Thurs., 3/26 Fri., 3/27	**Quiz** Ch 21, **Lab 9** Ampere's Law **22.1-2,10** Intro to Faraday's & Lenz's	 **RE26**
Mon., 3/30 Tues., 3/31	**22.3-4,.7** Faraday & Emf & Inductance	**RE27** **Lab 9 Write-up HW21:**RQ.12, 15,17; P.20, 23, 26

Equipment:

Inductance: Function Generator, nestable solenoids, oscilloscope.
VPython

Intro: Faraday's Law as Correlation, not Causation

- **Contextualizing Faraday's law**
  - **Stationary Charges.** In the first chapter of this book, we met the electric interaction of two *stationary* charges: $\vec{F}_{1\leftarrow 2} = \frac{1}{4pe_o}\frac{q_1 q_2}{r^3_{1\leftarrow 2}}\vec{r}_{1\leftarrow 2}$, for convenience sake, we broke this into two factors, q_1 and $\vec{E}_2(\vec{r}_1) = \frac{1}{4pe_o}\frac{q_2}{r^3_{1\leftarrow 2}}\vec{r}_{1\leftarrow 2}$. That allowed us to focus on the "electric field" of a source charge without having to worry about the other charge.

Look up the master force equation from the back of Griffiths

Moving Charges. A few chapters later, we considered two *moving* charges. The easiest way to transition from stationary to moving is to first analyze the situation in the frame of one of the two charges (where it appears to be stationary), and then transform into a frame in which it's moving. What you get has two distinct terms: $\vec{F}_{1\leftarrow 2} = \frac{q_1 q_2 |r|}{4pe_0(\vec{r}\cdot(c\hat{r}-\vec{v}_2))^3}\left\{\left[(c^2-v_2^2)(c\hat{r}-\vec{v}_2)\right]+\frac{\vec{v}_1}{c}\times\left[\hat{r}\times(c^2-v_2^2)(c\hat{r}-\vec{v}_2)\right]\right\}$

  - Where $\vec{r} = \vec{r}_1(t_{now}) - \vec{r}_2(t_{then})$, that is, the separation between the probe charge and where the source charge was back when it emitted the fields that the probe is experiencing now.

  - The first term in curly brackets depends only on the "source's " velocity, while the second term depends on the velocity of the test charge too. Identifying the first as an *electric* interaction and the second as a *magnetic* interaction, we have
  - $\vec{F}_{1\leftarrow 2} = q_1\vec{E}_2(\vec{r}_1) + q_1\vec{v}_1\times\vec{B}_2(\vec{r}_1)$
  - Doing some rephrasing, in terms of current locations of both charges gives

$$\vec{E}_2(\vec{r}_1) = \frac{1}{4pe_0}\left(\frac{1-(v_2/c)^2}{\left(1-(\hat{r}\times\vec{v}_2/c)^2\right)^{3/2}}\right)\frac{q_2}{r_{1\leftarrow 2}^{\ 2}}\hat{r}_{1\leftarrow 2}$$

$$\vec{B}_2(\vec{r}_1) = \frac{m_o}{4p}\left(\frac{1-(v_2/c)^2}{\left(1-(\hat{r}\times\vec{v}_2/c)^2\right)^{3/2}}\right)\frac{q_2}{r_{1\leftarrow 2}^{\ 2}}(\vec{v}_2\times\hat{r}_{1\leftarrow 2})$$

- **Accelerating Charges.** So, we started with stationary charges, then went for moving charges; the next natural step is *accelerating* charges. Now you get

