{\text{imaginary}}^2} \text{ and } \phi_A = \tan^{-1} \left(\frac{\text{Imaginary}}{\text{Real}} \right)$$

$$\vec{v}_{out} = \vec{v}_{in} + \Delta\vec{v}_C + \Delta\vec{v}_L$$

$$\vec{i}\vec{Z}_R = \vec{v}_{in} - \vec{i}\vec{Z}_C - \vec{i}\vec{Z}_L \Rightarrow \vec{i} = \frac{\vec{v}_{in}}{\vec{Z}_R + \vec{Z}_C + \vec{Z}_L}$$

This is essentially a demonstration that impedances in series add linearly just like resistances do.

$$\vec{v}_{out} = \frac{\vec{v}_{in}}{\vec{Z}_R + \vec{Z}_C + \vec{Z}_L} \vec{Z}_R = \frac{\vec{v}_{in}}{1 + \vec{Z}_C / \vec{Z}_R + \vec{Z}_L / \vec{Z}_R} = \frac{v_{in}}{1 + j\frac{-1}{RC\omega} + j\frac{L\omega}{R}} = \frac{v_{in}}{1 + j\left(\frac{L\omega}{R} - \frac{1}{RC\omega}\right)}$$

$$v_{out} e^{j\phi_{out}} = \frac{v_{in}}{\sqrt{1 + \left(\frac{L\omega}{R} - \frac{1}{RC\omega}\right)^2}} e^{j \tan^{-1}\left(\frac{L\omega}{R} - \frac{1}{RC\omega}\right)} = \frac{v_{in}}{\sqrt{1 + \left(\frac{L\omega}{R} - \frac{1}{RC\omega}\right)^2}} e^{j\left(-\tan^{-1}\left(\frac{L\omega}{R} - \frac{1}{RC\omega}\right)\right)}$$

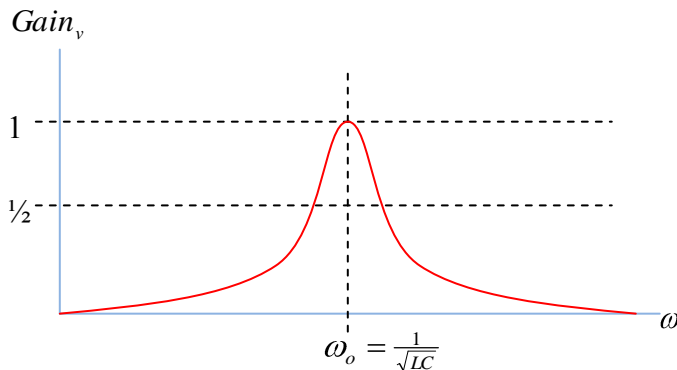
Looking at both the amplitude and the phase, something special happens when $\omega = \frac{1}{\sqrt{LC}}$.

Specifically, the phase shift goes away and the amplitude maximizes to v_{in} ... as if the capacitor and inductor weren't even there!

Functionally speaking, the voltage gain, $Gain_v = \frac{v_{out}}{v_{in}} = \frac{1}{\sqrt{1 + \left(\frac{L\omega}{R} - \frac{1}{RC\omega}\right)^2}}$, has a local

maximum at this frequency. It has two minima as well: plug in $\omega = 0$ or ∞ , and you get $Gain_v = 0$. So, without getting terribly quantitative about it, a plot of

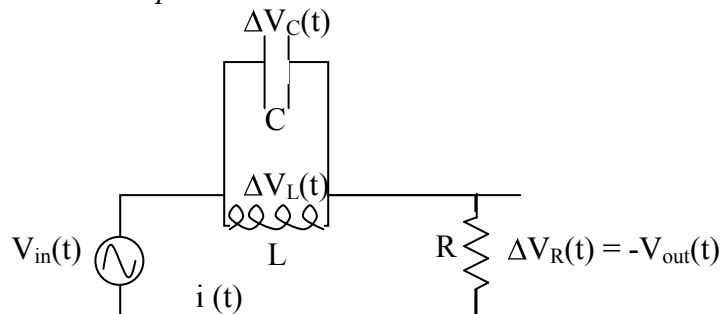
A plot of the voltage gain vs. the frequency looks something like



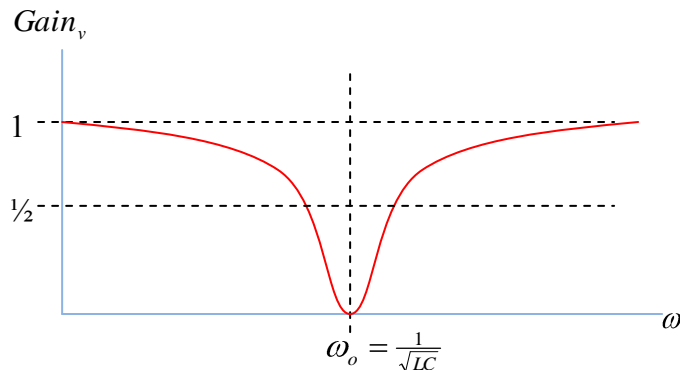
We can get just a little more specific and talk about how “sharp” the peak is. A common measure of that is the peak’s “full width at half max,” $\Delta\omega$. The expression for this is rather messy, but, it gets smaller the smaller $R\sqrt{\frac{C}{L}}$ gets, or, inversely, the bigger $\frac{1}{R}\sqrt{\frac{L}{C}}$ gets. So, the sharpness or “quality” of the peak is tied to this combination of the constants. Since $\omega_o = \frac{1}{\sqrt{LC}}$, and $X_{L_o} = L\omega_o$, this “quality factor” can be rephrased as $Q \equiv X_{L_o} / R$; the bigger it is, the sharper the peak is.

