Reanalyzing the Ampere-Maxwell Law

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In a recent Physics Teacher article,[1] I addressed a common miscommunication about Faraday's law, namely, that introductory texts often say the law expresses a causal relationship between the magnetic field's time variation and the electric field's circulation. In that article, I demonstrated that these field behaviors share a common cause in a time-varying current density. From that, many readers may have rightly guessed at a symmetric conclusion: while the Ampere-Maxwell law is commonly said to express a causal relation between the electric field's time variation and the magnetic field's circulation, these field behaviors share a distinct, common cause. Together, Faraday's law and the Ampere-Maxwell law constitute half of Maxwell's laws that form a foundation for almost all of electricity and magnetism. By misrepresenting these two laws, introductory texts not only present students with unnecessary conceptual hurdles early in their physics educations but also leave them with enduring misunderstandings about the very foundation of electricity and magnetism. Fortunately, compared to what is commonly taught, the actual cause of these field variations is conceptually simpler and more consistent with what the students will have already learned in the introductory texts' own earlier chapters.

Paralleling the "Rephrasing Faraday's Law" paper,[1] this paper demonstrates that the causality of the Ampere-Maxwell law is widely claimed, argues that it is impossible to deduce causality from this relation alone, and demonstrates that current densities and their variations are the causes of the circulating magnetic field and the time-varying electric field.

The Ampere-Maxwell law is presented in calculus-based introductory and advanced texts in one or both of these forms,

$$\oint \vec{B} \cdot d\vec{s} = \mu_o \int \vec{J} \cdot d\vec{A} + \mu_o \varepsilon_o \frac{d}{dt} \int \vec{E} \cdot d\vec{A} \qquad (1)$$

or

$$\vec{\nabla} \times \vec{B} = \mu_o \vec{J} + \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t}. \qquad (2)$$

Here, $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields, $\vec{J}$ is the current density, $d\vec{s}$ is a differential step along a path, and $d\vec{A}$ is a differential patch of area. In Eq. (1), the integration path for the magnetic field bounds the integration surface for the current density and electric field. The current density's integral is the current that pierces the surface while the electric field's integral defines the electric flux through the surface. All properties are evaluated at the same point in time. In Eq. (2), all three properties, $\vec{J}$, $\vec{E}$, and $\vec{B}$, are evaluated at the same point in time and space.

In either form, both the current and the time-varying electric field are often said to cause the circulating magnetic field. For example, in building up to Eq. (1), Halliday, Resnick, and Walker's introductory text reviews Faraday's law and then notes that "Because symmetry is often so powerful in physics, we should be tempted to ask whether induction can occur in the opposite sense; that is, can a changing electric flux induce a magnetic field? The answer is that it can…. The magnetic field $\vec{B}$ [is] produced by… a current and a changing electric field."[2] Young and Freedman's text also links its representation of Faraday's law to a similar representation of the Ampere-Maxwell law. Just after a final reiteration that changing magnetic fields source electric fields, the book offers that "this may seem strange, but it's the way nature behaves. What's more, we'll see… that a changing *electric* field acts as a source of *magnetic* field."[3] Similar statements are found in many of the texts that share or have shared the introductory physics market over the years.[4]

This understanding of the Ampere-Maxwell law is apt to remain with students as they move on to intermediate-, advanced-, and even graduate-level courses. Purcell's intermediate-level text first reminds students that a changing magnetic field "is accompanied by" an electric field before informing them that a changing electric field "can give rise to" a magnetic field.[5] Perhaps neither of these two phrases is intended to imply "causes," but that would be the likely interpretation by a student who's already learned of Faraday's and Ampere-Maxwell's laws from a typical introductory text. Griffith's advanced-level text says that a changing electric field "induces" a magnetic field.[6] To instructors, this may almost seem a literal translation of the Maxwell-Ampere law into English; given our long tradition of using "induces" when discussing this law and Faraday's law, the two laws have come to virtually define what physicists mean by the word. Still, to students, "induces" is apt to mean "causes."[7] Finally, Jackson's graduate-level text says nothing to either contradict or support this understanding when it introduces Maxwell's correction to Ampere's law.[8] Thus, the next generation of teachers and text writers, who have learned from such a sequence of texts, are likely to propagate the causal understanding of the Ampere-Maxwell law.

However, the Ampere-Maxwell law does not communicate a causal relation. Causality, rather than being inherent to Maxwell's four laws, is an additional condition that must be explicitly included in a physical model of electromagnetic fields.[9] That Maxwell's equations alone do not speak to causality is suggested by their instantaneity; all factors are evaluated at the same instant in time. So even if one factor were a cause of another, that fact could not be deduced from Maxwell's laws alone. Does the time-varying electric field help to cause the circulating magnetic field, or vice versa? Or does some off-stage actor (whose influence is introduced into a model via boundary or initial conditions) help cause them both?

