Rephrasing Faraday's Law

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As physics educators, we must often find the balance between simplicity and accuracy. Particularly in introductory courses, it can be a struggle to give students the level of understanding for which they're ready without misrepresenting reality. Of course, it's in these introductory courses that our students begin to construct the conceptual framework that they'll flesh out over a physics curriculum. So a misrepresentation at this early stage will seed difficulties and stubborn misconceptions that can persist or even strengthen through subsequent courses, especially since many upper-level texts focus more on *techniques* and would not directly challenge mistaken *concepts*. In the worst cases, our students retain misunderstandings past graduation, and even pass them on to their own students. One important case is the common representation of Faraday's Law as showing that a time-varying magnetic field *causes* a circulating electric field.

This paper demonstrates that this is a widely presented claim, argues that it is impossible to deduce causality from Faraday's Law, and demonstrates that the actual cause of both the circulating electric and time-varying magnetic fields is a time-varying current density. Being one of the fundamental laws of electricity and magnetism, its misinterpretation undermines the foundations for a student's understanding of the whole subject. Because electricity and magnetism is conceptually and technically challenging, even mystifying for introductory students, it is particularly important that we avoid seeding and reinforcing this misunderstanding.

In calculus-based introductory and advanced texts, one of the two following equations is usually dubbed "Faraday's Law,"[1]

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{A} \qquad (1)$$

or

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}. \qquad (2)$$

Here, $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields, $d\vec{s}$ is a differential step along a path, and $d\vec{A}$ is a differential patch of area. In Eq. (1), the path around which the electric field is evaluated bounds the area across which the magnetic field is evaluated; the integral on the right-hand side defines the magnetic flux. In either incarnation, Faraday's Law is often claimed to demonstrate that a time-varying magnetic field or flux produces a circulating electric field (sometimes referred to as the "non-coulombic" field). According to Halliday, Resnick, and Walker's introductory text, Eq. (1) "says simply that a

changing magnetic field induces an electric field." It continues with "Induced electric fields are produced not by static charges but by a changing magnetic flux."[2] Young and Freedman's text acknowledges that "this may be a little jarring; we are accustomed to thinking about electric fields as being caused by electric charges, and now we are saying that a changing magnetic field somehow acts as a source of electric field."[3] Similar statements are found in many of the texts that share or have shared the introductory physics market over the years.[4] A few texts say that the electric field is "associated with"[5] or "accompanied by,"[6] rather than "produced by," the changing magnetic field; however, without actually indicating what *does* cause the electric field, these texts allow students to reach the same conclusion – that the time varying magnetic field produces the electric field.

Unfortunately, such a conclusion would be supported as students moved on to intermediate and even advanced texts. Purcell's intermediate-level text replaces "produces" with "determines," but the subtle difference is likely lost on students.[7] The advanced undergraduate text by Griffith's does not offer an interpretation of its own, but it does offer Faraday's: "Faraday had an ingenious inspiration: A Changing magnetic field induces an electric field. It is this 'induced' electric field that …"[8] Jackson's revered and feared graduate-level text does the same.[9] Regardless of whether these statements are correct, they would reinforce a student's misunderstanding that was seeded by an introductory text.

Proof that this misunderstanding remains with a significant fraction of physics students beyond graduation and even graduate studies is the fact that it appears in generations of introductory texts and journal articles (the authors of which were once students themselves.) For example, a recent *American Journal of Physics* article began its abstract with "Electromagnetic radiation exists because changing magnetic fields induce changing electric fields and vice versa."[10] Lest the intended meaning of "induced" be unclear, the article goes on to talk of the "magnetic field produced by the induced electric field." It is surprising, given the wealth of literature that addresses difficulties in applying Faraday's Law,[11] that this author has found few sources that address this difficulty in understanding it.[12]

While the correlations presented in Faraday's law and the Ampere-Maxwell law do allow us to deduce that a changing magnetic field is accompanied by a changing electric field, it is not necessary to claim that one "induces" or causes the other. In fact, Faraday's Law cannot be used to establish the oft-claimed causal relationship between the electric and magnetic fields. To establish causality, it is necessary to establish a time lag between the cause and the effect. In the case of two events at different locations, the reason is obvious – it takes time for information to travel from one point to another. Neither representation of Faraday's Law, Eq. (1) or Eq. (2), incorporates time lags; in fact, Eq. (2) is explicitly local – it relates the time variation of a magnetic field to the curl of an electric field at the same point in space. In such a case, the time lag may vanish, but ambiguity replaces it – it is impossible to use that relationship to establish that the changing magnetic field causes the circulating electric field (or vice versa.) That is why Jefimenko's text pointedly says that Faraday's Law communicates a "correlation," [13] and others refer to it as a "kinematic,"[14] rather than a dynamic, relationship.

