

Chapter 11: EXPONENTIAL CURVE FITTING

11.1 INTRODUCTION

Many processes in nature have exponential dependences. The decay with time of the amplitude of a pendulum swinging in air, the decrease in time of the temperature of an object that is initially warmer than its surroundings, and the growth in time of an initially small bacterial colony are all processes that are well-modeled by exponential relationships.

To better consider the issues involved in dealing with such relationships, let's consider a very specific case. The absorption of radiation by a given thickness of some material can be modeled by the following simple exponential relationship:

$$R(x) = R_0 e^{\beta x} \quad (11.1)$$

Here $R(x)$ is the *count rate* of radiation particles (typically measured as the number of clicks on a Geiger counter that take place in some fixed time such as one minute), R_0 is the count rate with no shielding present, x is the thickness of the shielding material, and β is a negative constant that describes how rapidly the count rate decreases as the shielding thickness increases.

Some measurements on the rate at which radiation particles emitted by ^{55}Fe are detected when a Geiger counter is shielded one or more thin sheets of aluminum foil appear in Table 11.1 and Figure 11.1. Note that the count rate decreases as the thickness of the aluminum shielding increases. (Note also that the error bars in this case are just barely large enough to be visible.)

| Al thickness (cm) | Count rate (counts/min) |
|-------------------|-------------------------|
| 0.00162 | 1850 |
| 0.00324 | 1250 |
| 0.00486 | 800 |
| 0.00648 | 450 |
| 0.0081 | 350 |
| 0.00972 | 165 |

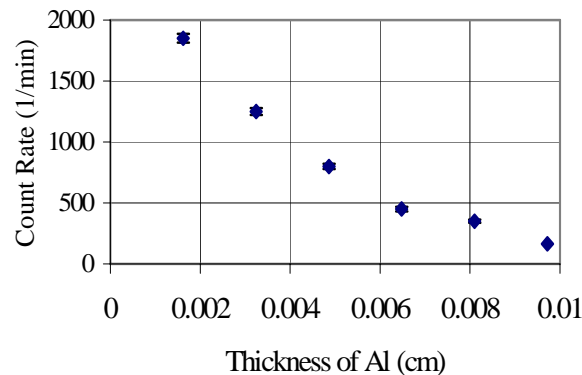


Figure 11.1: Count rate of radiation particles vs. thickness of aluminum shielding

While the count rate is clearly *not* a linear function of the shielding thickness x , you could not (just by looking at the graph) tell the difference between the exponential dependence of equation 11.1 and certain power laws. Finding the value of the constant β would be difficult as well. You *could* try different values of β and R_0 , calculate an $R(x)$ curve for each pair of values, and then see which pair best matches your experimental data, but this approach would clearly be very tedious. Exponential curve fitting, like power-law fitting, is a good example of a technique in which linearization *would* work if you already knew the exponent – but you don't.

11.2 LINEARIZING EXPONENTIAL RELATIONSHIPS

Fortunately, a better strategy exists. If we take the (natural) logarithm of both sides of equation 11.1, we get

$$\ln R(x) = \ln R_0 + \beta x \Rightarrow y(x) = \beta x + \ln R_0 \quad (11.2)$$

if we define $y(x) \equiv \ln R(x)$. This is the equation of a straight line. Therefore, if you graph $\ln R(x)$ vs. x , you should wind up with a straight line. Figure 11.2 shows such a graph.

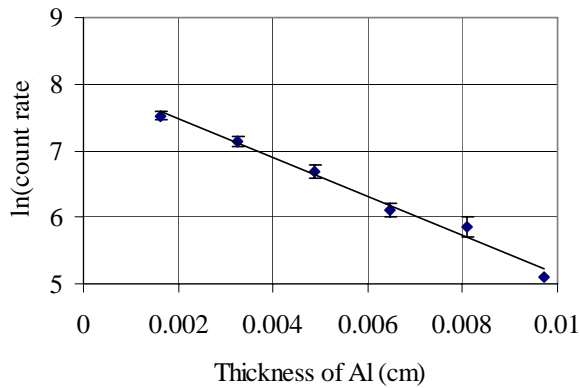


Figure 11.2: Plot of the natural log of the count rate vs. aluminum thickness

Furthermore, the slope of this line is β , the value of the constant in the original exponential equation. You find the value of β by calculating the slope in the usual way. That is,

$$\beta = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\ln R(x_2) - \ln R(x_1)}{x_2 - x_1} \quad (11.3)$$

Now, the data graphed above don't lie perfectly on a straight line, which is the result of experimental uncertainty. The line, however, looks like a good approximation to the data. The best fit line, however, seems to pass through the points (0 cm, 8.0) and (0.01 cm, 5.1). [Remember that, strictly speaking, both the y values should have the unit terms of

