22. False. Note that \( p \) is the density function for the population, not the cumulative density function. Thus \( p(10) = 1/2 \) means that the probability of \( x \) lying in a small interval of length \( \Delta x \) around \( x = 10 \) is about \( (1/2)\Delta x \).

23. True. This follows directly from the definition of the cumulative density function.

24. True. The interval from \( x = 9.98 \) to \( x = 10.04 \) has length 0.06. Assuming that the value of \( p(x) \) is near 1/2 for \( 9.98 < x < 10.04 \), the fraction of the population in that interval is \( \int_{9.98}^{10.04} p(x) \, dx \approx (1/2)(0.06) = 0.03 \).

25. False. Note that \( p \) is the density function for the population, not the cumulative density function. Thus \( p(10) = p(20) \) means that \( x \) values near 10 are as likely as \( x \) values near 20.

26. True. By the definition of the cumulative distribution function, \( P(20) - P(10) = 0 \) is the fraction of the population having \( x \) values between 10 and 20.

Solutions for Chapter 8 Review

Exercises

1. Vertical slices are circular. Horizontal slices would be similar to ellipses in cross-section, or at least ovals (a word derived from ovum, the Latin word for egg).

![Figure 8.139](image)

2. The limits of integration are 0 and \( b \), and the rectangle represents the region under the curve \( f(x) = h \) between these limits. Thus,

\[
\text{Area of rectangle} = \int_{0}^{b} h \, dx = hx \bigg|_{0}^{b} = hb.
\]

3. The circle \( x^2 + y^2 = r^2 \) cannot be expressed as a function \( y = f(x) \), since for every \( x \) with \(-r < x < r\), there are two corresponding \( y \) values on the circle. However, if we consider the top half of the circle only, as shown below, we have \( x^2 + y^2 = r^2 \), or \( y^2 = r^2 - x^2 \), and taking the positive square root, we have that \( y = \sqrt{r^2 - x^2} \) is the equation of the top semicircle.

![Equation](image)

Then

\[
\text{Area of Circle} = 2(\text{Area of semicircle}) = 2 \int_{-r}^{r} \sqrt{r^2 - x^2} \, dx
\]
We evaluate this using integral table formula 30.

\[ 2 \int_{x=-r}^{x=r} \sqrt{r^2 - x^2} \, dx = 2 \left[ \frac{1}{2} \left( x \sqrt{r^2 - x^2} + r^2 \arcsin \frac{x}{r} \right) \right]_{-r}^{r} \]

\[ = r^2 (\arcsin 1 - \arcsin (-1)) \]

\[ = r^2 \left( \frac{\pi}{2} - \left( -\frac{\pi}{2} \right) \right) = \pi r^2. \]

4. Name the slanted line \( y = f(x) \). Then the triangle is the region under the line \( y = f(x) \) and between the lines \( y = 0 \) and \( x = b \). Thus,

\[ \text{Area of triangle} = \int_{0}^{b} f(x) \, dx. \]

Since \( f(x) \) is a line of slope \( h/b \) which passes through the origin, its equation is \( f(x) = hx/b \). Thus,

\[ \text{Area of triangle} = \int_{0}^{b} \frac{hx}{b} \, dx = \frac{hx^2}{2b} \bigg|_{0}^{b} = \frac{hb^2}{2b} = \frac{hb}{2}. \]

5. We slice the region vertically. Each rotated slice is approximately a cylinder with radius \( y = x^2 + 1 \) and thickness \( \Delta x \). See Figure 8.140. The volume of a typical slice is \( \pi (x^2 + 1)^2 \Delta x \). The volume, \( V \), of the object is the sum of the volumes of the slices:

\[ V \approx \sum \pi (x^2 + 1)^2 \Delta x. \]

As \( \Delta x \to 0 \) we obtain an integral:

\[ V = \int_{0}^{4} \pi (x^2 + 1)^2 \, dx = \pi \int_{0}^{4} (x^4 + 2x^2 + 1) \, dx = \pi \left( \frac{x^5}{5} + \frac{2x^3}{3} + x \right) \bigg|_{0}^{4} = \frac{3772\pi}{15} = 790.006. \]

6. We slice the region vertically. Each rotated slice is approximately a cylinder with radius \( y = \sqrt{x} \) and thickness \( \Delta x \). See Figure 8.141. The volume of a typical slice is \( \pi (\sqrt{x})^2 \Delta x \). The volume, \( V \), of the object is the sum of the volumes of the slices:

\[ V \approx \sum \pi (\sqrt{x})^2 \Delta x. \]
As $\Delta x \to 0$ we obtain an integral.

$$V = \int_1^2 \pi (\sqrt{x})^2 \, dx = \pi \int_1^2 x \, dx = \pi \left( \frac{x^2}{2} \right)_1^2 = \frac{3\pi}{2} = 4.712.$$

7. We slice the region vertically. Each rotated slice is approximately a cylinder with radius $y = e^{-2x}$ and thickness $\Delta x$. See Figure 8.142. The volume of a typical slice is $\pi (e^{-2x})^2 \Delta x$. The volume, $V$, of the object is the sum of the volumes of the slices:

$$V \approx \sum \pi (e^{-2x})^2 \Delta x.$$

As $\Delta x \to 0$ we obtain an integral.

$$V = \int_0^1 \pi (e^{-2x})^2 \, dx = \pi \int_0^1 e^{-4x} \, dx = \pi \left( \frac{1}{4} \right) \left( e^{-4x} \right)_0^1 = -\frac{\pi}{4} (e^{-4} - 1) = 0.771.$$
8. We slice the region vertically. Each rotated slice is approximately a cylinder with radius \( y = 4 - x^2 \) and thickness \( \Delta x \). See Figure 8.143. The volume, \( V \), of a typical slice is \( \pi (4 - x^2)^2 \Delta x \). The volume of the object is the sum of the volumes of the slices:

\[
V \approx \sum \pi (4 - x^2)^2 \Delta x.
\]

As \( \Delta x \to 0 \) we obtain an integral. Since the region lies between \( x = -2 \) and \( x = 2 \), we have:

\[
V = \int_{-2}^{2} \pi (4 - x^2)^2 \, dx = \pi \int_{-2}^{2} (16 - 8x^2 + x^4) \, dx = \pi \left[ 16x - \frac{8x^3}{3} + \frac{x^5}{5} \right]_{-2}^{2} = \frac{512\pi}{15} = 107.233.
\]

9. We divide the region into vertical strips of thickness \( \Delta x \). As a slice is rotated about the \( x \)-axis, it creates a disk of radius \( r_{\text{out}} \) from which has been removed a smaller circular disk of inside radius \( r_{\text{in}} \). We see in Figure 8.144 that \( r_{\text{out}} = 2x \) and \( r_{\text{in}} = x \). Thus,

\[
\text{Volume of a slice } \approx \pi (r_{\text{out}})^2 \Delta x - \pi (r_{\text{in}})^2 \Delta x = \pi (2x)^2 \Delta x - \pi x^2 \Delta x.
\]

To find the total volume, \( V \), we integrate this quantity between \( x = 0 \) and \( x = 3 \):

\[
V = \int_{0}^{3} (\pi (2x)^2 - \pi x^2) \, dx = \pi \int_{0}^{3} (4x^2 - x^2) \, dx = \pi \int_{0}^{3} 3x^2 \, dx = \pi x^3 \bigg|_{0}^{3} = 27\pi = 84.823.
\]
10. The two functions intersect at \((0, 0)\) and \((8, 2)\). We slice the volume with planes perpendicular to the \(x\)-axis. This divides the solid into thin washers with

\[
\text{Volume of slice } = \pi r_{\text{out}}^2 \Delta x - \pi r_{\text{in}}^2 \Delta x.
\]

The inner radius is the vertical distance from the \(x\)-axis to the curve \(y = \frac{1}{4}x\). Similarly, the outer radius is the vertical distances from the \(x\)-axis to the curve \(y = \sqrt{x}\). Integrating from \(x = 0\) to \(x = 8\) we have

\[
V = \int_0^8 \left[ \pi (\sqrt{x})^2 - \pi \left(\frac{1}{4}x\right)^2 \right] \, dx.
\]

11. The region is bounded by \(y = 2\), the \(y\)-axis and \(y = x^{1/3}\). The two functions \(y = 2\) and \(y = x^{1/3}\) intersect at \((8, 2)\). We slice the volume with planes that are perpendicular to the \(y\)-axis. This divides the solid into thin cylinders with

\[
\text{Volume } \approx \pi r^2 \Delta y.
\]

The radius is the distance from the \(y\)-axis to the curve \(x = y^3\). Integrating from \(y = 0\) to \(y = 2\) we have

\[
V = \int_0^2 \pi (y^3)^2 \, dy.
\]

12. The region is bounded by \(y = 2\), the \(y\)-axis and \(y = x^{1/3}\). The two functions \(y = 2\) and \(y = x^{1/3}\) intersect at \((8, 2)\). We slice the volume with planes that are perpendicular to the line \(y = -2\). This divides the solid into thin washers with

\[
\text{Volume } \approx \pi r_{\text{out}}^2 \Delta x - \pi r_{\text{in}}^2 \Delta x.
\]