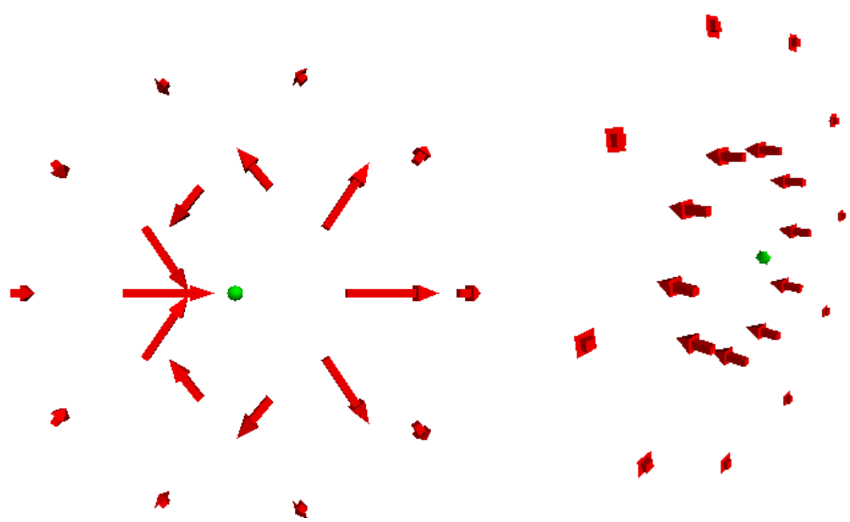
$$\vec{F}_{1\leftarrow 2} = \frac{q_1 q_2 |r|}{4\pi\epsilon_0 \left(\vec{r}\cdot(c\hat{r}-\vec{v}_2)\right)^3}\left\{\left[(c^2-v_2^2)(c\hat{r}-\vec{v}_2)+\vec{r}\times((c\hat{r}-\vec{v}_2)\times\vec{a}_2)\right]+\frac{\vec{v}_1}{c}\times\left[\hat{r}\times\left[(c^2-v_2^2)(c\hat{r}-\vec{v}_2)+\vec{r}\times((c\hat{r}-\vec{v}_2)\times\vec{a}_2)\right]\right]\right\}$$

(Griffiths 10.67)

Again, the first clump of terms does not depend on the test charge's velocity, and the second does. So the first set is the *electric* interaction and the second is the *magnetic* interaction. Notice that the *electric* interaction includes a term pointing in a direction that is distinctly *not* along r (since it's got an r×a in it). Furthermore, the first term points from where the source charge *would now be* if it were going at a constant speed, but not from where it actually is (since its' accelerating). So there are explicit and implicit dependences on the acceleration of the charge.

Notice that we have one term of the electric field that points radially (our good, old "Coulombic" field) but we *also* have a term that points in some direction dependent on the *acceleration* of the sources. *This* is the Faraday field.

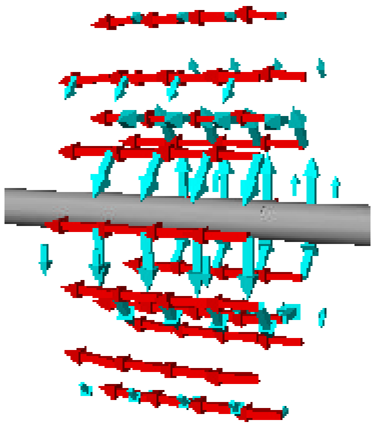
Here's what it looks like for just one accelerating charge.



Generalize to currents

- Now, if we have whole currents of charges, then these relations get rephrased, and most generally in terms of current densities and time varying current densities (instead of velocities and accelerations). It is the latter, time varying current densities that are responsible for the Faraday Electric Field, a.k.a., the "non-coulombic" field.

  Here's what it looks like for a current of accelerating charges. (the current is moving and accelerating to the right).



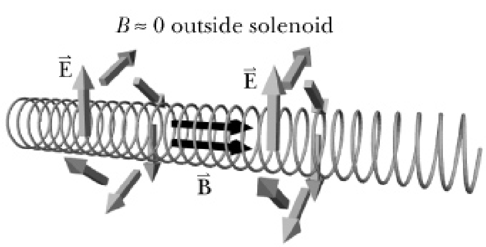
Notice that this electric field *opposes* the change in current. This is a very general characteristic of the Faraday Electric Field.

Contradicting book

- The book, along with most introductory texts, slips and says that the "non-coulombic" electric field is *produced* by the time varying magnetic field. That may well have been how Faraday interpreted what he observed, but over a century has passed and we've learned a few things – both this electric field *and* a time varying magnetic field are produced by the same thing – time varying current densities.
- Faraday's Law is undeniably correct, but it does not represent a *causal* relation, it represents a correlation between two effects of a shared cause.

Getting Familiar with Faraday's Electric Field

The simplest example for seeing the Faraday effect is a solenoid with a changing current. The magnetic field points along the axis inside the solenoid and is approximately zero outside as shown below.