While the detailed solution to this question would be beyond introductory students, an outline of it may be valuable for instructors and its conclusion should significantly strengthen our students' understanding of Electricity and Magnetism. [10]

Based on different nuances in defining "cause," different answers can be supported, and over 2000 years of philosophy and science have not produced a definition that's unambiguous enough for a physicist's tastes.[11] Still, the students are likely to poses a basic understanding of "cause" that is consistent with the Principle of Causality, and, prior to their meeting either Faraday's law or the Maxwell-Ampere law, introductory students will have already learned that static charges and currents are the causes static electric and magnetic fields. Therefore, these points should be the foundation for an appropriate answer to the question of what causes an electric field's time variation or a magnetic field's circulation.

The Principle of Causality simply asserts that a *cause* event is necessary for the *effect* event, and if the events are separated by space they must also be separated by enough time for the required information to propagate from the cause location to the effect location. Adding the fact that information propagates through electromagnetic fields at the speed of light, allows us to say that an effect at location $\vec{r}$ and time *t* follows from a cause at location $\vec{r}\,'$ and time

$$t_r \equiv t - \mathscr{r}/c, \tag{3}$$

where $\vec{\mathscr{r}} \equiv \vec{r} - \vec{r}\,'$. The retarded time that is defined in Eq. (3) plays a key role in introducing causality into electricity and magnetism that, incidentally, brings with it the otherwise-absent arrow of time.[12]

Imposing this condition of causality when describing the scalar and vector potential allows us to generalize beyond the statements that static charges and currents cause static fields (the Coulomb and Biot-Savart laws) to statements that encompass the effects of *non*-static charges and currents as well (the *generalized* Coulomb and Biot-Savart laws) – statements that satisfy the five conditions of Maxwell's four laws plus Causality:[13,14, 15]

$$\vec{E}(\vec{r},t) = \frac{1}{4\pi\varepsilon_o} \int \left[\frac{\rho(\vec{r}\,',t_r)}{\mathscr{r}^2}\hat{\mathscr{r}} + \frac{\dot{\rho}(\vec{r}\,',t_r)}{c\mathscr{r}}\hat{\mathscr{r}} - \frac{\dot{\vec{J}}(\vec{r}\,',t_r)}{c^2\mathscr{r}} \right] d\tau' \tag{4a}$$

and

$$\vec{B}(\vec{r},t) = \frac{\mu_o}{4\pi} \int \left[\frac{\vec{J}(\vec{r}\,',t_r)}{\mathscr{r}^2} + \frac{\dot{\vec{J}}(\vec{r}\,',t_r)}{c\mathscr{r}} \right] \times \hat{\mathscr{r}}\, d\tau'. \tag{4b}$$

The electric and magnetic fields at location $\vec{r}$ and time *t* follow from the integrals of expressions containing the charge density, ρ, and its time derivative, $\dot{\rho}$, as well as the current density, $\vec{J}$, and its time derivative, $\dot{\vec{J}}$, at all locations $\vec{r}\,'$ throughout the volume of space, τ', at each location's appropriate retarded time, t_r. These equations have the appropriate time-lags for causal relations. Though it may not be immediately apparent, these relations are quite general since the current densities can be taken to include both free and bound currents (such as the atomic-scale "currents" that are associated with magnetization) and changes in polarization over time.[15, 16]

From Eqs 4(a) and 4(b), the time derivative of the electric field and the curl of the magnetic field can easily be obtained. The former comes from taking the time derivative of Eq. 4(a)'s integrand. In principle, the latter could come from taking the curl of Eq. 4(b)'s integrand, but since the retarded times themselves have spatial dependence, it is far simpler to invoke the Ampere-Maxwell law once the electric field's time derivative has been obtained. Thus

$$\frac{\partial}{\partial t}\vec{E}(\vec{r},t)=\frac{1}{4\pi\varepsilon_o}\int\left[\frac{\dot{\rho}(\vec{r}\,',t_r)}{\boldsymbol{\mathscr{r}}^2}\hat{\boldsymbol{\mathscr{r}}}+\frac{\ddot{\rho}(\vec{r}\,',t_r)}{c\boldsymbol{\mathscr{r}}}\hat{\boldsymbol{\mathscr{r}}}-\frac{\ddot{\vec{J}}(\vec{r}\,',t_r)}{c^2\boldsymbol{\mathscr{r}}}\right]d\tau' \tag{5}$$