The inner radius is the distance from the line \(y = -2\) to the curve \(y = x^{1/3}\) and the outer radius is the distance from the line \(y = -2\) to the line \(y = 2\). Integrating from \(x = 0\) to \(x = 8\) we have

\[
V = \int_0^8 \left[ \pi (-2)^2 - \pi (x^{1/3} - (-2))^2 \right] \, dx.
\]

13. The region is bounded by \(x = 4y\), the \(x\)-axis and \(x = 8\). The two lines \(x = 4y\) and \(x = 8\) intersect at \((8, 2)\). We slice the volume with planes that are perpendicular to the line \(x = 10\). This divides the solid into thin washers with

\[
\text{Volume } \approx \pi r_{\text{out}}^2 dy - \pi r_{\text{in}}^2 dy.
\]

The inner radius is the distance from the line \(x = 10\) to the line \(x = 8\) and the outer radius is the distance from the line \(x = 10\) to the line \(x = 4y\). Integrating from \(y = 0\) to \(y = 2\) we have

\[
V = \int_0^2 \left[ \pi (10 - 4y)^2 - \pi (2)^2 \right] \, dy.
\]

14. The region is bounded by \(y = \frac{1}{4}x\), the \(x\)-axis and \(x = 8\). The two lines \(y = \frac{1}{4}x\) and \(x = 8\) intersect at \((8, 2)\). We slice the volume with planes that are perpendicular to the line \(y = 3\). This divides the solid into thin washers with

\[
\text{Volume } \approx \pi r_{\text{out}}^2 \Delta x - \pi r_{\text{in}}^2 \Delta x.
\]

The inner radius is the distance from the line \(y = 3\) to the line \(y = \frac{1}{4}x\) and the outer radius is the distance from the line \(y = 3\) to the \(x\)-axis. Integrating from \(x = 0\) to \(x = 8\) we have

\[
V = \int_0^8 \left[ \pi (1 - \frac{1}{4}x)^2 - \pi (3)^2 \right] \, dx.
\]

15. The region is bounded by \(x = 4x\) and \(x = y^3\). The two functions intersect at \((0, 0)\) and \((8, 2)\). We slice the volume with planes that are perpendicular to the line \(x = -3\). This divides the solid into thin washers with

\[
\text{Volume } = \pi r_{\text{out}}^2 \Delta y - \pi r_{\text{in}}^2 \Delta y.
\]

The inner radius is the distance from the line \(x = -3\) to the line \(x = y^3\) and the outer radius is the distance from the line \(x = -3\) to the line \(x = 4y\). Integrating from \(y = 0\) to \(y = 2\) we have

\[
V = \int_0^2 \left[ \pi (4y + 3)^2 - \pi (y^3 + 3)^2 \right] \, dx.
\]
16. Each slice is a circular disk. The radius, $r$, of the disk increases with $h$ and is given in the problem by $r = \sqrt{h}$. Thus

Volume of slice $\approx \pi r^2 \Delta h = \pi h \Delta h$.

Summing over all slices, we have

$$\text{Total volume} \approx \sum \pi h \Delta h.$$ 

Taking a limit as $\Delta h \to 0$, we get

$$\text{Total volume} = \lim_{\Delta h \to 0} \sum \pi h \Delta h = \int_0^{12} \pi h \, dh.$$ 

Evaluating gives

$$\text{Total volume} = \frac{\pi h^2}{2}\bigg|_0^{12} = 72\pi.$$ 

17. Slice parallel to the base of the cone, or, equivalently, rotate the line $x = (3 - y)/3$ about the $y$-axis. (One can also slice the other way.) See Figure 8.145. The volume $V$ is given by

$$V = \int_{y=0}^{y=3} \pi x^2 \, dy = \int_0^3 \pi \left(\frac{3-y}{3}\right)^2 \, dy$$

$$= \pi \int_0^3 \left(1 - \frac{2y}{3} + \frac{y^2}{9}\right) \, dy$$

$$= \pi \left[y - \frac{y^3}{3} + \frac{y^4}{27}\right]_0^3 = \pi.$$ 

18. (a) We slice the pyramid horizontally. See Figure 8.146. Each slice is a square slab of thickness $\Delta h$, so the volume of a slice at height $h$ is $s^2 \Delta h$, where $s$ is the length of a side. We use the similar triangles in Figure 8.147 to write $s$ as a function of $h$:

$$\frac{s}{10 - h} = \frac{8}{10} \quad \text{so} \quad s = 0.8(10 - h).$$

The volume of the slice at height $h$ is $(0.8(10 - h))^2 \Delta h$. To find the total volume, we integrate this quantity from $h = 0$ to $h = 10$.

$$V = \int_0^{10} (0.8(10 - h))^2 \, dh = 0.64 \int_0^{10} (h - 10)^2 \, dh = \frac{16}{75} (h - 10)^3 \bigg|_0^{10} = \frac{640}{3} = 213.333 \, m^3.$$ 

(b) As in part (a),

Volume of a slice at height $h \approx s^2 \Delta h = (0.8(10 - h))^2 \Delta h$.

The height $h$ ranges from $h = 0$ to $h = 6$. We have

$$V = \int_0^{6} (0.8(10 - h))^2 \, dh = 0.64 \int_0^{6} (h - 10)^2 \, dh = \frac{16}{75} (h - 10)^3 \bigg|_0^{6} = \frac{4992}{25} = 199.680 \, m^3.$$
19. We slice the tank horizontally. There is an outside radius \( r_{\text{out}} \) and an inside radius \( r_{\text{in}} \), and, at height \( h \),

\[
\text{Volume of a slice } \approx \pi (r_{\text{out}})^2 \Delta h - \pi (r_{\text{in}})^2 \Delta h.
\]

See Figure 8.148. We see that \( r_{\text{out}} = 3 \) for every slice. We use similar triangles to find \( r_{\text{in}} \) in terms of the height \( h \):

\[
r_{\text{in}} = \frac{3}{6} \quad \text{so} \quad r_{\text{in}} = \frac{1}{2} h.
\]

At height \( h \),

\[
\text{Volume of slice } \approx \pi (3)^2 \Delta h - \pi \left( \frac{1}{2} h \right)^2 \Delta h.
\]

To find the total volume, we integrate this quantity from \( h = 0 \) to \( h = 6 \).

\[
V = \int_{0}^{6} \left( \pi (3)^2 - \pi \left( \frac{1}{2} h \right)^2 \right) dh = \pi \int_{0}^{6} \left( 9 - \frac{1}{4} h^2 \right) dh = \pi \left| 9h - \frac{h^3}{12} \right|_{0}^{6} = 36\pi = 113.097 \text{ m}^3.
\]

![Figure 8.148](image)

20. Since \( f(x) = \sin x \), \( f'(x) = \cos(x) \), so

\[
\text{Arc Length} = \int_{0}^{\pi} \sqrt{1 + \cos^2 x} \, dx.
\]

21. We’ll find the arc length of the top half of the ellipse, and multiply that by 2. In the top half of the ellipse, the equation \((x^2/a^2) + (y^2/b^2) = 1\) implies

\[
y = \pm b \sqrt{1 - \frac{x^2}{a^2}}.
\]

Differentiating \((x^2/a^2) + (y^2/b^2) = 1\) implicitly with respect to \( x \) gives us

\[
\frac{2x}{a^2} + \frac{2y}{b^2} \frac{dy}{dx} = 0,
\]

so

\[
\frac{dy}{dx} = -\frac{b^2 x}{a^2 y}.
\]

Substituting this into the arc length formula, we get

\[
\text{Arc Length} = \int_{-a}^{a} \sqrt{1 + \left( -\frac{b^2 x}{a^2 y} \right)^2} \, dx
\]

\[
= \int_{-a}^{a} \sqrt{1 + \frac{b^4 x^2}{a^4 (b^2)(1 - \frac{x^2}{a^2})}} \, dx
\]

\[
= \int_{-a}^{a} \sqrt{1 + \frac{b^2 x^2}{a^2 (a^2 - x^2)}} \, dx.
\]
Hence the arc length of the entire ellipse is
\[2 \int_{-a}^{a} \sqrt{1 + \left(\frac{b^2x^2}{a^2(a^2 - x^2)}\right)} \, dx.\]

22. Since \(f'(x) = \cos x\), we have
\[L = \int_{0}^{3} \sqrt{1 + (\cos x)^2} \, dx = 3.621.\]
We see in Figure 8.149 that the length of the curve is slightly longer than the length of the \(x\)-axis from \(x = 0\) to \(x = 3\), so the answer of 3.621 makes sense.

![Figure 8.149](image)

23. Since \(f'(x) = 10x\), we have
\[L = \int_{0}^{3} \sqrt{1 + (10x)^2} \, dx = 45.230.\]
We see in Figure 8.150 that the length of the curve is definitely longer than 45 and slightly longer than \(\sqrt{45^2 + 3^2} = 45.10\), so the answer of 45.230 is reasonable.

![Figure 8.150](image)

24. The arc length of \(\sqrt{1 - x^2}\) from \(x = 0\) to \(x = 1\) is one quarter of the perimeter of the unit circle. Hence the length is
\[\frac{2\pi}{4} = \frac{\pi}{2}.\]

25. The arc length is given by
\[L = \int_{1}^{2} \sqrt{1 + e^{2x}} \, dx \approx 4.785.\]
Note that \(\sqrt{1 + e^{2x}}\) does not have an obvious elementary antiderivative, so we use an approximation method to find an approximate value for \(L\).