Notch Filter

So, if you put a capacitor and an inductor in series, then the only signal they pass is that which is *both* too high for the capacitor to block *and* too low for the inductor to block. The result is that signals of only a narrow range of frequencies squeak through. Now, what if you put the capacitor and inductor in *parallel* instead?



In that case, low frequencies could propagate across the inductor without being seriously impeded and high frequencies could propagate across the capacitor without being seriously impeded, but *medium* frequencies have trouble either way. The result is called a “Notch Filter.”



The math's fairly straight-forward. Impedances in series add like resistances in series, and impedances in parallel add like resistances in parallel,

$$\frac{1}{\vec{Z}_{LC}} = \frac{1}{\vec{Z}_L} + \frac{1}{\vec{Z}_C} = \frac{1}{j\omega L} + \frac{1}{-j} = \left(\frac{1}{j\omega L} - \frac{\omega C}{j} \right) = \left(\frac{\frac{1}{\omega L} - \omega C}{j} \right) \Rightarrow \vec{Z}_{LC} = \frac{j}{\frac{1}{\omega L} - \omega C}$$

Then Kirchhoff's loop gives us

$$v_{in} = -\Delta v_{LC} - \Delta v_R \quad \text{and} \quad v_{out} = -\Delta v_R$$

thus

$$Gain_v = \left| \frac{v_{out}}{v_{in}} \right| = \left| \frac{-\Delta v_R}{-j\omega L + \Delta v_R} \right| = \left| \frac{iR}{j\omega L + iR} \right| = \left| \frac{R}{j\omega L + R} \right| = \frac{R}{\left| \frac{j}{\omega L} + R \right|} = \frac{R}{\sqrt{\left(\frac{1}{\omega L} - \omega C \right)^2 + R^2}}$$

Notice what happens for $\omega_o = \frac{1}{\sqrt{CL}}$, the denominator blows up, so the whole gain dies. That's the "notch" in the plot; signals of these frequency don't get passed through the circuit.

4-5 Complex Waveforms & Fourier Transforms

So, we've learned a bit about how Inductors, Capacitors, and Resistors (and combinations thereof) effect simple sinusoidal inputs. What about not-so-simple inputs? Fourier's Theorem gives us a way to address that. It states (and can be proved, but we won't bother) that any mathematical function can be expressed as a sum of cosines and sines of different amplitudes and frequencies. In the most general case, the sum is an integral, but in the simpler case, where the function is repetitive, i.e., repeats itself with some frequency, ω_o , then the sum is following discrete sum

$$v(t) = \frac{A_o}{2} + \sum_{n=1}^{\infty} \left(A_n \cos(n\omega_o t) + B_n \sin(n\omega_o t) \right)$$

Of course, since $\cos(n\omega_o t) = \sin(n\omega_o t + \frac{\pi}{2})$, this could be rewritten as

$$v(t) = \frac{A_o}{2} + \sum_{n=1}^{\infty} \left(A_n \sin(n\omega_o t + \frac{\pi}{2}) + B_n \sin(n\omega_o t) \right)$$

As a side note, the book gives the coefficients for repeating square, triangle, and sawtooth shapes (some of the common functions that a function generator er, um, generates.) What's important for us in the context of R's, C's, and L's, is that any complex function is equivalent to the sum of simple sinusoidal functions, and the output of, say a low pass filter would look like the sum of outputs of hat low pass filter for each input sinusoidal function. More specifically, say the input is of the form

$$v_{in}(t) = \sum_{n=1}^{\infty} \left(v_{inA_n} \sin(n\omega_o t + \frac{\pi}{2}) + v_{inB_n} \sin(n\omega_o t) \right)$$

Then the output would be of the form

$$v_{out}(t) = \sum_{n=1}^{\infty} \left(v_{outA_n} \sin(n\omega_o t + \frac{\pi}{2} + \phi_n) + v_{outB_n} \sin(n\omega_o t + \phi_n) \right)$$

where

$$v_{outA_n} = \frac{v_{inA_n}}{\left(\sqrt{RCn\omega_o^2 + 1} \right)}, \quad v_{outB_n} = \frac{v_{inB_n}}{\left(\sqrt{RCn\omega_o^2 + 1} \right)} \quad \text{and} \quad \phi_n = -\tan^{-1}(RC\omega_n)$$