Let's say a counter clockwise circulating current increases, the referencing the illustration of an accelerated current's field, we should get a *clockwise* electric field.

Induction. Meanwhile, if the counter clockwise current is increasing, then the magnetic field it produces is also increasing. So, correlated to the curled electric field is the increasing magnetic field. The common language is to say that the changing magnetic field "induces" the curled electric field, and this process is called *magnetic induction*. Unfortunately, this is historic, but inaccurate language (much like "heat flow") – it's the changing current density that causes both magnetic and electric effects. It is sometimes called a non-Coulomb electric field $\vec{E}_{NC}$.

That said, the correlation between the fields is far simpler to express than is the causation between the currents and the fields.

Direction of the Curly Electric Field:

The right hand rule is similar to the one used to determine the direction of the magnetic field produced by a current:

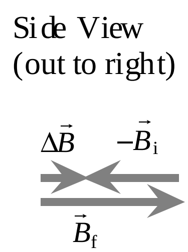
> With your right thumb pointed in the direction of $-d\vec{B}/dt$, your fingers will curl in the direction of $\vec{E}_{NC}$. This is the opposite of the direction of the change in the current density that produces the change in magnetic field.

To use this rule, it is useful to know that the direction of the change in the magnetic field for a small time $\Delta\vec{B}$ is the same direction as $d\vec{B}/dt$.

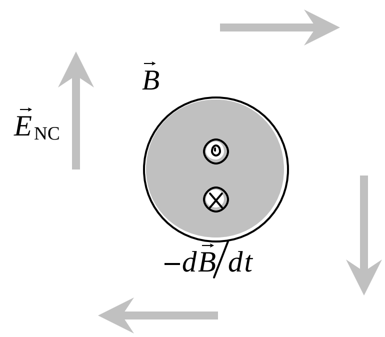
Examples:

(1) $\vec{B}$ is out and increasing

The direction of the change in the magnetic field $\Delta\vec{B} = \vec{B}_f - \vec{B}_i$ is outward as shown below.

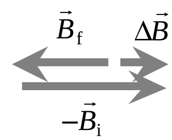


The direction of $-d\vec{B}/dt$ is inward (because of the minus sign), so $\vec{E}_{NC}$ curls clockwise.



(2) $\vec{B}$ is in and decreasing (Exercise for students)

The direction of the change in the magnetic field $\Delta\vec{B} = \vec{B}_f - \vec{B}_i$ is outward as shown below.



The direction of $-d\vec{B}/dt$ is inward (because of the minus sign), so $\vec{E}_{NC}$ curls clockwise.

Faraday's Law:

Magnetic flux is defined as:

$$\Phi_{mag} = \int \vec{B} \cdot d\vec{A},$$

We already encountered this in Ch 21; however then we were thinking of the flux out through a closed surface (like a ball), and that summed to 0. Now we're just considering the flux through an open surface (like sheet of paper).

The induced emf for a closed loop is defined as:

$$\text{emf}_F = \oint \vec{E}_{NC} \cdot d\vec{\ell}.$$

Faraday's law: The induced emf for a closed loop is equal to the rate of change of the magnetic flux on the area enclosed by the loop,

$$\text{emf}_F = -\left.\frac{\partial \Phi_{mag}}{\partial t}\right|_A.$$

Notation: the subscript "F" indicates that we're just talking about the emf due to Faraday's effect and the "A" indicates that we're holding Area constant. We'll later see that holding B constant and allowing A to vary also gives rise to an emf – what we've already come to call "motional" emf.

The minus sign is to remind you how to determine the direction of $\vec{E}_{\text{NC}}$ from $-d\vec{B}/dt$.

This is a new law, which cannot be derived from what *we* have done previously. Mind, to the extent that all magnetic phenomenon can be derived from electric ones and the proper relativistic transformations, this *can* be reasoned (I've seen a number of papers do so); however, while that is, on some philosophical level, comforting, practically, it's too much of a pain to do.

