Chapter 9: LINEARIZING A NON-LINEAR RELATIONSHIP

" 'It does not come to me in quite so direct a line as that; it takes a bend or two, but nothing of consequence.'"
--- *Persuasion*

9.1 INTRODUCTION

The purpose of many physics experiments is to explore how one measured quantity depends on another. For example, in this course, you will explore how the force exerted by the end of a spring depends on the length of the spring, how the period of a pendulum depends on its length, and so on. Research physicists at present explore things such as the temperature at which a material becomes superconducting depends on the strength of the magnetic field permeating the material, how the period of orbiting neutron stars depends on time, how the number of subatomic particles scattered from a certain kind of target depends on the angle of scattering, and so on.

The most useful way to present the results from an experiment like this is to plot a graph showing how one variable depends on the other. In some cases, the relationship between the variables is linear (meaning that the graph will show a straight line), as we have seen in several experiments in this course. The method of linear regression discussed in Chapter 8 provides a powerful tool for extracting information about such linear relationships.

But in many cases, the relationship between the experimental variables is *not* linear. As we will see in the next section, a straightforward graph of such a nonlinear relationship does not tell us very much, and we have no easy way to determine the "best fit" to a nonlinear graph the way that we do with a linear graph.

The purpose of this chapter is to open your mind to possible tricks for creating a linear graph of quantities that we expect to be related nonlinearly. If we can artificially create a linear graph of a normally nonlinear relationship, we can use linear regression to extract information about the relationship that would otherwise be hard to obtain.

9.2 AN EXAMPLE

For example, imagine an experiment where we want to determine an object's acceleration as it slides down a frictionless incline by measuring its displacement as a function of time. We will find in Unit *N* that such an object should experience a constant acceleration of magnitude $a = g\sin\theta$ (where θ is the angle the incline makes with the horizontal) directed down the incline. Let's define the x-axis to point down the incline, so that $a_x = +a$. If the object is released with zero initial velocity, then theoretically the displacement d of the object should be related to the time t since it was released as follows:

$$d = x(t) - x_0 = \tfrac{1}{2}a_x t^2 = \tfrac{1}{2}at^2 \qquad (9.1)$$

Now, if one were to draw a graph of *d* versus *t*, one would expect a parabola (as shown in Figure 9.1) instead of a straight line. This is fine, except that (particularly with the uncertainties shown) it is difficult to distinguish a graph of a $d \propto t^2$ relationship from one showing $d \propto t^3$ or $d \propto t^{4/3}$ or any one of many other relationships. Since there are a large number of relationships between *d* and *t* that could produce similar-looking curves, it is difficult to verify by just looking at the graph that our assumption that the object has constant acceleration is reasonable. Moreover, there is no simple way to compute the value of *a* from such a graph.

We can address both of these problems by plotting *d* as a function of t^2, not *t*. (See Figure 9.2.). In this case, if the object's acceleration really is constant, then the graph will be a straight line, and if the object's acceleration is *not* constant, the graph will be curved. Therefore, a mere glance at the graph helps us check whether our basic assumption is correct. If the graph does turn out to be a straight line, then the *slope* of such a graph, whose value and uncertainty can be easily determined using *LinReg*, will be *a*/2, making it easy to determine the value and uncertainty of *a*. (Note that according to Chapter 7, *a* will have the same percent uncertainty as *a*/2. One also can use the weakest-link rule discussed in that chapter to find the uncertainty in t^2 given the uncertainty in *t*. We need to know the uncertainty in t^2 to draw the uncertainty bars in Figure 9.2.)

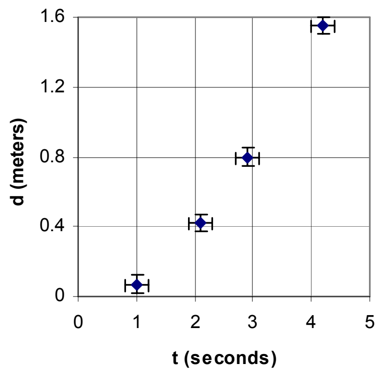

Figure 9.1: Graph of displacement vs. time for a hypothetical experiment involving an object sliding down a frictionless incline. Is the curve sketched by these data a parabola or not? What is the value of *a* consistent with these data?