The reason for the perfect correlation between a circulating electric field and a time-varying magnetic field is that they share a common cause, a time-varying current density. While the proof of this would not be accessible to introductory students, it should be accessible to instructors. It is simplest if we

begin with the time-dependent generalizations of Coulomb's Law and the Biot-Savart Law.[15] When Griffiths presents them in his text, he stresses that they are "the *causal* solutions to Maxwell's equations."[16] They give the electric and magnetic fields as

$$\vec{E}(\vec{r},t) = \frac{1}{4\pi\varepsilon_o}\int\left[\frac{\rho(\vec{r}',t_r)}{\mathcal{r}^2}\hat{\mathcal{r}} + \frac{\dot{\rho}(\vec{r}',t_r)}{c\mathcal{r}}\hat{\mathcal{r}} - \frac{\dot{\vec{J}}(\vec{r}',t_r)}{c^2\mathcal{r}}\right]d\tau', \tag{4a}$$

and

$$\vec{B}(\vec{r},t) = \frac{\mu_o}{4\pi}\int\left[\frac{\vec{J}(\vec{r}',t_r)}{\mathcal{r}^2} + \frac{\dot{\vec{J}}(\vec{r}',t_r)}{c\mathcal{r}}\right]\times\hat{\mathcal{r}}d\tau' \tag{4b}$$

Here, the electric field ($\vec{E}$) and magnetic field ($\vec{B}$) are evaluated at location $\vec{r}$ and time *t*. They are found by integrating expressions containing the charge density, ρ, and its time derivative, $\dot{\rho}$, as well as the current density, $\vec{J}$, and its time derivative, $\dot{\vec{J}}$, at all locations $\vec{r}'$ throughout the volume of space, τ'. Since a change in electric and magnetic fields propagates at speed *c*, it is necessary that the effect on $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$ of a source at point $\vec{r}'$, which is a distance $\mathcal{r} = |\vec{r} - \vec{r}'|$ away, be dependent on the charges and currents at the previous time, $t_r = t - \mathcal{r}/c$. It's important to note that the current densities can be taken to include both free and bound currents (such as the atomic-scale "currents" that are associated with magnetization) and changes in polarization over time;[17] thus these relationships are quite general.[18] Plugging the expression for the magnetic field into Equation 2 and taking the time derivative yields

$$\vec{\nabla}\times\vec{E}(r,t) = -\frac{\partial}{\partial t}\vec{B}(r,t) = -\frac{\mu_o}{4\pi}\int\left[\frac{\dot{\vec{J}}(\vec{r}',t_r)}{\mathcal{r}^2} + \frac{\ddot{\vec{J}}(\vec{r}',t_r)}{c\mathcal{r}}\right]\times\hat{\mathcal{r}}d\tau' \tag{5}$$

.

Thus the cause of the curl in the electric field and time variation of the magnetic field is a time-varying current density.[19]

Accordingly, introductory texts should offer this revised insight into Faraday's Law: "This equation says simply that a changing magnetic field is accompanied by a circulating electric field. (Both are generated by a time varying current density.)" This statement strikes the appropriate balance between simplicity and accuracy, and is far less "jarring" to students' intuition and understanding than is the mistaken statement that is common found in texts. Strictly speaking, the parenthetical statement has little bearing on the correct interpretation of Faraday's law; however, it is necessary to prevent students from inferring the common incorrect interpretation. This rephrasing should significantly demystify electric and magnetic fields by relating them back to their physical sources, rather than teaching students (or just allowing them to assume) that the fields have the unphysical capacity to source each other. We

may hope that, if students of Electricity and Magnetism begin on a firmer foundation, they will have fewer conceptual and technical difficulties later, and so will their students.

