Calculus I: Derivatives and Smooth Airplane Take-off

A small airplane takes off from a level runway and climbs to an altitude of 1 mile, where it continues to fly in the same direction and at the same altitude. If we assume the airplane takes off in a certain direction, such as due east, and continues to fly in that direction, we can chart its flight path using *x*- and *y*-coordinates in a two-dimensional coordinate system, with *x* measuring the airplane's horizontal position, in miles, and *y* its vertical position, also in miles. Assuming the airplane leaves the ground at the point (0,0) and levels out at the point (1,1), let's try to find a function $y = f(x)$ that describes the flight path of the airplane.

1) (a) Graph the continuous piecewise linear function $f_1(x) = \left\{ \begin{array}{cc} 0, & x < 0 \\ x, & 0 \le x \le 1 \\ 1, & x > 1 \end{array} \right\}$ for $-1 \le x \le 2$. Your graph should contain the points (0,0) and (1,1).

   (b) Is the flight described by this graph smooth at the points (0,0) and (1,1)?

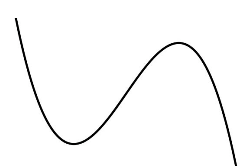
   (c) Does the derivative $f_1'(x)$ exist at the points (0,0) and (1,1)? If so, what is its value?

2) (a) Graph the continuous piecewise defined function $f_2(x) = \left\{ \begin{array}{cc} 0, & x < 0 \\ x^2, & 0 \le x \le 1 \\ 1, & x > 1 \end{array} \right\}$ for $-1 \le x \le 2$. Does this graph contain the points (0,0) and (1,1)?

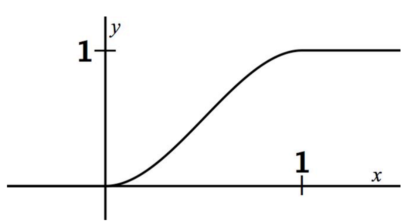
   (b) Is the flight described by this graph smooth at the points (0,0) and (1,1)?

   (c) Does the derivative $f_2'(x)$ exist at the points (0,0) and (1,1)? If so, what is its value?

3) In order to smooth out the flight at the point (1,1), we might try a piecewise defined function of the form $f_3(x) = \left\{ \begin{array}{cc} 0, & x < 0 \\ ax^3 + bx^2 + cx + d, & 0 \le x \le 1 \\ 1, & x > 1 \end{array} \right\}$. This is because cubic functions for which $a < 0$ have the general shape shown at right.



We want to select *a, b, c,* and *d* so that, for $-1 \le x \le 2$, the graph of $f_3(x)$ has the shape shown at left.



Since the graph of $f_3(x)$ must contain the points (0,0) and (1,1), we must have $f_3(0) = 0$ and $f_3(1) = 1$. In order to have $f_3(1) = 1$, we must have

$a \cdot 1^3 + b \cdot 1^2 + c \cdot 1 + d = 1$, which simplifies to $a + b + c + d = 1$. What does $f_3(0) = 0$, together with $f_3(x) = ax^3 + bx^2 + cx + d$ for $0 \le x \le 1$, tell you?

Fill in the following table for the function $f(x) = ax^3 + bx^2 + cx + d$ (defined for every *x,* or at least for $-1 < x < 2$), then use the four equations in the third column to solve for *a, b, c,* and *d*.

Flight requires …	$f(x) = ax^3 + bx^2 + cx + d$	Simplified equation
$f(0) = 0$		
$f(1) = 1$	$a \cdot 1^3 + b \cdot 1^2 + c \cdot 1 + d = 1$	$a + b + c + d = 1$
	$f'(x) =$	
$f'(0) =$		
$f'(1) =$		

Therefore, the function $f_3(x)$ describing the flight of the airplane is as follows:

$$f_3(x) = \begin{Bmatrix} 0, & x < 0 \\ \_\_x^3 + \_\_x^2 + \_\_x + \_\_, & 0 \le x \le 1 \\ 1, & x > 1 \end{Bmatrix} \text{ (fill in the blanks).}$$

Double-check your function $f_3(x)$:
Does the graph of $f_3(x)$ contain the points (0,0) and (1,1)?
Is the flight described by this graph smooth at the points (0,0) and (1,1)?
What is the value of the derivative $f_3'(x)$ at the points (0,0) and (1,1)?