26. The arc length is given by
\[L = \int_{1}^{2} \sqrt{1 + \left(x^4 + \frac{1}{16x^4} - \frac{1}{2}\right) x^2} \, dx = \int_{1}^{2} \sqrt{x^2 + \left(\frac{1}{4x^2}\right)^2} \, dx = \int_{1}^{2} x^2 + \frac{1}{4x^2} \, dx = \frac{59}{24}.\]
27. We have \(\frac{dx}{dt} = -3\sin t, \frac{dy}{dt} = 2\cos t\), so, evaluating the integral numerically, we have
\[
\text{Arc length} = \int_0^{2\pi} \sqrt{9\cos^2 t + 4\sin^2 t} \, dt = 15.865.
\]
The curve is an ellipse.

28. We have \(\frac{dx}{dt} = -2\sin(2t), \frac{dy}{dt} = 2\cos(2t)\), so, simplifying the integrand, we have
\[
\text{Arc length} = \int_0^\pi \sqrt{4\sin^2(2t) + 4\cos^2(2t)} \, dt = 2\int_0^\pi \, dt = 2\pi.
\]
The curve is a circle of radius 1.

29. (a) The points of intersection are 
- The graph of \(f\) is concave up and passes through the points (0, 0) and (1, 1), so it lies below the line \(y = x\).
- The area under \(y = x\) from 0 to 1 is half the area of a square of side 1, or \(1/2\). Thus, \(\int_0^1 f(x) \, dx < \frac{1}{2}\).
- Since \(f(0) = 0\), the Fundamental Theorem gives \(\int_0^{0.5} f'(x) \, dx = f(0.5) - f(0) = f(0.5)\).
- The graph of \(f\) is concave up and passes through the points (0, 0) and (1, 1), so it lies below the line \(y = x\). This means \(f(x) < x\) for \(0 < x < 1\).
- Since \(f(x) < x\), we have \(f(0.5) < 0.5\). Hence \(\int_0^{0.5} f'(x) \, dx < \frac{1}{2}\).

30. (a) The volume of the region formed by rotating the graph of \(f\) on \(0 \leq x \leq 1\) about the \(x\)-axis.
- The graph of \(f\) is concave up and contains (0, 0) and (1, 1), so it lies below the line \(y = x\) on \(0 < x \leq 1\).
- This means the region formed by rotating the graph of \(f\) lies within the region formed by rotating the line segment \(y = x\), which is a line of base \(r = 1\) and height \(h = 1\). The volume of this cone is \(\frac{1}{3}\pi r^2 h = \pi/3\).
- Since this cone contains the region formed by rotating the graph of \(f\), we have \(\int_0^1 \pi (f(x))^2 \, dx < \frac{\pi}{3}\).

31. (a) The line segment between (0, 0) and (1, 1) is shorter than the arc length of \(f\), so \(\int_0^1 \sqrt{1 + (f'(x))^2} \, dx > \sqrt{2}\).

Problems

34. (a) The points of intersection are \(x = 0\) to \(x = 2\), so we have
\[
\text{Area} = \int_0^2 (2x - x^2) \, dx = x^2 - \frac{x^3}{3}\bigg|_0^2 = \frac{4}{3} = 1.333.
\]

(b) The outside radius is \(2x\) and the inside radius is \(x^2\), so we have
\[
\text{Volume} = \int_0^2 (\pi(2x)^2 - \pi(x^2)^2) \, dx = \pi \int_0^2 (4x^2 - x^4) \, dx = \frac{\pi}{15}(20x^3 - 3x^5)\bigg|_0^2 = \frac{64\pi}{15} = 13.404.
\]
(c) The length of the perimeter is equal to the length of the top plus the length of the bottom. Using the arclength formula, and the fact that the derivative of $2x$ is $2$ and the derivative of $x^2$ is $2x$, we have

$$L = \int_0^2 \sqrt{1 + 2^2} \, dx + \int_0^2 \sqrt{1 + (2x)^2} \, dx = 4.4721 + 4.6468 = 9.119.$$ 

35. There are at least two possible answers. Since $\sqrt{4 - x^2} - (-\sqrt{4 - x^2}) \geq 0$ when $0 \leq x \leq 2$, one possibility is that the integral gives the area between the curve $y = 2\sqrt{4 - x^2}$ and the line $y = 0$ as shown in Figure 8.151.

Alternatively, since $\sqrt{4 - x^2} \geq -\sqrt{4 - x^2}$ when $0 \leq x \leq 2$, the integral gives the area between the quarter circles $y = \sqrt{4 - x^2}$ and $y = -\sqrt{4 - x^2}$, as shown in Figure 8.152.

36. The two functions intersect at $(0, 0)$ and $(5, 25)$. We slice the volume with planes perpendicular to the horizontal line $y = 30$. This divides the solid into thin washers with volume

$$\text{Volume of slice} = \pi ((r_{\text{out}})^2 - (r_{\text{in}})^2) \, \Delta x.$$ 

The outer radius is the vertical distance from the line $y = 30$ to the curve $y = x^2$, so $r_{\text{out}} = 30 - x^2$. Similarly, the inner radius is the vertical distance from the line $y = 30$ to the curve $y = 5x$, so $r_{\text{in}} = 30 - 5x$. Integrating from $x = 0$ to $x = 5$ we have

$$V = \int_0^5 \pi ((30 - x^2)^2 - (30 - 5x)^2) \, dx.$$ 

37. The two functions intersect at $(0, 0)$ and $(5, 25)$. We slice the volume with planes perpendicular to the vertical line $x = 8$. This divides the solid into thin washers with volume

$$\text{Volume of slice} = \pi ((r_{\text{out}})^2 - (r_{\text{in}})^2) \, \Delta y.$$ 

The outer radius is the horizontal distance from the line $x = 8$ to the curve $x = y/5$, so $r_{\text{out}} = 8 - y/5$. Similarly, the inner radius is the horizontal distance from the line $x = 8$ to the curve $x = \sqrt{y}$, so $r_{\text{in}} = 8 - \sqrt{y}$. Integrating from $y = 0$ to $y = 25$ we have

$$V = \int_0^{25} \pi ((8 - y/5)^2 - (8 - \sqrt{y})^2) \, dy.$$
38. (a) See Figure 8.153

![Figure 8.153: Rotated Region](image1)

(b) Divide [0,1] into \(N\) subintervals of width \(\Delta x = \frac{1}{N}\). The volume of the \(i^{th}\) disc is \(\pi (\sqrt{x_i})^2 \Delta x = \pi x_i \Delta x\). So, \(V \approx \sum_{i=1}^{N} \pi x_i \Delta x\). See Figure 8.154

(c) 

\[
\text{Volume} = \int_0^1 \pi x \, dx = \left[ \frac{\pi x^2}{2} \right]_0^1 = \frac{\pi}{2} \approx 1.57.
\]

39. (a) See Figure 8.155.

![Figure 8.155](image2)

Slice the figure perpendicular to the \(x\)-axis. One gets washers of inner radius \(1 - \sqrt{x}\) and outer radius 1. Therefore,

\[
V = \int_0^1 \left( \pi 1^2 - \pi (1 - \sqrt{x})^2 \right) \, dx \\
= \pi \int_0^1 \left( 1 - \left[ 2\sqrt{x} + x \right] \right) \, dx \\
= \pi \left[ \frac{4}{3} x^{3/2} - \frac{1}{2} x^2 \right]_0^1 = \frac{5\pi}{6} \approx 2.62.
\]

(b) See Figure 8.156. Note that \(x = y^2\). We now integrate over \(y\) instead of \(x\), slicing perpendicular to the \(y\)-axis. This gives us washers of inner radius \(x\) and outer radius 1. So

\[
V = \int_{y=0}^{y=1} \left( \pi 1^2 - \pi x^2 \right) \, dy \\
= \int_0^1 \pi (1 - y^4) \, dy \\
= \left[ \pi y - \frac{\pi}{5} y^5 \right]_0^1 = \pi - \frac{\pi}{5} = \frac{4\pi}{5} \approx 2.51.
\]
40. (a) Since $y = ax^2$ is non-negative, we integrate to find the area:

$$\text{Area} = \int_0^1 (ax^2) \, dx = \frac{a^3}{3}.$$  

(b) Each slice of the object is approximately a cylinder with radius $ax^2$ and thickness $\Delta x$. We have

$$\text{Volume} = \int_0^1 \pi(ax^2)^2 \, dx = \pi a^3 \frac{1}{3} = \frac{32}{5}a^2 \pi.$$

41. (a) Since $y = e^{-bx}$ is non-negative, we integrate to find the area:

$$\text{Area} = \int_0^1 (e^{-bx}) \, dx = -\frac{1}{b}e^{-bx} \bigg|_0^1 = \frac{1}{b} \left(1 - e^{-b}\right).$$

(b) Each slice of the object is approximately a cylinder with radius $e^{-bx}$ and thickness $\Delta x$. We have

$$\text{Volume} = \int_0^1 \pi(e^{-bx})^2 \, dx = \pi \int_0^1 e^{-2bx} \, dx = -\frac{\pi}{2b}e^{-2bx} \bigg|_0^1 = \frac{\pi}{2b} \left(1 - e^{-2b}\right).$$