$\ln(\text{counts/minute})$ added, but those terms will cancel when you do the subtraction.] The slope of this line is

$$\beta = \frac{5.1 - 8.0}{0.01 \text{ cm} - 0 \text{ cm}} = -290 \text{ cm}^{-1} \quad (11.4)$$

The point (0 cm, 8.0) where the line intersects the y axis, is (of course) the y intercept of the line. So the value of $y(0)$ is the constant $\ln R_0$ in equation 11.4. Therefore, R_0 is the antilog (exponential) of 8.0, or 2980 counts/minute. Therefore, we can use the $\ln R$ vs. x graph to determine both the constants in equation 11.2; that equation now reads:

$$R(x) = (2980 \text{ counts/min})e^{\beta x}, \text{ where } \beta = -290 \text{ cm}^{-1} \quad (11.5)$$

In the expression above, x must be measured in cm, since the value of β has units of cm^{-1} . If you wanted to measure x in mm instead, β would have to have the value -29 mm^{-1} .

11.3 CREATING SEMI-LOG GRAPHS USING LINREG

One can easily create a semi-log graph using the program *LinReg*: simply enter the basic data into the data table and then check the “Show Ln(Vertical Data)” box. The program will then calculate and plot the natural logarithms (and the uncertainties in those logarithms) of the measurements displayed on your vertical axis. (See Chapter 8 for a more detailed description of *LinReg*.)

11.4 SEMI-LOG GRAPH PAPER

People have also invented special graph paper that can be used to create semi-log plots without the help of a calculator or computer. This paper is called **semi-log paper**, and we have included a sample semi-log plot of the radiation shielding data of Table 11.1 in Figure 11.5, and a blank sheet of semi-log paper at the end of this chapter.

Fundamentally, the vertical axis on a piece of semi-log graph paper shows the values of y corresponding to an implicit linear scale for $\ln y$. To illustrate this, we have drawn the implicit linear scale on the right edge of Figure 11.5. Therefore when we plot a point on the graph according to the value of y shown on the left hand scale, we are really implicitly plotting the value of $\ln y$ displayed on the right hand scale. This makes it easy to plot measured values directly without having to compute the logarithms.

It is a property of such logarithmic scales that the *pattern* of y -marks on the left-hand scale can be made to repeat exactly after y increases by a power of 10. The printers of semi-log paper do repeat the pattern of marks for each power of 10, and make the paper more flexible by not specifying the power of 10 that you will use at the bottom edge of the axis. (The numerical labels also simply repeat for each power of 10.) You adapt the paper to the range of your own data by relabeling the 1's and 10's on the axis by a sequence of increasing powers of 10 that span your particular data. This is illustrated in Figure 11.5.

It really doesn't matter what power of 10 you assign to the bottom edge of the axis. Changing the definition of the power of 10 at the bottom simply shifts the logarithmic scale up or down. Since when we compute the slope we are only really interested in the *differences* between logarithmic values, shifting the logarithmic scale up or down doesn't affect anything.

11.5 A PROCEDURE FOR EXPLORING EXPONENTIAL RELATIONSHIPS

Semi-log graphs are most useful when you suspect (for one reason or another) your data has an exponential dependence of the form $y = ke^{\beta x}$. To test your suspicion, do the following:

1. Plot a low-level semi-log graph of your data either using semi-log paper or by calculating values of $\ln y$ and plotting your results using Cartesian graph paper. If the resulting graph is in either case a pretty good straight line, this reinforces your suspicion that the data might reflect an exponential relationship, and it is worth continuing to the next step.
2. Find the constant multiplier k , by extrapolating your best fit line back to $x = 0$ and reading either the value $\ln k$ off the vertical axis (if you used Cartesian graph paper) or the value of k itself (if you used semi-log graph paper).
3. Calculate the value of β from the slope $\Delta(\ln y) / \Delta x$ as discussed in section 11.2. (If you used semi-log graph paper, you will have to compute $\ln y$ by hand for the two points that you use to define the slope.)
4. Finally, refine these values for intercept and slope by entering the data into *LinReg*. Be sure to check your results against what you found using your low-level plot.