4) Re-do Problem 3 for an airplane that takes off from the point (0,0) and reaches its cruising altitude of 1 mile at the point (2,1).

5) Re-do Problem 3 using the piecewise defined function

$$f_4'(x) = \begin{Bmatrix} 0, & x < 0 \\ ax^2, & 0 \le x < ½ \\ 1 - b(x-1)^2, & ½ \le x \le 1 \\ 1, & x > 1 \end{Bmatrix}.$$

Select *a* and *b* so that $f_4(x)$ is continuous at *x* = 0, *x* = ½, and *x* = 1, so that $f_4(½) = ½$, and so that the flight described by the graph is smooth at *x* = 0, *x* = ½, and *x* = 1.

6) Use a sine or cosine function instead of a cubic function to describe the flight path of the airplane for $0 \le x \le 1$. Which function has the smallest maximum slope for $0 \le x \le 1$, the cubic function from Problem 3, the piecewise quadratic function from Problem 5, or the sine (or cosine) function from this problem? Which function describes the most comfortable flight path for a pilot or passenger?

1) (c) To determine whether or not the derivative $f_1'(x)$ exists at the points (0,0) and (1,1), one can consider the *slopes* of the tangent lines at or near the points.

More formally, since $f_1(x)$ is continuous in a neighborhood and differentiable in a deleted neighborhood of $x = 0$, one can consider the limits

$\lim_{x\to 0^-} f_1'(x) = \lim_{x\to 0^-} 0 = 0$ and $\lim_{x\to 0^+} f_1'(x) = \lim_{x\to 0^+} 1 = 1$. Similarly for $x = 1$.

Even more formally, one can consider the limits

$$\lim_{h\to 0^-} \frac{f_1(0+h)-f_1(0)}{h} = \lim_{h\to 0^-} \frac{0-0}{h} = \lim_{h\to 0^-} 0 = 0 \text{ and}$$

$$\lim_{h\to 0^+} \frac{f_1(0+h)-f_1(0)}{h} = \lim_{h\to 0^+} \frac{h-0}{h} = \lim_{h\to 0^+} 1 = 1. \text{ Similarly for } x = 1.$$

The derivative at every other point on the graph is given by

$$f_1'(x) = \begin{Bmatrix} 0, & x<0 \\ 1, & 0<x<1 \\ 0, & x>1 \end{Bmatrix}.$$

2) (c) To determine whether or not the derivative $f_2'(x)$ exists at the points (0,0) and (1,1), one can consider the *slopes* of the tangent lines at or near the points.

More formally, since $f_2(x)$ is continuous in a neighborhood and differentiable in a deleted neighborhood of $x = 0$, one can consider the limits

$\lim_{x\to 0^-} f_2'(x) = \lim_{x\to 0^-} 0 = 0$ and $\lim_{x\to 0^+} f_2'(x) = \lim_{x\to 0^+} 2x = 0$. Similarly for $x = 1$.

Even more formally, one can consider the limits

$$\lim_{h\to 0^-} \frac{f_2(0+h)-f_2(0)}{h} = \lim_{h\to 0^-} \frac{0-0}{h} = \lim_{h\to 0^-} 0 = 0 \text{ and}$$

$$\lim_{h\to 0^+} \frac{f_2(0+h)-f_2(0)}{h} = \lim_{h\to 0^+} \frac{h^2-0}{h} = \lim_{h\to 0^+} h = 0. \text{ Similarly for } x = 1.$$

The derivative at every other point on the graph is given by

$$f_2'(x) = \begin{Bmatrix} 0, & x<0 \\ 2x, & 0<x<1 \\ 0, & x>1 \end{Bmatrix}.$$