42. (a) We divide the region into vertical strips of thickness $\Delta x$. As a slice is rotated about the $x$-axis, it creates a disk of radius $r_{\text{out}}$ from which has been removed a smaller circular disk of radius $r_{\text{in}}$. We see in Figure 8.157 that $r_{\text{out}} = \sin x$ and $r_{\text{in}} = 0.5x$. Thus,

$$\text{Volume of a slice} \approx \pi (r_{\text{out}})^2 \Delta x - \pi (r_{\text{in}})^2 \Delta x = \pi (\sin x)^2 \Delta x - \pi (0.5x)^2 \Delta x.$$  

To find the total volume, we integrate this quantity between the points of intersection $x = 0$ and $x = 1.9$:

$$V = \int_0^{1.9} (\pi (\sin x)^2 - \pi (0.5x)^2) \, dx = \pi \left[-\frac{\sin x \cos x}{2} - \frac{x^3}{12} + x\right]_0^{1.9} = 1.669.$$  

(b) We see in Figure 8.158 that $r_{\text{out}} = 5 - 0.5x$ and $r_{\text{in}} = 5 - \sin x$. Thus,

$$\text{Volume of a slice} \approx \pi (r_{\text{out}})^2 \Delta x - \pi (r_{\text{in}})^2 \Delta x = \pi (5 - 0.5x)^2 \Delta x - \pi (5 - \sin x)^2 \Delta x.$$  

To find the total volume, $V$, we integrate this quantity between the points of intersection $x = 0$ and $x = 1.9$:

$$V = \int_0^{1.9} \left(\pi (5 - 0.5x)^2 - \pi (5 - \sin x)^2\right) \, dx = \pi \left[\frac{1}{12} (6(\sin x - 20) \cos x + x(x^2 - 30x - 6))\right]_0^{1.9} = 11.550.$$
We divide the region into vertical strips of thickness $\Delta x$. As a slice is rotated about the $x$-axis, it creates a disk of radius $r_{\text{out}}$ from which has been removed a disk of radius $r_{\text{in}}$. We see in Figure 8.159 that $r_{\text{out}} = 5 + 2x$ and $r_{\text{in}} = 5$. Thus, the volume of a slice is

$$\approx \pi (r_{\text{out}}^2 - r_{\text{in}}^2) \Delta x = \pi (5 + 2x)^2 \Delta x - \pi (5)^2 \Delta x.$$ 

To find the total volume, $V$, we integrate this quantity between $x = 0$ and $x = 4$:

$$V = \int_0^4 (\pi (5 + 2x)^2 - \pi (5)^2) \, dx = \pi \int_0^4 ((5 + 2x)^2 - 25) \, dx = \pi \left( \frac{4}{3} x^3 + 10x^2 \right) \bigg|_0^4 = \frac{736\pi}{3} = 770.737.$$ 

![Figure 8.159](attachment:image1.png)

We divide the region into vertical strips of thickness $\Delta x$. As a slice is rotated about the $x$-axis, it creates a disk of radius $r_{\text{out}}$ from which has been removed a disk of radius $r_{\text{in}}$. We see in Figure 8.160 that $r_{\text{out}} = 2 + x^2$ and $r_{\text{in}} = 2$. Thus, the volume of a slice is

$$\approx \pi (r_{\text{out}}^2 - r_{\text{in}}^2) \Delta x = \pi (2 + x^2)^2 \Delta x - \pi (2)^2 \Delta x.$$ 

To find the total volume, $V$, we integrate this quantity between $x = 0$ and $x = 3$:

$$V = \int_0^3 (\pi (2 + x^2)^2 - \pi (2)^2) \, dx = \pi \int_0^3 ((2 + x^2)^2 - 4) \, dx = \pi \left( \frac{3}{10} x^5 + 20x^3 \right) \bigg|_0^3 = \frac{423\pi}{5} = 265.778.$$ 

![Figure 8.160](attachment:image2.png)

(b) We see in Figure 8.161 that $r_{\text{out}} = 10$ and $r_{\text{in}} = 10 - x^2$. Thus, the volume of a slice is

$$\approx \pi (r_{\text{out}}^2 - r_{\text{in}}^2) \Delta x = \pi (10)^2 \Delta x - \pi (10 - x^2)^2 \Delta x.$$ 

To find the total volume, $V$, we integrate this quantity between $x = 0$ and $x = 3$:

$$V = \int_0^3 (\pi (10)^2 - \pi (10 - x^2)^2) \, dx = \pi \int_0^3 (100 - (10 - x^2)^2) \, dx = \pi \left( \frac{10}{15} (100x^3 - 3x^5) \right) \bigg|_0^3 = \frac{657\pi}{5} = 412.805.$$ 

![Figure 8.161](attachment:image3.png)
45. Slice the object into disks vertically, as in Figure 8.162. A typical disk has thickness $\Delta x$ and radius $y = \sqrt{1 - x^2}$. Thus

Volume of disk $\approx \pi y^2 \Delta x = \pi (1 - x^2) \Delta x$.

Volume of solid = $\lim_{\Delta x \to 0} \sum \pi (1 - x^2) \Delta x = \int_0^1 \pi (1 - x^2) \, dx = \pi \left( x - \frac{x^3}{3} \right) \bigg|_0^1 = \frac{2\pi}{3}$

Note: As we expect, this is the volume of a half sphere.

46. Slice the object into rings horizontally, as in Figure 8.163. A typical ring has thickness $\Delta y$, inner radius $2 + x = 2 + \sqrt{1 - y^2}$, and outer radius $x = 2 + \sqrt{1 - y^2}$. Thus,

Volume of ring $\approx \pi (2 + \sqrt{1 - y^2})^2 \Delta y - \pi 2^2 \Delta y = \pi (4\sqrt{1 - y^2} + 1 - y^2) \Delta y$.

Volume of solid = $\int_0^1 \pi (4\sqrt{1 - y^2} + 1 - y^2) \, dy$

= $4\pi \int_0^1 \sqrt{1 - y^2} \, dy + \pi \int_0^1 \, dy - \pi \int_0^1 y^2 \, dy$

= $4\pi \left( \frac{1}{2} \left( y \sqrt{1 - y^2} \right)_0^1 + \int_0^1 \frac{1}{\sqrt{1 - y^2}} \, dy \right) + \pi \left( y \right)_0^1 - \pi \frac{y^3}{3} \bigg|_0^1$

= $2\pi y \sqrt{1 - y^2} + 2\pi \arcsin y + \pi y - \frac{\pi y^3}{3} \bigg|_0^1$

= $0 + 2\pi \arcsin 1 + \pi - \frac{\pi}{3} - 0 - 2\pi \arcsin 0 + 0 + 0$

= $\pi^2 + \frac{2\pi}{3} = 11.964$. 

Figure 8.162: Cross-section of solid
47. Slice the object into rings horizontally, as in Figure 8.164. A typical ring has thickness $\Delta y$, outer radius 1, and inner radius $1 - x = 1 - \sqrt{1 - y^2}$. Thus,

$$\text{Volume of ring } \approx \pi x^2 \Delta y - \pi (1 - \sqrt{1 - y^2})^2 \Delta y = \pi (2\sqrt{1 - y^2} - (1 - y^2)) \Delta y.$$ 

Volume of solid

$$\begin{align*}
\text{Volume of solid} &= \int_0^1 \pi (2\sqrt{1 - y^2} - 1 + y^2) \, dy \\
&= 2\pi \int_0^1 \sqrt{1 - y^2} \, dy - \pi \int_0^1 1 \, dy + \pi \int_0^1 y^2 \, dy \\
&= \left[\pi y \sqrt{1 - y^2}\right]_0^1 + \pi y^3[1]_0^1 \\
&= \pi \sqrt{1 - y^2} + \pi y^3|_0^1 \\
&= 0 + \frac{\pi^2}{2} - \pi + \frac{\pi^3}{3} - 0 - 0 + 0 - 0 \\
&= \frac{\pi^2}{2} - \frac{2\pi}{3} = 2.840.
\end{align*}$$

![Figure 8.164: Cross-section of solid](image)

48. Slicing perpendicularly to the $x$-axis gives squares whose thickness is $\Delta x$ and whose side is $y = \sqrt{1 - x^2}$. See Figure 8.165. Thus,

$$\text{Volume of square slice } \approx (\sqrt{1 - x^2})^2 \Delta x = (1 - x^2) \Delta x.$$ 

Volume of solid

$$\begin{align*}
\text{Volume of solid} &= \int_0^1 (1 - x^2) \, dx = x - \frac{x^3}{3}\bigg|_0^1 = \frac{2}{3}.
\end{align*}$$

![Figure 8.165: Base of solid](image)
49. Slicing perpendicularly to the \( y \)-axis gives semicircles whose thickness is \( \Delta y \) and whose diameter is \( x = \sqrt{1-y^2} \). See Figure 8.166. Thus

\[
\text{Volume of semicircular slice} \approx \frac{\pi}{2} \left( \frac{\sqrt{1-y^2}}{2} \right)^2 \Delta y = \frac{\pi}{8} (1-y^2) \Delta y.
\]

\[
\text{Volume of solid} = \int_0^1 \frac{\pi}{8} (1-y^2) \, dy = \frac{\pi}{8} \left( \frac{y^3}{3} \right) \bigg|_0^1 = \frac{\pi}{8} \cdot \frac{2}{3} = \frac{\pi}{24}.
\]

Figure 8.166: Base of Solid

50. An isosceles triangle with legs of length \( s \) has

\[
\text{Area} = \frac{1}{2} s^2.
\]

Slicing perpendicularly to the \( y \)-axis gives isosceles triangles whose thickness is \( \Delta y \) and whose leg is \( x = \sqrt{1-y^2} \). See Figure 8.167. Thus

\[
\text{Volume of triangular slice} \approx \frac{1}{2} \sqrt{1-y^2} \Delta y = \frac{1}{2} (1-y^2) \Delta y.
\]

\[
\text{Volume of solid} = \int_0^1 \frac{1}{2} (1-y^2) \, dy = \frac{1}{2} \left( \frac{y^3}{3} \right) \bigg|_0^1 = \frac{1}{3}.
\]

51. The curve \( y = x(x-3)^2 \) has \( x \)-intercepts at \( x = 0, 3 \) and lies above the \( x \)-axis on this interval. Thus, \( \int_0^3 x(x-3)^2 \, dx \) gives the area under the graph of \( f \) from \( x = 0 \) to \( x = 3 \).