Example: A loop around a solenoid

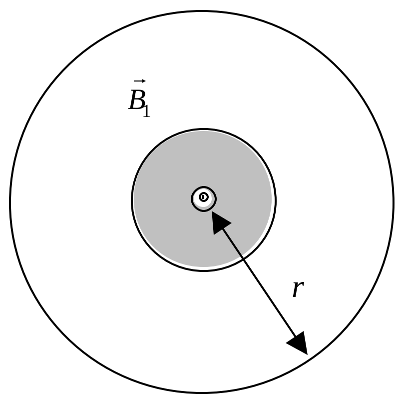
Suppose the solenoid has a radius r_1 and $\vec{B}_1$ of the solenoid is out and increasing at a constant rate $d\vec{B}_1/dt$. Consider a loop of radius r centered on the solenoids axis. By symmetry, the non-Coulomb electric field $\vec{E}_{\text{NC}}$ must be the same size everywhere on the loop, so:

$$\text{emf} = \oint \vec{E}_{\text{NC}} \cdot d\vec{\ell} = E_{\text{NC}}(2pr)$$

The flux on the surface surrounded by the loop depends on the size of the loop:

(1) $r > r_1$ – magnetic flux inside whole solenoid contributes, so:

$$\Phi_{\text{mag}} = B_1(pr_1^2)$$



In this case:

$$|\text{emf}| = \left|\frac{d\Phi_{\text{mag}}}{\text{dt}}\right|$$

$$E_{\text{NC}}(2pr) = \frac{dB_1}{dt}(pr_1^2)$$

$$E_{\text{NC}} = \frac{r_1^2(dB_1/dt)}{2r},$$

which falls off as 1/*r*.

(2) $r < r_1$ – only magnetic flux inside loop contributes, so:

$$\Phi_{\text{mag}} = B_1(pr^2)$$

In this case:

$$E_{\mathrm{NC}}(2pr) = \frac{dB_1}{dt}\left(pr^2\right)$$

$$E_{\mathrm{NC}} = \frac{r\left(dB_1/dt\right)}{2},$$

which increases as *r*.

Two Solenoids.

Returning to $\mathrm{emf}_F = \left.\frac{\partial \Phi_{\mathrm{mag}}}{\partial \mathrm{t}}\right|_A$, If we have just one loop, it has area A, and flux BA. If we have two loops, it has area 2A, and flux 2BA,… N loops means flux NBA, and corresponding

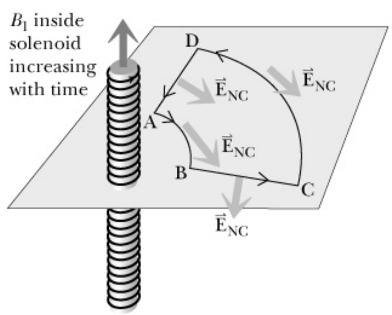
$$\mathrm{emf}_F = -N\left.\frac{\partial \Phi_{\mathrm{mag}}}{\partial \mathrm{t}}\right|_A$$

While the emf around one loop may be too small to measure. The emf along a coil of 500 loops, i.e., along a solenoid, may be quite measurable.

Demo: One solenoid nested inside another.

Loop outside solenoid

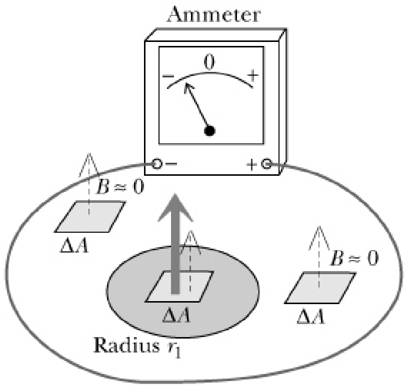
What about a loop that does not surround the solenoid with a changing current?

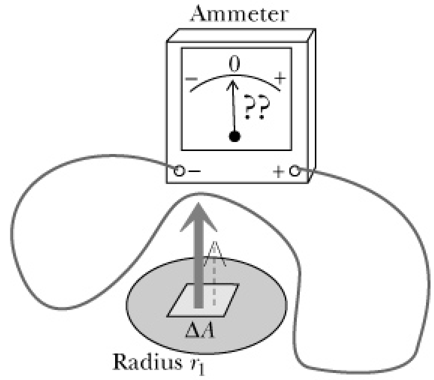


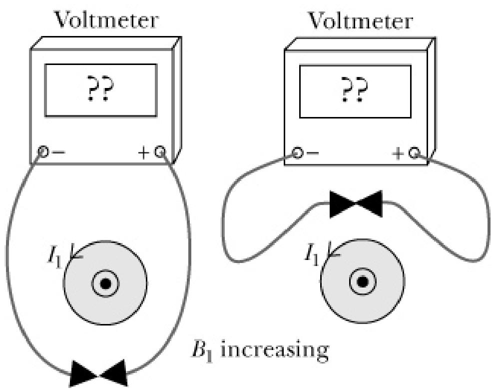
There are two ways to explain why the emf for the loop is zero:

