Groups of Order 3 and 4

1. Use the "each element appears exactly once in each row and each column" property of group tables to fill in the group table for $G = \{e, a, b\}$, where e is the identity element for the group G. Since there is just one way to fill in this group table, there is essentially just one group of order 3 – that is, just one group with 3 elements. But wait! The set $\mathbf{Z}_3 = \{0, 1, 2\}$ under the operation of addition modulo 3 is a group with 3 elements. Write its group table. Which element in Z_3 plays the role of e in G? Which element plays the role of a? Of b?

*	***e***	***a***	***b***
e			
a			
b			

$+_3$	0	1	2
0			
1			
2			

2. Use the "each element appears exactly once in each row and each column" property of group tables to fill in the group table for $G = \{e, a, b, c\}$. Note that we can fill in the diagonal entry with the heavy border with e, b, or c. First, fill in this box with c and show how to fill in the rest of the table. Then, fill in the box with b and show how to fill in the rest of the table. Finally, fill in the box with e and show the <u>two</u> ways to fill in the rest of the table. (Hint: Note that there are two possibilities for the next diagonal entry.)

*	***e***	***a***	***b***	***c***
e	*e*	*a*	*b*	*c*
a	*a*			
b	*b*			
c	*c*			

*	***e***	***a***	***b***	***c***
e	*e*	*a*	*b*	*c*
a	*a*			
b	*b*			
c	*c*			

*	***e***	***a***	***b***	***c***
e	*e*	*a*	*b*	*c*
a	*a*			
b	*b*			
c	*c*			

*	***e***	***a***	***b***	***c***
e	*e*	*a*	*b*	*c*
a	*a*			
b	*b*			
c	*c*			

3. We have seen group tables for several groups of order 4, including the following groups. To which one of the four group tables from part 2 does each of these groups correspond?

   $U(5) = \mathbf{Z}_5^* = \{1, 2, 3, 4\}$ under multiplication modulo 5

   $U = \{1, -1, i, -i\}$ under ordinary multiplication

   $\mathbf{Z}_4 = \{0, 1, 2, 3\}$ under addition modulo 4

   Group of symmetries of the rectangle, $W = \{R_0, R_{180}, H, V\}$

4. To which one of the four group tables from part 2 does each of the following groups correspond?

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5. The groups in part 4 are, of course, the same group! How many nonisomorphic (= not of the same form) groups of order 4 are there? Which of your group tables from part 2 actually are of the same form?

6. What is an easy way to tell the two nonisomorphic groups of order 4 apart using their group tables?

Parts 1 and 2 are from *Laboratory Experiences in Group Theory,* by Ellen Parker.