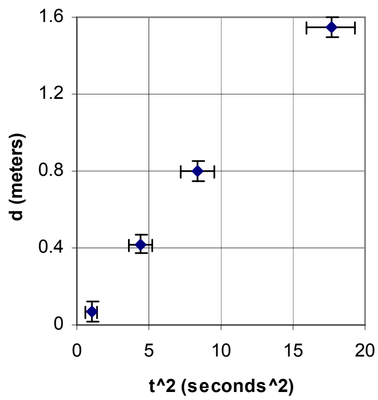

Figure 9.2: Graph of displacement vs. t^2 for the same experimental data. Note that the data now lie on a straight line, the slope and intercept of which can be calculated. (Note also that the uncertainties of the t^2 values must be calculated from the uncertainties of the corresponding *t* values.)

Alternatively, we could take the base-ten logarithm of both sides of equation 9.1, getting:

$$\log d = \log\left(\tfrac{1}{2}at^2\right) = \log\left(\tfrac{1}{2}a\right) + 2\log t \qquad (9.2)$$

If we were to plot log d versus log t, then we would get a straight line with slope = 2, which is the power to which t is raised in equation 1, and an intercept equal to $\log(a/2)$. This means that we can calculate the value and uncertainty of a from the value of the intercept and its uncertainty as calculated by *LinReg*. This is the ***power-law fitting*** technique discussed in more detail in Chapter 10.

The point here is that we have come up with two different methods of creating a linear graph of the inherently nonlinear relationship expressed by equation 9.1.This does not exhaust the possibilities by any means (we could plot $\sqrt{d}$ vs. t or d^2 versus t^4 for example), though the two methods discussed are the most natural in this case. Having a linear graph makes it possible interpret and analyze the experimental data more easily.

The only real cost is some added work in computing the size of uncertainty bars. If we were to plot d versus t^2 (for example), we have to compute the uncertainty of t^2 from the uncertainty in t using one of the methods described in Chapter 7. This is usually not much of a problem. In the case at hand, the weakest link rule tells us that the fractional uncertainty of t^2 is simply twice the fractional uncertainty of t.

9.3 A GENERAL APPROACH TO LINEARIZATION

So a general procedure for creating linear graphs of nonlinear relations might be summarized as follows. Assuming that you have a hypothesis about the nature of the relationship between the experimental quantities being graphed,

1. Find a function of one variable that when plotted against the other variable (or a function of that second variable) will yield a straight line *if* the hypothetical relationship is true.
2. Compute the uncertainties in the plotted quantities (if different from the measured quantities) using the appropriate method described in Chapter 7.
3. Draw a rough linearized graph of your experimental data to make sure that the plotted quantities really do seem to lie on a roughly straight line. (If not, try to determine whether your original hypothesis is correct or whether some of your measurements and/or calculations may have been in error.)
4. If the rough graph does look pretty linear, then enter the *plotted* quantities (with their uncertainties) into *LinReg*, and have it compute the slope and intercept of the best-fit line.
5. If the value of the slope and/or intercept of the graph are linked to interesting quantities, compute the value and uncertainty of any such quantities using the value and uncertainty of the slope or intercept computed by *LinReg*.

EXERCISES

Exercise 9.1
Consider the hypothetical incline experiment discussed above. If we determined the acceleration a for various angles θ, how might we display the relationship between these quantities in an appropriately linearized graph? How would the slope of your graph be related to the magnitude g of the gravitational field vector?

Exercise 9.2
Imagine you are doing an experiment to measure the magnitude of the electrostatic potential energy $V(r)$ between two charged objects as a function their separation r (see chapter C7). Describe at least *two* ways that you could create a graph that would display the expected relation between these quantities as a straight line.