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[1] These two are not strictly equivalent; for an exact equivalence see G. Monsivais, "The relativistic Ampere-Maxwell law in integral form", *Am. J. Phys.* **72** (9), 1178-1182 (Sept. 2004).

[2] D. Halliday, R. Resnick, and J. Walker, Fundamentals of Physics, 6th ed. (Wiley, New York, 2001), p.722.
[3] H. Young and R. Freedman, Sears and Zemansky's University Physics: with Modern Physics, 12th ed. (Pearson Addison-Wesley, San Francisco, 2007), p 1009.

[4] D. Giancoli, Physics for Scientists & Engineers with Modern Physics, 4th ed. (Pearson Prentice Hall, Upper Saddle River, New Jersey, 2008), p. 773; D. Halliday and R. Resnick, Fundamentals of Physics, (Wiley, New York, 1970), p. 584; R. Knight, Physics for Scientists and Engineers: a Strategic Approach, 2nd ed. (Pearson Addison-Wesley, San Fransico, 2008), p. 1059; H. Ohanian, Physics, 2nd ed. (Norton, New York, 1989) p. 792.;R. Serway and J. Jewett Jr., Physics for Scientists and Engineers, 6th ed. (Brooks/Cole – Thomas Learning, Belmont, CA, 2004), p. 981; R. Wolfson and J. Pasachoff, Physics for Scientists and Engineers, 3rd ed. (Addison-Wesley, San Fransico, 1999), p. 820.

[5] P. Tipler, G. Mosca, Physics for Scientists and Engineers, 5th ed. (Freeman, New York, 2004), p. 900.

[6] E. Hecht, Physics: Calculus, 2nd ed. (Brooks/Cole, Pacific Grave CA, 2000), p. 817.

[7] E. Purcell, Electricity and Magnetism Berkeley Physics Course, Volume 2, 2nd ed. (McGraw Hill, New York, 1985), p. 274.

[8] D. Griffiths, Introduction to Electrodynamics, 3rd ed. (Prentice Hall, Upper Saddle River, New Jersey, 1999), p. 302.

[9] J. D. Jackson, Classical Electrodynamics, 2nd ed. (Wiley, New York, 1975), p. 210.

[10] B. Batell and A. Ferstl, "Electrically induced magnetic fields; a consistent approach", *Am. J. Phys.* **71**, 925-929 (Sept. 2003).

[11] For example, F. Sears, "Faraday's Law and Ampere's Law", *Am. J. Phys.* **31**, 439-443 (June 1963); N. Rosen and D. Schieber, "Some remarks on Faraday's Law", *Am. J. Phys.* **50**, 974-975 (Nov. 1982); F. Munley, "Challenges to Faraday's flux rule", *Am. J. Phys.* **72**, 1478-1483, (Dec. 2004); I Gallili, D. Kaplan, and Y. Lehavi, "Teaching Faraday's law of electromagnetic induction in an introductory physics course", *Am. J. Phys.* **74**, 337-343,(April 2006).

[12] J. Roche, "Explaining electromagnetic induction: a critical re-examination," *Phys. Educ.* **22**, 91-97 (1987); W. G. V. Rosser, Interpretation of Classical Electromagnetism, (Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996), pp. 124-126.

[13] O. Jefimenko, Electricity and Magnetism, 2nd ed. (Electret Scientific Company, Star City, WV, 1966), p. 498.

[14] D. H. Kobe, "Derivation of Maxwell's equations from the gauge invariance of classical mechanics", *Am. J. Phys.* **48**, 348-353 (May 1980).

[15] Ref 13, pp. 515-516.

[16] D. Griffiths, pp. 427-428. This text is recommended for a clear derivation of Eqs. (4).

[17] D. Griffiths and M. Heald, "Time-dependent generalizations of the Biot-Savart and Coulomb laws," *Am. J. Phys*. **59,** 111-117 (Feb. 1991).

[18] O. Jefimenko, "Solution of Maxwell's equations for electric and magnetic fields in arbitrary media," *Am. J. Phys.* **60,** 899-902 (Oct. 1992).

[19] Alternatively, one can directly take the curl of the Eq. (4a). Since derivatives involving retarded times are not straightforward, particularly useful are the relations A11, A12, and A13 of J. A. Heras "Can Maxwell's equations be obtained from the continuity equation?", *Am. J. Phys.* **75,** 652-657 (July 2007).