52. The curve \( y = x(x-3)^2 \) has \( x \)-intercepts at \( x = 0, 3 \) and lies above the \( x \)-axis on this interval. Rotating the curve about the \( x \)-axis forms a solid of revolution with

\[
\text{Volume} = \int_0^3 \pi f(x)^2 \, dx = \int_0^3 \pi \left( x(x-3)^2 \right)^2 \, dx = \int_0^3 \pi x^2(x-3)^4 \, dx.
\]

Thus, this expression represents a volume of revolution about the \( x \)-axis between \( x = 0 \) and \( x = 3 \).

53. Since \( y = (e^x + e^{-x})/2, \ y' = (e^x - e^{-x})/2 \). The length of the catenary is

\[
\int_{-1}^1 \sqrt{1+(y')^2} \, dx = \int_{-1}^1 \sqrt{1+\left(\frac{e^x-e^{-x}}{2}\right)^2} \, dx = \int_{-1}^1 \sqrt{\frac{e^{2x}}{4} - \frac{1}{2} + \frac{e^{-2x}}{4}} \, dx
\]

\[
= \int_{-1}^1 \sqrt{\left(\frac{e^x+e^{-x}}{2}\right)^2} \, dx = \int_{-1}^1 \frac{e^x+e^{-x}}{2} \, dx
\]

\[
= \left[ \frac{e^x-e^{-x}}{2} \right]_{-1}^1 = e - e^{-1}.
\]
54. (a) Slice the headlight into $N$ disks of height $\Delta x$ by cutting perpendicular to the $x$–axis. The radius of each disk is $y$; the height is $\Delta x$. Therefore, the Riemann sum approximating the volume of the headlight is

$$\sum_{i=1}^{N} \pi y_i^2 \Delta x = \sum_{i=1}^{N} \frac{9\pi y_i^2}{4} \Delta x.$$ (b) 

$$\pi \int_0^4 \frac{9x^2}{4} dx = \frac{9\pi}{8} x^2 \bigg|_0^4 = 18\pi.$$ 

55. (a) The line $y = ax$ must pass through $(l, b)$. Hence $b = al$, so $a = b/l$. (b) Cut the cone into $N$ slices, slicing perpendicular to the $x$–axis. Each piece is almost a cylinder. The radius of the $i$th cylinder is $r(x_i) = \frac{bx_i}{l}$, so the volume

$$V \approx \sum_{i=1}^{N} \pi \left( \frac{bx_i}{l} \right)^2 \Delta x.$$ 

Therefore, as $N \to \infty$, we get

$$V = \int_0^l \pi b^2 l^{-2} x^2 dx = \frac{\pi b^2}{l^2} \left[ \frac{x^3}{3} \right]_0^l = \left( \frac{\pi b^2}{l^2} \right) \left( \frac{l^3}{3} \right) = \frac{1}{3} \pi b^2 l.$$ 

56. (a) If you slice the apple perpendicular to the core, you expect that the cross section will be approximately a circle.

If $f(h)$ is the radius of the apple at height $h$ above the bottom, and $H$ is the height of the apple, then

$$\text{Volume} = \int_0^H \pi f(h)^2 \, dh.$$ 

Ignoring the stem, $H \approx 3.5$. Although we do not have a formula for $f(h)$, we can estimate it at various points. (Remember, we measure here from the bottom of the apple, which is not quite the bottom of the graph.)

<table>
<thead>
<tr>
<th>$h$</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>3.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(h)$</td>
<td>1</td>
<td>1.5</td>
<td>2</td>
<td>2.1</td>
<td>2.3</td>
<td>2.2</td>
<td>1.8</td>
<td>1.2</td>
</tr>
</tbody>
</table>

Now let $g(h) = \pi f(h)^2$, the area of the cross-section at height $h$. From our approximations above, we get the following table.
We can now take left- and right-hand sum approximations. Note that $\Delta h = 0.5$ inches. Thus

LEFT(9) = (3.14 + 7.07 + 12.57 + 13.85 + 16.62 + 13.85 + 10.18)(0.5) = 38.64.


Thus the volume of the apple is $\approx 39$ cu in.

(b) The apple weighs $0.03 \times 39 \approx 1.17$ pounds, so it costs about 94¢.

\[
\int_{-1}^{1} 12\pi\sqrt{1-x^2} \, dx = 12\pi\int_{-1}^{1} \sqrt{1-x^2} \, dx.
\]

But $\int_{-1}^{1} \sqrt{1-x^2} \, dx$ is the area of a semicircle of radius 1, which is $\frac{\pi}{2}$. So we get $12\pi \cdot \frac{\pi}{2} = 6\pi^2 \approx 59.22$. (Or, you could use

\[
\int \sqrt{1-x^2} \, dx = \left[x\sqrt{1-x^2} + \arcsin(x)\right],
\]

by VI-30 and VI-28.)

58. The arc length of the curve $y = f(t)$ from $t = 3$ to $t = 8$ is $\int_{3}^{8} \sqrt{1 + (f'(t))^2} \, dt$. Thus, we want a function $f$ such that

\[
\int_{3}^{8} \sqrt{1 + (f'(t))^2} \, dt = \int_{3}^{8} \sqrt{1 + e^{6t}} \, dt.
\]

Thus, we have

\[
(f'(t))^2 = e^{6t}.
\]

One possibility is

\[
f'(t) = e^{3t}; \quad f(t) = \frac{1}{3}e^{3t} + C.
\]

For any constant $C$, the original integral is the arc length of the curve $y = \frac{1}{3}e^{3t} + C$ from $t = 3$ to $t = 8$.

Another solution to $(f'(t))^2 = e^{6t}$ is $f'(t) = -e^{3t}$, which gives $f(t) = -\frac{1}{3}e^{3t} + C$. 

<table>
<thead>
<tr>
<th>$h$</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>3.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y(h)$</td>
<td>3.14</td>
<td>7.07</td>
<td>12.57</td>
<td>13.85</td>
<td>16.62</td>
<td>13.85</td>
<td>10.18</td>
<td>4.52</td>
</tr>
</tbody>
</table>

Figure 8.168: The Torus
Figure 8.169: Slice of Torus
59. We take a cross-section of the pipe and cut it up in such a way that the speed of the water is nearly uniform on each slice. See Figure 8.170.

We use thin rings around the pipe’s center; if a given ring is narrow enough, all points on it will be roughly equidistant from the center. Since the water speed is a function of the distance from the center, the speed is nearly constant on the entire ring.

Let \( r \) be the distance from the center to the inner boundary of the ring, and let \( \Delta r \) be the width of the ring, as in Figure 8.170. By straightening the ring into a thin rectangle, we find that its area is approximately given by the quantity \( 2\pi r \Delta r \). The speed across a part of the ring is roughly equal to the speed across the inner boundary, \( 10(1 - r^2) \) inches per second. The flow is defined as the speed times the area; thus on any given ring we have

\[
\text{Flow} \approx 10(1 - r^2) \cdot 2\pi r \Delta r.
\]

The total flow across the pipe cross-section is approximated by a Riemann sum incorporating all of the rings:

\[
\text{Total Flow} \approx 20\pi \sum (1 - r^2) r \Delta r,
\]

where \( r \) is in between 0 and 1. Letting \( \Delta r \to 0 \), we obtain the exact solution:

\[
\text{Total Flow} = 20\pi \int_0^1 (1 - r^2) r \, dr = 20\pi \left( \frac{r^2}{2} - \frac{r^4}{4} \right) \bigg|_0^1 = 5\pi \text{ cubic inches/second}.
\]

Figure 8.170

60. Multiplying \( r = 2a \cos \theta \) by \( r \), converting to Cartesian coordinates, and completing the square gives

\[
\begin{align*}
\frac{1}{2}x^2 & = 2ax \\
x^2 + y^2 & = 2ax \\
x^2 - 2ax + a^2 + y^2 & = a^2 \\
(x - a)^2 + y^2 & = a^2.
\end{align*}
\]

This is the standard form of the equation of a circle with radius \( a \) and center \((x, y) = (a, 0)\).

To check the limits on \( \theta \) note that the circle is in the right half plane, where \(-\pi/2 \leq \theta \leq \pi/2\). Rays from the origin at all these angles meet the circle because the circle is tangent to the \( y \)-axis at the origin.