- The magnetic field outside the solenoid is approximately zero, so there is no change in the magnetic flux as the current changes.
- The length of the arcs is proportional to *r* and the size of the non-Coulomb electric field $\vec{E}_{\mathrm{NC}}$ is proportional to 1/*r* (outside the solenoid). Notice that $\vec{E}_{\mathrm{NC}}$ points along $d\vec{\ell}$ for one arc and opposite to $d\vec{\ell}$ for the other arc. $\vec{E}_{\mathrm{NC}}$ is perpendicular to the straight segments.

The shape of the loop doesn't matter (any loop can be approximated as arcs and radial segments). For example, the readings on ammeters or voltmeters just depends on whether or not the loop is around the region where the magnetic flux is changing.


Ammeter
0
$B \approx 0$
ΔA
ΔA
$B \approx 0$
ΔA
Radius r_1


Ammeter
0
??
ΔA
Radius r_1


Voltmeter
??
I_1
Voltmeter
??
I_1
B_1 increasing

What follows are background notes for the instructor.

(Note: Griffiths illustrates the equivalence of the two approaches: using Faraday's law with retarded potentials and using Coulomb's law with relativistic transformations for the simple case of constant velocity. The derivation of Eq'n 10.68 uses V and A (and thus Faraday's) and retarded potential while the derivation of the same relation, now dubbed Eq'n 12.92 uses just Coulomb's and relativistic transformations.)

Prior to this chapter, we'd seen that an electric field is generated by charges. For a simple, stationary charge, Coulomb's law gives:

$$\vec{E}_o = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r_{o1}^{\ 2}}\hat{r}_{o1},$$

Where the *o* subscript stands for "observation location."

If we have a distribution of point charges, the field is found by the Superposition Principle:

$$\vec{E}_o = \sum_i \frac{1}{4\pi\varepsilon_0}\frac{q_{oi}}{r_{oi}^{\ 2}}\hat{r}_{oi} \Rightarrow \int \frac{1}{4\pi\varepsilon_0}\frac{\rho(r_q)}{r_{oq}^{\ 2}}\hat{r}_{oq}dVol$$

where ρ is the charge density, which is a function of position.

Later, we noted that the electric field due to a moving charge is a little different. One way to find this is to transform from the frame in which the charge is stationary to one in which it is moving at a constant speed v. Then, the electric field of a point charge sensed by the observer is

$$\vec{E}_o = \frac{1}{4\pi\varepsilon_0}\left(\frac{1-(v/c)^2}{\left(1-(v\sin\theta/c)^2\right)^{3/2}}\right)\frac{q_1}{r_{o1}^{\ 2}}\hat{r}_{o1} \quad \text{(Griffiths 3}^{\text{rd}}\text{ Ed. 12.92)}$$

where θ is the angle between v and r.

Note: one might be able to generalize this for an *accelerating* charge without appealing to General Relativity – by successively transforming to frames in which the charge is moving at incrementally different velocities.

If we have a distribution of point charges, each moving at its own velocity, I imagine that this would look like

$$\vec{E}_o = \int \frac{1}{4\pi\varepsilon_0}\left(\frac{1-(v_q/c)^2}{\left(1-(v_q\sin\theta_q/c)^2\right)^{3/2}}\right)\frac{\rho(r_q)}{r_{oq}^{\ 2}}\hat{r}_{oq}dVol$$

Whether the charge is moving or not, Gauss's law holds:

$$\oint \vec{E}\cdot\hat{n}\,dA = \frac{\sum q_{\text{inside}}}{e_0} \quad \text{or} \quad \text{div}\left(\vec{E}\right) = \vec{\nabla}\cdot\vec{E} = \frac{r}{e_0}$$