and

$$\vec{\nabla}\times\vec{B}(\vec{r},t)=\mu_o\vec{J}(\vec{r},t)+\frac{\mu_o}{4\pi}\int\left[\frac{\dot{\rho}(\vec{r}\,',t_r)}{\boldsymbol{\mathscr{r}}^2}\hat{\boldsymbol{\mathscr{r}}}+\frac{\ddot{\rho}(\vec{r}\,',t_r)}{c\boldsymbol{\mathscr{r}}}\hat{\boldsymbol{\mathscr{r}}}-\frac{\ddot{\vec{J}}(\vec{r}\,',t_r)}{c^2\boldsymbol{\mathscr{r}}}\right]d\tau' \ . \tag{6}$$

From Eq. (5), we can conclude that the electric field varies with time in response to changing charge and current densities. Similarly, Eq. (6) communicates that the magnetic field's circulation is caused by these as well as by a current density. While the $\mu_o\vec{J}(\vec{r},t)$ term in Eq. (6) may be causally ambiguous (being evaluated at the same time and location as is the magnetic field's circulation), it can be derived by taking the curl of Eq. 4(b), which is itself more obviously causal.[17]

A conceptual simplification, though mathematical complication, comes from invoking the continuity equation which equates time-varying charge densities with spatially varying current densities. [18] Thereby, an equivalent conclusion to that suggested by Eq. (5) is that the electric field's time variation is caused by current densities' time and spatial variations; similarly, the magnetic field's circulation is caused by current densities and their time and spatial variation.

Should introductory texts say *this* rather than that the field variations cause each other, they would engender a far better understanding of the nature of electric and magnetic fields. Correcting both their presentation of Faraday's law[1] and the Ampere-Maxwell law would give students a more accurate and, fortuitously, simpler and more consistent understanding of the causes of electric and magnetic fields – charges and currents, their magnitudes, locations, and variations. We may hope that, after a few generations of students being taught this, popular confusions, such as light's being a cycle of electric and magnetic field variations that cause each other,[19] will vanish.

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[1] S. Eric Hill, "Rephrasing Faraday's Law", *Phys. Teach., 410-412* (Sept. 2010).

[2] D. Halliday, R. Resnick, and J. Walker, *Fundamentals of Physics,* 6th ed. (Wiley, New York, 2001), p.758-759.

[3] Young and R. Freedman, *Sears and Zemansky's University Physics: with Modern Physics,* 12th ed. (Pearson Addison-Wesley, San Francisco, 2007), p 1010.

[4] N. H. Frank, *Introduction to Electricity and Optics*, 1st ed. (McGraw-Hill, New Yourk, 1940), p.157; D. Halliday and R. Resnick, *Fundamentals of Physics*, (Wiley, New York, 1970), p. 633; H. Ohanian, *Physics*, 2nd ed. (Norton, New York, 1989) p. 849; R. Wolfson and J. Pasachoff, *Physics for Scientists and Engineers,* 3rd ed. (Addison-Wesley, San Fransico, 1999), p. 887; E. Hecht, *Physics: Calculus,* 2nd ed. (Brooks/Cole, Pacific Grave CA, 2000), p. 885; P. Tipler, G. Mosca, *Physics for Scientists and Engineers,* 5th ed. (Freeman, New York, 2004), p. 973; R. Serway and J. Jewett Jr., *Physics for Scientists and Engineers,* 6th ed. (Brooks/Cole – Thomas Learning, Belmont, CA, 2004), p. 944; D. Giancoli, *Physics for Scientists & Engineers with Modern Physics,* 4th ed. (Pearson Prentice Hall, Upper Saddle River, New Jersey, 2008), p. 813; R. Knight, *Physics for Scientists and Engineers: a Strategic Approach,* 2nd ed. (Pearson Addison-Wesley, San Fransico, 2008), p. 1094.

[5] E. Purcell, *Electricity and Magnetism Berkeley Physics Course, Volume 2,* 2nd ed. (McGraw Hill, New York, 1985), p. 326.

[6] D. Griffiths, *Introduction to Electrodynamics,* 3rd ed. (Prentice Hall, Upper Saddle River, New Jersey, 1999), p. 323.

[7] The verb "Induce" may be as pedagogically problematic and inescapable as is the noun "heat."

[8] J. D. Jackson, *Classical Electrodynamics,* 2nd ed. (Wiley, New York, 1975), p. 218.