61. The area is given by

\[
\int_{-\pi/2}^{\pi/2} \frac{1}{2} r^2 \, d\theta = \int_{-\pi/2}^{\pi/2} \left( 2a \cos \theta \right)^2 \, d\theta = 4a^2 \int_{-\pi/2}^{\pi/2} \cos^2 \theta \, d\theta = 2a^2 \left( \frac{1}{2} \cos \theta \sin \theta + \frac{\theta}{2} \right) \bigg|_{-\pi/2}^{\pi/2} = \pi a^2.
\]

(We have used formula IV-18 from the integral table. The integral can also be done using a calculator or integration by parts.)

62. See Figure 8.171. The circles meet where

\[
\begin{align*}
2a \cos \theta & = a \\
\cos \theta & = \frac{1}{2} \\
\theta & = \pm \frac{\pi}{3}.
\end{align*}
\]
SOLUTIONS to Review Problems for Chapter Eight

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The area is obtained by subtraction:

\[
\text{Area} = \int_{-\pi/3}^{\pi/3} \left( \frac{1}{2} (2a \cos \theta)^2 - \frac{1}{2} a^2 \right) d\theta
\]

\[
= \int_{-\pi/3}^{\pi/3} \left( 2a^2 \cos^2 \theta - \frac{1}{2} a^2 \right) d\theta
\]

\[
= \left( 2a^2 \left( \frac{1}{2} \cos \theta \sin \theta + \frac{\theta}{2} \right) - \frac{a^2 \theta}{2} \right) \bigg|_{-\pi/3}^{\pi/3}
\]

\[
= \left( \frac{\pi}{3} + \frac{\sqrt{3}}{2} \right) a^2.
\]

Since

\[
\frac{\left( \frac{\pi}{3} + \frac{\sqrt{3}}{2} \right) a^2}{\pi a^2} = 61\%
\]

the shaded region covers 61% of circle \( C \).

![Figure 8.171](image)

63. (a) Writing \( C \) in parametric form gives

\[
x = 2a \cos^2 \theta \quad \text{and} \quad y = 2a \cos \theta \sin \theta,
\]

so the slope is given by

\[
\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-2a \sin^2 \theta + 2a \cos^2 \theta}{-4a \cos \theta \sin \theta} = \frac{\sin^2 \theta - \cos^2 \theta}{2 \cos \theta \sin \theta}.
\]

(b) The maximum y-value occurs where \( dy/dx = 0 \), so

\[
\sin^2 \theta - \cos^2 \theta = 0 \quad \Rightarrow \quad \theta = \pm \frac{\pi}{4}
\]

The value \( \theta = \pi/4 \) gives the maximum y-value; \( \theta = -\pi/4 \) gives the minimum y-value.

64. Writing \( C \) in parametric form gives

\[
x = 2a \cos^2 \theta \quad \text{and} \quad y = 2a \cos \theta \sin \theta,
\]

so

\[
\text{Arc length} = \int_{-\pi/2}^{\pi/2} \sqrt{(-4a \cos \theta \sin \theta)^2 + (-2a \sin^2 \theta + 2a \cos^2 \theta)^2} \, d\theta
\]

\[
= 2a \int_{-\pi/2}^{\pi/2} \sqrt{4 \cos^2 \theta \sin^2 \theta + \sin^4 \theta - 2 \sin^2 \theta \cos^2 \theta + \cos^4 \theta} \, d\theta
\]

\[
= 2a \int_{-\pi/2}^{\pi/2} \sqrt{\sin^4 \theta + 2 \sin^2 \theta \cos^2 \theta + \cos^4 \theta} \, d\theta
\]

\[
= 2a \int_{-\pi/2}^{\pi/2} \sqrt{(\sin^2 \theta + \cos^2 \theta)^2} \, d\theta
\]

\[
= 2a \int_{-\pi/2}^{\pi/2} d\theta = 2\pi a.
\]
65. This function has zeros at \( x = -2 \) and \( x = 1 \). The bounded region lies between these two zeros. Thus,

\[
\text{Volume} = \int_{-2}^{1} \pi \left((x - 1)^2(x + 2)\right)^2 dx.
\]

66. The total mass is 12 gm, so the center of mass is located at 
\[
\mathbf{r} = \frac{1}{12}(-5 \cdot 3 - 3 \cdot 3 + 2 \cdot 3 + 7 \cdot 3) = \frac{1}{4}.
\]

67. (a) Since the density is constant, the mass is the product of the area of the plate and its density.

Area of the plate = \( \int_{0}^{1} (\sqrt{x} - x^2) \, dx = \left(\frac{2}{3} x^{3/2} - \frac{1}{3} x^3\right)\bigg|_{0}^{1} = \frac{1}{3} \text{cm}^2.\)

Thus the mass of the plate is \( 2 \cdot 1/3 = 2/3 \) gm.

(b) See Figure 8.172. Since the region is “fatter” closer to the origin, \( \bar{x} \) is less than \( 1/2 \).

(c) To find \( \bar{x} \), we slice the region into vertical strips of width \( \Delta x \). See Figure 8.172.

Area of strip = \( A_s(x) \Delta x \approx (\sqrt{x} - x^2) \Delta x \text{ cm}^2 \).

Then we have

\[
\mathbf{r} = \frac{\int x A_s(x) \, dx}{\text{Mass}} = \frac{\int_{0}^{1} 2x (\sqrt{x} - x^2) \, dx}{2/3} = \frac{3}{2} \int_{0}^{1} 2(x^{3/2} - x^3) \, dx = \frac{3}{2} \left(\frac{2}{3} x^{5/2} - \frac{1}{4} x^4\right)\bigg|_{0}^{1} = \frac{9}{20} \text{ cm}.
\]

This is less than \( 1/2 \), as predicted in part (b). So \( \bar{x} = \bar{y} = 9/20 \text{ cm} \).

68. Let \( x \) be the height from ground to the weight. It follows that \( 0 \leq x \leq 20 \). At height \( x \), to lift the weight \( \Delta x \) more, the work needed is \( 200 \Delta x + 2(20 - x) \Delta x = (240 - 2x) \Delta x \). So the total work is

\[
W = \int_{0}^{20} (240 - 2x) \, dx = (240x - x^2)\bigg|_{0}^{20} = 240(20) - 20^2 = 4400 \text{ ft-lbs}.
\]

69. Imagine the pole is divided into \( n \) segments of length \( \Delta x \). The heights of the segments are given by \( x_1, x_2, \ldots, x_n \).

A segment of length \( \Delta x \) weighs \( \frac{20 \text{ lb}}{10 \text{ ft}} \cdot \Delta x = 2 \Delta x \). The work required to raise a segment a vertical distance of \( x_i \) ft is

Work to raise segment \( x_i, ft = \text{Weight} \cdot \text{Distance} = 2x_i \Delta x \).

The total work is therefore

\[
\text{Total work} = \lim_{n \to \infty} \sum_{i=1}^{n} 2x_i \Delta x
= \int_{0}^{10} 2x \, dx = x^2\bigg|_{0}^{10} = 100 \text{ ft-lbs}.
\]

To check our answer, notice that the work required to raise the entire 20 lb pole so that it is suspended horizontally 10 ft above the ground is:

\[
\text{Work} = \frac{20 \text{ lb}}{10 \text{ ft}} \cdot \text{Distance} = 200 \text{ ft-lbs}.
\]

This is more than 100 lbs, because it should take more work to raise the entire pole 10 ft than to stand it upright.
70. Let \( x \) be the distance from the bucket to the surface of the water. It follows that \( 0 \leq x \leq 40 \). At \( x \) feet, the bucket weighs \((30 - \frac{1}{4}x)\), where the \( \frac{1}{4}x \) term is due to the leak. When the bucket is \( x \) feet from the surface of the water, the work done by raising it \( \Delta x \) feet is \((30 - \frac{1}{4}x) \Delta x \). So the total work required to raise the bucket to the top is

\[
W = \int_0^{40} (30 - \frac{1}{4}x) \, dx
\]

\[
= \left[ 30x - \frac{1}{8}x^2 \right]_0^{40}
\]

\[
= 30(40) - \frac{1}{8}40^2 \approx 1000 \text{ ft-lb}.
\]

71. Consider lifting a rectangular slab of water \( h \) feet from the top up to the top. See Figure 8.173. The area of such a slab is \((10)(20) = 200 \text{ square feet}\); if the thickness is \( \Delta h \), then the volume of such a slab is \( 200 \Delta h \text{ cubic feet}\). This much water weighs 62.4 pounds per \( \text{ft}^3 \), so the weight of such a slab is \((200 \Delta h)(62.4) = 12480 \Delta h \text{ pounds}\). To lift that much water \( h \) feet requires \( 12480h \Delta h \) foot-pounds of work. To lift the whole tank, we lift one plate at a time; integrating over the slabs yields

\[
\int_0^h 12480h \, dh = \frac{12480h^2}{2} \bigg|_0^h = \frac{12480 \cdot h^2}{2} = 1,404,000 \text{ foot-pounds}.
\]

72. We begin by slicing the oil into slabs at a distance \( h \) below the surface with thickness \( \Delta h \). We can then calculate the volume of the slab and the work needed to raise this slab to the surface, a distance of \( h \).