EXERCISES

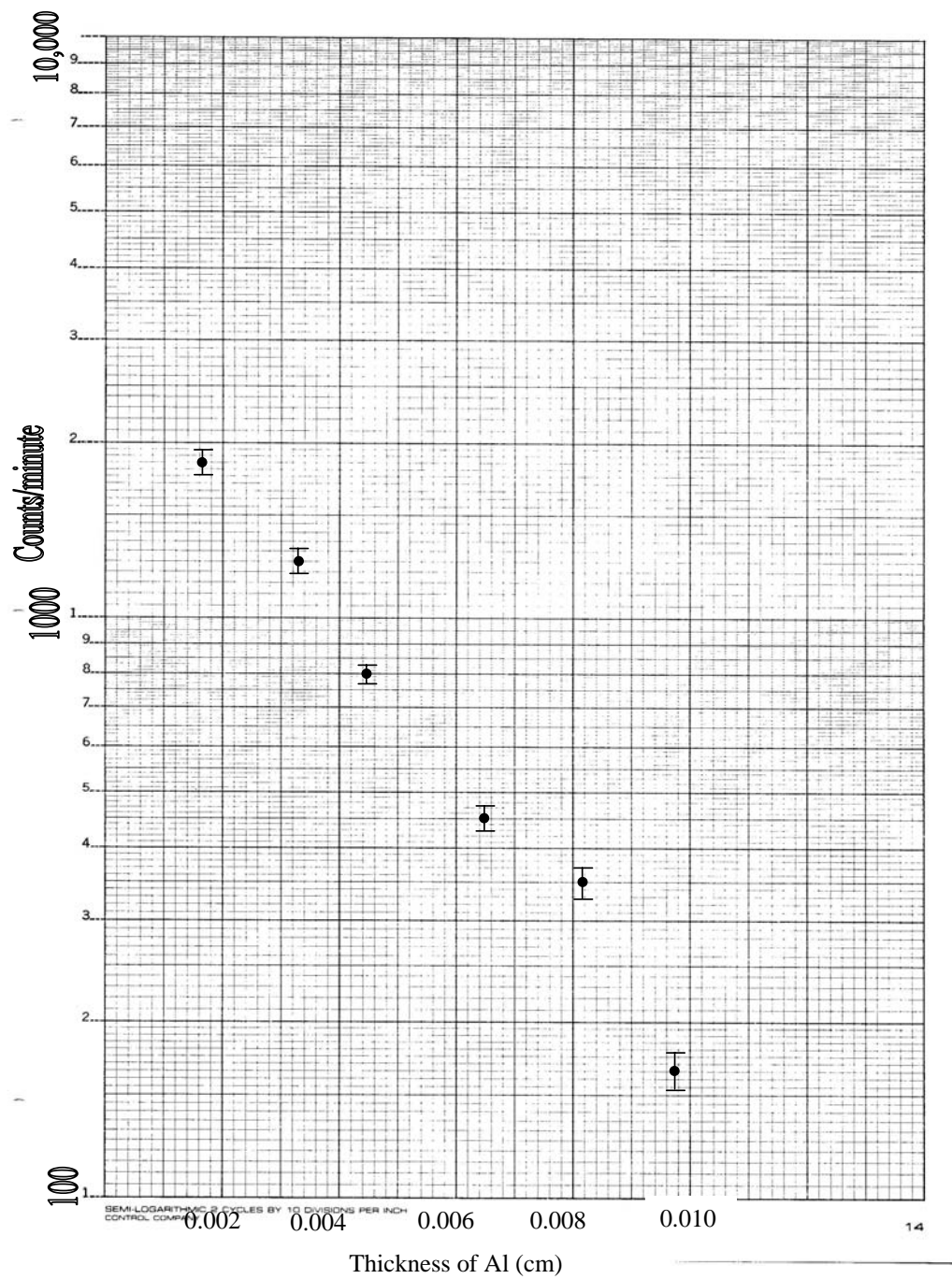
Exercise 11.1

Verify that if $\beta = 290 \text{ cm}^{-1}$, then β also equals 29 mm^{-1} , as claimed below equation 11.7.

Exercise 11.2

Figure 11.5 on the next page shows a semi-log graph of the radiation data, with error bars. Draw the line that you think best fits the data, and then find values of β and R_0 from your line. Since Figure 11.5 is larger than Figure 11.2, your new estimates of β and R_0 will probably be better than the earlier estimates summarized in equation 11.7. [Hint: You will have to calculate $\ln R$ for two points on the line to accurately determine the slope, but you can read R_0 right from the diagram.]

Figure 11.5: Semi-log plot of data from Table 11.1.



Exercise 11.3

On the blank semi-log paper provided in Figure 11.6, plot the data given in the table to the right. Determine whether this data seems to reflect an exponential relationship

$N = N_0 e^{\beta t}$, and if so, find the values of β and N_0 that best fit this data from both graphs. Also, plot in your lab notebook a graph of $\ln N$ versus t on ordinary graph paper and do the same analysis. (You can use equation 9.16 in Chapter 9 of this manual to compute the uncertainty in $\ln N$.)

| time t (min) | Number of bacteria N |
|----------------|------------------------|
| 10 | $149,000 \pm 15,000$ |
| 20 | $215,000 \pm 20,000$ |
| 30 | $335,000 \pm 35,000$ |
| 40 | $477,000 \pm 45,000$ |
| 50 | $769,000 \pm 75,000$ |

Figure 11.6: Blank piece of 3-cycle semi-log paper