[9] Ref. 6, pp. 422-425; F. Rohrlich, "Causality, the Coulomb field, and Newton's law of gravitation," *Am. J. Phys.* **70** (4), 411-414 (2002). Be aware that a series of Comments and Replies appear in **70** (9), **71** (7), and **72** (3) of the same journal. Equivalent to requiring causality, one can require that finite energy is initially invested in the fields; see Ref. 12.

[10] O. D. Jefimenko, "Presenting electromagnetic theory in accordance with the principle of causality," *Eur. J. Phys.* **25** (2), 287–296 (2004).

[11] R. Jones, "Resource Letter CD-1: Causality and determinism in physics," *Am. J. Phys.* **64** (3), 208-215 (1996).

[12] J. L. Anderson, "Why we use retarded potentials," *Am. J. Phys.* **60** (5), 465-467 (1992).

[13] O. Jefimenko, *Electricity and Magnetism,* 2nd ed. (Electret Scientific Company, Star City, West Virginia, 1966), pp. 515-516; An equivalent set of equations were independently produced by Panofsky and Phillips. K. T. McDonald, "The relation between expressions for time-dependent electromagnetic fields given by Jefimenko and by Panofsky and Phillips," *Am. J. Phys.* **65** (11), 1074-1076 (1997).

[14] Ref. 6, p. 302.

[15] D. J. Griffiths and M. A. Heald, "Time-Dependent Generalizations of the Biot-Savart and Coulombs Laws," *Am. J. Phys.* **59**, 111-117 (1991).

[16] O. Jefimenko, "Solution of Maxwell's equations for electric and magnetic fields in arbitrary media," *Am. J. Phys.* **60,** 899-902 (1992).

[17] J. A. Heras, "The exact relation between the displacement current and the conduction current: Comment on 'Time-dependent generalizations of the Biot-Savart and Coulomb laws' by D. J. Griffiths and M. A. Heald [Am. J. Phys. 59 (2), 111-117 (1991)]," *Am. J. Phys.* **76** (6), 592-595 (2008).

[18] *Ibid*. In the notation of this paper, Heras' equations 3 and 8 take the forms

$$\frac{\partial}{\partial t}\vec{E}(\vec{r},t) = -\frac{\vec{J}(\vec{r},t)}{3\varepsilon_o} + \frac{1}{4\pi\varepsilon_o}\int\left[\frac{3\hat{\boldsymbol{\imath}}\{\hat{\boldsymbol{\imath}}\cdot\vec{J}(\vec{r}\,',t_r)\} - \vec{J}(\vec{r}\,',t_r)}{\boldsymbol{\imath}^3} + \frac{3\hat{\boldsymbol{\imath}}\{\hat{\boldsymbol{\imath}}\cdot\dot{\vec{J}}(\vec{r}\,',t_r)\} - \dot{\vec{J}}(\vec{r}\,',t_r)}{c\boldsymbol{\imath}^2} + \frac{\hat{\boldsymbol{\imath}}\times\{\hat{\boldsymbol{\imath}}\times\ddot{\vec{J}}(\vec{r}\,',t_r)\}}{c^2\boldsymbol{\imath}}\right]d\tau' \text{ and}$$

$$\vec{\nabla}\times\vec{B}(\vec{r},t) = \mu_o\,\tfrac{2}{3}\,\vec{J}(\vec{r},t) + \frac{\mu_o}{4\pi}\int\left[\frac{3\hat{\boldsymbol{\imath}}\{\hat{\boldsymbol{\imath}}\cdot\vec{J}(\vec{r}\,',t_r)\} - \vec{J}(\vec{r}\,',t_r)}{\boldsymbol{\imath}^3} + \frac{3\hat{\boldsymbol{\imath}}\{\hat{\boldsymbol{\imath}}\cdot\dot{\vec{J}}(\vec{r}\,',t_r)\} - \dot{\vec{J}}(\vec{r}\,',t_r)}{c\boldsymbol{\imath}^2} + \frac{\hat{\boldsymbol{\imath}}\times\{\hat{\boldsymbol{\imath}}\times\ddot{\vec{J}}(\vec{r}\,',t_r)\}}{c^2\boldsymbol{\imath}}\right]d\tau'\,.$$

[19] For example: B. Batell and A. Ferstl, "Electrically induced magnetic fields; a consistent approach", *Am. J. Phys.* **71**, 925-929 (2003); H. Young and R. Freedman, p 1016; R. Wolfson and J. Pasachoff, pp. 887-888; R. Knight, pp. 1097-1098; <u>The Mechanical Universe…and Beyond</u> Part II (39/40), video produced by the California Institute of Technology and the Southern California Consurtium (distributed by Intellimation, P.O.Box 1922, Santa Barbara,CA 93116-1922; released 1985), VHS, color, 60 min.