Volume of \( \Delta h \) disk = \( \pi r^2 \Delta h = 25\pi \Delta h \)

Weight of \( \Delta h \) disk = \((25\pi)(50)\Delta h \)

Distance to raise = \( h \)

Work to raise = \((25\pi)(50)\Delta h \).

Integrating the work over all such slabs, we have

\[
\text{Work} = \int_{19}^{25} (50)(25\pi)(h) \, dh
\]

\[
= 625\pi h^2 \bigg|_{19}^{25}
\]

\[
= 390,625\pi - 225,025\pi
\]

\[
\approx 518,363 \text{ ft-lbs}.
\]
A diagram of this tank is shown in Figure 8.174.

73. We slice the gasoline horizontally. At a distance $h$ feet below the surface, the horizontal slab is a cylinder with radius $r$ and thickness $\Delta h$, so

$$\text{Volume of one slab} \approx \pi r^2 \Delta h.$$ 

To find the radius $r$ at a depth $h$ from the top as in Figure 8.175, we note that $h^2 + r^2 = 5^2$, so $r = \sqrt{25 - h^2}$. At depth $h$

$$\text{Volume of one slice} \approx \pi (\sqrt{25 - h^2})^2 \Delta h = \pi (25 - h^2) \Delta h \text{ ft}^3.$$ 

The gasoline at depth $h$ must be lifted a distance of $h$ ft, so

$$\text{Work to move one slice} = \rho \cdot \text{Volume} \cdot \text{Distance lifted} \approx \rho \pi (25 - h^2) \Delta h (h) \text{ ft-lb.}$$

The work done, $W$, to lift all the gasoline is the sum of the work done on the pieces:

$$W \approx \sum \rho \pi (25 - h^2) \Delta h \text{ ft-lb.}.$$ 

As $\Delta h \to 0$, we obtain a definite integral. Since $h$ varies from $h = 0$ to $h = 5$ and $\rho = 42$, we have:

$$W = \int_0^5 \rho \pi (25h - h^3) dh = 42\pi \left(\frac{25h^2}{2} - \frac{h^4}{4}\right) \bigg|_0^5 = \frac{13125\pi}{2} = 20,617 \text{ ft-lb.}$$

The work to pump all the gasoline out is 20,617 ft-lbs.

74. Let $h$ be height above the bottom of the dam. Then

$$\text{Water force} = \int_0^{25} (62.4)(25 - h)(60) \, dh \bigg|_0^{25} = (62.4)(60)\left(25 - \frac{h^2}{2}\right) \bigg|_0^{25} = (62.4)(60)(312.5) = 1,170,000 \text{ lbs.}$$

75. If the weight of the chain were negligible, the work required would be $1000 \cdot 20 = 20,000 \text{ ft-lbs}$. Because of the chain, the total work is slightly more than 20,000 ft-lbs. When the object is $h$ ft off the ground, the length of chain is $50 - h$ so the total weight being lifted is $1000 + 2(50 - h)$ lb. See Figure 8.176. Thus

$$\text{Work to lift the weight an addition} \Delta h \text{ higher} = \text{Weight} \cdot \text{Distance lifted} \approx (1000 + 2(50 - h)) \Delta h \text{ ft-lb.}$$
To find the total work, we integrate this quantity from $h = 0$ to $h = 20$:

$$W = \int_{0}^{20} (1000 + 2(50 - h)) dh = \int_{0}^{20} (1100 - 2h) dh = \left(1100h - h^2\right)_{0}^{20} = 21,600 \text{ft-lbs}.$$  

76. Future Value = \int_{0}^{15} 3000e^{0.06(15-t)} dt = 3000e^{0.9} \int_{0}^{15} e^{-0.06t} dt

\approx 87,980.16

Present Value = \int_{0}^{15} 3000e^{-0.06t} dt = 3000\left(\frac{1}{-0.06}e^{-0.06t}\right)_{0}^{15} = 3000e^{-0.06} + \frac{1}{-0.06}e^{0} \approx 29,671.52.

There’s a quicker way to calculate the present value of the income stream, since the future value of the income stream is (as we’ve shown) $72,980.16, the present value of the income stream must be the present value of $72,980.16. Thus,

Present Value = $72,980.16(e^{-0.06 \cdot 15})

\approx $29,671.52,

which is what we got before.

77. We divide up time between 1971 and 1992 into intervals of length $\Delta t$, and calculate how much of the strontium-90 produced during that time interval is still around.

Strontium-90 decays exponentially, so if a quantity $S_0$ was produced $t$ years ago, and $S$ is the quantity around today, $S = S_0e^{-kt}$. Since the half-life is 28 years, $S = e^{-k(28)}$, giving $k = -\ln(1/2)/28 \approx 0.025$.

We measure $t$ in years from 1971, so that 1992 is $t = 21$.

Since strontium-90 is produced at a rate of 3 kg/year, during the interval $\Delta t$, a quantity $3\Delta t$ kg was produced. Since this was $(21-t)$ years ago, the quantity remaining now is $(3\Delta t)e^{-0.025(21-t)}$. Summing over all such intervals gives

Strontium remaining in 1992 \approx \int_{0}^{21} 3e^{-0.025(21-t)} dt = \frac{3e^{-0.025(21-t)}}{0.025} \bigg|_{0}^{21} = 49 \text{ kg}.

[Note: This is like a future value problem from economics, but with a negative interest rate.]
78. (a) Slice the mountain horizontally into \( N \) cylinders of height \( \Delta h \). The sum of the volumes of the cylinders will be

\[
\sum_{i=1}^{N} \pi r_i^2 \Delta h = \sum_{i=1}^{N} \pi \left( \frac{3.5 \cdot 10^5}{\sqrt{h + 600}} \right)^2 \Delta h.
\]

(b) Volume \( = \int_{400}^{14400} \pi \left( \frac{3.5 \cdot 10^5}{\sqrt{h + 600}} \right)^2 \, dh = 1.23 \cdot 10^{11} \pi \int_{400}^{14400} \frac{1}{(h + 600)} \, dh = 1.23 \cdot 10^{11} \pi \, \ln(h + 600) \bigg|_{400}^{14400} = 1.23 \cdot 10^{11} \pi \, \ln(15000) \approx 1.05 \cdot 10^{12} \) cubic feet.

79. Look at the disc-shaped slab of water at height \( y \) and of thickness \( \Delta y \) in Figure 8.177. The rate at which water is flowing out when it is at depth \( y \) is \( k \sqrt{y} \) (Torricelli’s Law, with \( k \) constant). Then, if \( x = g(y) \), we have

\[
\Delta t = \frac{\text{Time for water to drop by this amount}}{\text{Rate}} = \frac{\pi (g(y))^2 \Delta y}{k \sqrt{y}}.
\]

\[\begin{array}{c}
\text{Figure 8.177} \\
\end{array}\]

If the rate at which the depth of the water is dropping is constant, then \( dy/dt \) is constant, so we want

\[
\frac{\pi (g(y))^2}{k \sqrt{y}} = \text{constant},
\]

so \( g(y) = c \sqrt[4]{y} \), for some constant \( c \). Since \( x = 1 \) when \( y = 1 \), we have \( c = 1 \) and so \( x = \sqrt[4]{y} \), or \( y = x^4 \).

80. The statement \( P(70) = 0.76 \) means that 76% of the population has ages less than 70.

81. Graph \( B \) is more spread out to the right, and so it represents a gas in which more of the molecules are moving at faster velocities. Thus the average velocity in gas \( B \) is larger.

82. Every photon which falls a given distance from the center of the detector has the same probability of being detected. This suggests that we divide the plate up into concentric rings of thickness \( \Delta r \). Consider one such ring having inner radius \( r \) and outer radius \( r + \Delta r \). For this ring,

Number of photons hitting ring per unit time \( \approx N \cdot \text{Area of ring} \approx N \cdot 2\pi r \Delta r \).
Then,

\[ \text{Number of photons detected on ring per unit time } \approx \text{Number hitting } \cdot S(r) \approx N \cdot 2\pi r \Delta r \cdot S(r). \]

Summing over all rings gives us

\[ \text{Total number of photons detected per unit time } \approx \sum 2\pi N r S(r) \Delta r. \]

Taking the limit as \( \Delta r \to 0 \) gives

\[ \text{Total number of photons detected per unit time } = \int_0^R 2\pi N r S(r) dr. \]

83. The number of houses in a ring of width \( \Delta r \) a distance \( r_i \) from the city center is given by:

\[ \text{Number houses} = 1000 \text{ houses/mi}^2 \times \text{Area of ring} = 1000 \cdot 2\pi r_i \Delta r = 2000\pi r_i \Delta r. \]

The value of the houses in this ring is given by:

\[ \text{Value} = \text{Price per house} \times \text{Number of houses} = p(r_i) \cdot 1000 \cdot 2\pi r_i \Delta r = 2000\pi r_i p(r_i) \Delta r. \]

The total value of the houses within 7 miles of the city center is therefore

\[ \text{Total value} = \lim_{n \to \infty} \sum_{i=1}^{n} 2000\pi r_i p(r_i) \Delta r \]
\[ = \int_0^7 2000\pi r p(r) \; dr \]
\[ = 2000\pi \int_0^7 400 e^{-0.2r^2} r \; dr \]
\[ = 800,000\pi \int_{r=0}^7 e^{w^2} (-2.5) \; dw \quad \text{let } w = -0.2r^2, dw = -0.4r \; dr \]
\[ = -2,000,000\pi \int_{r=0}^7 e^w \; dw \]
\[ = -2,000,000\pi e^{-0.2r^2} \bigg|_{r=0}^{r=7} \]
\[ = -2,000,000\pi \left( e^{-0.2(7)^2} - e^{-0.2(0)^2} \right) = 6,282,837. \]

This figure is in $1000s, so the total value of homes is approximately $6.2828 billion.