Interestingly, Griffith's also arrives at 12.92 by a *different* rout (labeling the equation 10.68). The two main features of this other rout are that it relates the electric field *now* to the charge distribution *back then*, in particular, back when the field was emitted that, traveling at the speed of light, gets to us now: $t_{then} = t_{now} - r/c$. But in this case, Coulomb's law is not enough, a new law / a new mechanism for generating electric field is required: Faraday's Law. One way of phrasing this law is

$$E_{Faraday}(t_{now}) = -\frac{1}{4pe_o}\int \frac{1}{c^2\left|r_{oq}\right|}\frac{d\vec{J}\left(r_q, t_{then}\right)}{dt}dVol \qquad \text{(from Griffiths 3}^{\text{rd}}\text{ 10.27, .28, .29)}$$

Or, putting it in slightly more familiar form:

$$\nabla\times E_{Faraday}(t_{now}) = -\nabla\times\left[\frac{1}{4pe_o}\int\frac{1}{c^2\left|r_{oq}\right|}\frac{d\vec{J}\left(r_q, t_{then}\right)}{dt}dVol\right]$$

$$\nabla\times E_{Faraday}(t_{now}) = -\frac{\partial}{\partial t}\left[\frac{1}{4pe_o c^2}\int\left[\frac{\vec{J}\left(r_q, t_{then}\right)}{r_{oq}{}^2} + \frac{1}{c\left|r_{oq}\right|}\frac{d\vec{J}\left(r_q, t_{then}\right)}{dt}\right]\times\hat{r}dVol\right]$$

$$\nabla\times E_{Faraday}(t_{now}) = -\frac{\partial\vec{B}(t_{now})}{\partial t}$$

For a point charge source, rather than a continuous current, this is

$$E_{Faraday}(t_{now}) = -\frac{1}{4pe_o}\frac{qc}{\left(\left|r\right|c - \vec{r}\cdot\vec{v}\right)^3}\left[\left(\left|r\right|c - \vec{r}\cdot\vec{v}\right)\left(\left|r\right|\vec{a}/c - \vec{v}\right) + \frac{\left|r\right|}{c}\left(c^2 - v^2 + \vec{r}\cdot\vec{a}\right)\vec{v}\right] \qquad (10.63)$$

For comparison, the Colombic term is

$$E_{Coul}(t_{now}) = -\frac{1}{4pe_o}\frac{qc}{\left(\left|r\right|c - \vec{r}\cdot\vec{v}\right)^3}\left[\left(\left|r\right|c - \vec{r}\cdot\vec{v}\right)\vec{v} - \left(c^2 - v^2 + \vec{r}\cdot\vec{a}\right)\vec{r}\right] \qquad (10.62)$$

So the total electric field of a moving point charge is their sum:

$$E(t_{now}) = \frac{1}{4pe_o}\frac{qc}{\left(\left|r\right|c - \vec{r}\cdot\vec{v}\right)^3}\left[\left(\left|r\right|c - \vec{r}\cdot\vec{v}\right)\vec{v} - \left(c^2 - v^2\right)\left(c\hat{r} - \vec{v}\right) + \vec{r}\times\left(\left(c\hat{r} - \vec{v}\right)\times\vec{a}\right)\right] \qquad (10.65,\ \text{w/ .64})$$

Which reduces to 12.92 / 10.68 if there is no acceleration.

This is a rare *causal* phrasing of Faraday's Law – that an electric field is generated by a changing current density. In essence, the *total* electric field depends not just on the charge distribution (Coulomb's Law), but how the *flow* of charges is *changing* (Faraday's Law).

Recall that current is the source of magnetic field. So this relationship between electric field and current *implies* one between electric field and time varying magnetic field. That is the much more common way of phrasing Faraday’s law, but, from a philosophical stand point, it’s worth noting that the relationship between the electric and time varying magnetic fields is a *correlation* while the one between electric field and time varying current is a *causation* – time varying magnetic field does not *cause* an electric field, rather, time varying current causes *both* electric and time varying magnetic fields.

It’s also interesting to note that, from the perspective of any one of the source charges, it itself is not moving, so it itself is exerting a force on another charge only via a coulombic electric field.