84. (a) Divide the cross-section of the blood into rings of radius \( r \), width \( \Delta r \). See Figure 8.178.

Then

\[ \text{Area of ring } \approx 2\pi r \Delta r. \]
The velocity of the blood is approximately constant throughout the ring, so

\[ \text{Rate blood flows through ring} \approx \text{Velocity} \cdot \text{Area} \]

\[ = \frac{P}{4\eta} (R^2 - r^2) \cdot 2\pi r \Delta r. \]

Thus, summing over all rings, we find the total blood flow:

\[ \text{Rate blood flowing through blood vessel} \approx \sum \frac{P}{4\eta} (R^2 - r^2) 2\pi r \Delta r. \]

Taking the limit as \( \Delta r \to 0 \), we get

\[ \text{Rate blood flowing through blood vessel} = \int_0^R \frac{\pi P}{2\eta} (R^2 - r^2) dr. \]

\[ = \frac{\pi P}{2\eta} \left( \frac{R^4}{2} - \frac{r^4}{4} \right) \bigg|_0^R = \frac{\pi P R^4}{8\eta}, \]

(b) Since

\[ \text{Rate of blood flow} = \frac{\pi P R^4}{8\eta}, \]

if we take \( k = \frac{\pi P}{(8\eta)} \), then we have

\[ \text{Rate of blood flow} = k R^4, \]

that is, rate of blood flow is proportional to \( R^4 \), in accordance with Poiseuille’s Law.

85. Pick a small interval of time \( \Delta t \) which takes place at time \( t \). Fuel is consumed at a rate of \((25 + 0.1v)^{-1}\) gallons per mile. In the time \( \Delta t \), the car moves \( v \Delta t \) miles, so it consumes \( v \Delta t/(25 + 0.1v) \) gallons during the instant \( \Delta t \). Since \( v = 50 \frac{\text{mph}}{\text{hr}} \), the car consumes

\[ \frac{v \Delta t}{25 + 0.1v} = \frac{50 \frac{\text{mph}}{\text{hr}} \cdot \Delta t}{25 + 0.1 (50 \frac{\text{mph}}{\text{hr}})} = \frac{50 \Delta t}{25(t + 1) + 5t} = \frac{10t \Delta t}{6t + 5} \]

gallons of gas, in terms of the time \( t \) at which the instant occurs. To find the total gas consumed, sum up the instants in an integral:

\[ \text{Gas consumed} = \int_2^3 10t \frac{\text{gal}}{6t + 5} dt \approx 1.25 \text{ gallons.} \]

86. (a) Slicing horizontally, as shown in Figure 8.179, we see that the volume of one disk-shaped slab is

\[ \Delta V \approx \pi x^2 \Delta y = \frac{\pi y}{a} \Delta y. \]

Thus, the volume of the water is given by

\[ V = \int_0^h \frac{\pi y}{a} dy = \frac{\pi y^2}{2a} \bigg|_0^h = \frac{\pi h^2}{2a}. \]

(b) The surface of the water is a circle of radius \( x \). Since the surface, \( y = h \), we have \( h = ax^2 \). Thus, at the surface, \( x = \sqrt{(h/a)} \). Therefore the area of the surface of water is given by

\[ A = \pi x^2 = \frac{\pi h}{a}. \]

(c) If the rate at which water is evaporating is proportional to the surface area, we have

\[ \frac{dV}{dt} = -kA. \]

(The negative sign is included because the volume is decreasing.) By the chain rule, \( \frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt} \). We know \( \frac{dh}{dt} = \frac{4h}{a} \) and \( A = \frac{\pi h}{a} \) so

\[ \frac{\pi h dh}{a} \cdot \frac{dh}{dt} = -k \frac{\pi h}{a} \]

giving \( \frac{dh}{dt} = -k. \)
87. (a) The volume of water in the centrifuge is \( \pi (1^2) \cdot 1 = \pi \) cubic meters. The centrifuge has total volume \( 2\pi \) cubic meters, so the volume of the air in the centrifuge is \( \pi \) cubic meters. Now suppose the equation of the parabola is \( y = h + bx^2 \).

We know that the volume of air in the centrifuge is the volume of the top part (a cylinder) plus the volume of the middle part (shaped like a bowl). See Figure 8.180.

To find the volume of the cylinder of air, we find the maximum water depth. If \( x = 1 \), then \( y = h + b \). Therefore the height of the water at the edge of the bowl, 1 meter away from the center, is \( h + b \). The volume of the cylinder of air is therefore \( [2 - (h + b)] \cdot \pi \cdot (1)^2 = [2 - h - b]\pi \).

To find the volume of the bowl of air, we note that the bowl is a volume of rotation with radius \( x \) at height \( y \), where \( y = h + bx^2 \). Solving for \( x^2 \) gives \( x^2 = (y - h)/b \). Hence, slicing horizontally as shown in the picture:

\[
\text{Bowl Volume} = \int_{h}^{h+b} \pi x^2 \, dy = \int_{h}^{h+b} \pi y - h \frac{y - h}{b} \, dy = \left[ \frac{\pi (y - h)^2}{2b} \right]_{h}^{h+b} = \frac{b\pi}{2}.
\]

We know that the volume of the air in the centrifuge is \( \pi \) cubic meters, so the volume of the air in the centrifuge is \( \pi \) cubic meters. Now suppose the equation of the parabola is \( y = h + bx^2 \).

We know that the volume of air in the centrifuge is the volume of the top part (a cylinder) plus the volume of the middle part (shaped like a bowl). See Figure 8.180.

To find the volume of the cylinder of air, we find the maximum water depth. If \( x = 1 \), then \( y = h + b \). Therefore the height of the water at the edge of the bowl, 1 meter away from the center, is \( h + b \). The volume of the cylinder of air is therefore \( [2 - (h + b)] \cdot \pi \cdot (1)^2 = [2 - h - b]\pi \).

To find the volume of the bowl of air, we note that the bowl is a volume of rotation with radius \( x \) at height \( y \), where \( y = h + bx^2 \). Solving for \( x^2 \) gives \( x^2 = (y - h)/b \). Hence, slicing horizontally as shown in the picture:

\[
\text{Bowl Volume} = \int_{h}^{h+b} \pi x^2 \, dy = \int_{h}^{h+b} \pi y - h \frac{y - h}{b} \, dy = \left[ \frac{\pi (y - h)^2}{2b} \right]_{h}^{h+b} = \frac{b\pi}{2}.
\]

So the volume of both pieces together is \( [2 - h - b]\pi + b\pi/2 = (2 - h - b/2)\pi \). But we know the volume of air should be \( \pi \), so \( (2 - h - b/2)\pi = \pi \), hence \( h + h/2 = 1 \) and \( b = 2 - 2h \). Therefore, the equation of the parabolic cross-section is \( y = h + (2 - 2h)x^2 \).

(b) The water spills out the top when \( h + b = h + (2 - 2h) = 2 \), or when \( h = 0 \). The bottom is exposed when \( h = 0 \).

Therefore, the two events happen simultaneously.

88. Any small piece of mass \( \Delta M \) on either of the two spheres has kinetic energy \( \frac{1}{2}v^2 \Delta M \). Since the angular velocity of the two spheres is the same, the actual velocity of the piece \( \Delta M \) will depend on how far away it is from the axis of revolution. The further away a piece is from the axis, the faster it must be moving and the larger its velocity \( v \). This is because if \( \Delta M \) is at a distance \( r \) from the axis, in one revolution it must trace out a circular path of length \( 2\pi r \) about the axis. Since every piece in either sphere takes 1 minute to make 1 revolution, pieces farther from the axis must move faster, as they travel a greater distance.

Thus, since the thin spherical shell has more of its mass concentrated farther from the axis of rotation than does the solid sphere, the bulk of it is traveling faster than the bulk of the solid sphere. So, it has the higher kinetic energy.

89. Any small piece of mass \( \Delta M \) on either of the two hoops has kinetic energy \( \Delta v^2 \Delta M \). Since the angular velocity of the two hoops is the same, the actual velocity of the piece \( \Delta M \) will depend on how far away it is from the axis of revolution. The further away a piece is from the axis, the faster it must be moving and the larger its velocity \( v \). This is because if \( \Delta M \) is at a distance \( r \) from the axis, in one revolution it must trace out a circular path of length \( 2\pi r \) about the axis. Since every piece in either hoop takes 1 minute to make 1 revolution, pieces farther from the axis must move faster, as they travel a greater distance.

The hoop rotating about the cylindrical axis has all of its mass at a distance \( R \) from the axis, whereas the other hoop has a good bit of its mass close (or on) the axis of rotation. So, since the bulk of the hoop rotating about the cylindrical axis is traveling faster than the bulk of the other hoop, it must have the higher kinetic energy.