9	Mon., 10/27 Tues. 10/28 Wed., 10/29 Fri., 10/31	**4.3** Angular Momentum 6pm More Math/Physics Research Talks – AHoN 116 **4.4.1-.2** Spin ½ & Magnetic Fields (Q5.5,6.1-.2, 8.5) **4.4.3** Addition of Angular Momenta	**Daily 9.M** **Weekly 9** **Daily 9.W** **Daily 9.F**
10	Mon. 11/3 Wed., 11/5 Fri., 11/7	**Review** (Ch 3-4) **Exam** (Ch 3-4) **5.1** Identical Particles: 2-Particle Systems (Q8.6, 11.5)	**Weekly 10** **Daily 10.F**

Equipment

- Griffith's text
- Printout of roster with what pictures I have
- Spherical Schrodinger handout
- P plot.py

Check dailies

Daily 9.M Monday 10/27 Griffiths **4. 3** Angular Momentum

4.3 Angular Momentum

Back when we were looking at the radial equation, we'd gotten to the point of

$$ER \equiv -\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}R+V(r)R,$$

and I'd paused to appeal to Classical Mechanics an attempt to bolster our understanding of that second term, what Grifiths calls the "centripetal" term. Working in spherical coordinates, the momentum breaks down into a term for the motion radially in and out and a term for the motion around and around.

$$\vec{p}=m\vec{v}=m\left(\frac{dr}{dt}\hat{r}+\frac{rd\theta}{dt}\hat{\theta}+\frac{r\sin\theta d\phi}{dt}\hat{\phi}\right)$$

The latter can be rephrased in terms of angular momentum. Squaring it and writing out the energy expression gives

$$E=K+V(r)=\frac{p^2}{2m}+V(r)=\frac{p_r^2}{2m}+\frac{L^2}{2mr^2}+V(r)$$

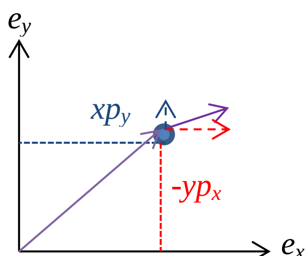
So I suggested that the term with $l(l+1)$ might be understood to represent the angular momentum of the particle, or more specifically, of the electron orbiting the proton.

Now it's time for us to focus more explicitly on angular momentum in its own right.

Classical Angular Momentum Refresher

Angular Momentum quantifies how much an object is moving *about* a point. That's defined to include both where the object is relative to the point and how quickly and in what direction it's moving. By convention, we use the right-hand-rule to designate a direction for the angular momentum as being *along* the axis about which object is moving.

For example if it were momentarily crossing the x-axis moving in the y direction it would simply have angular momentum of xp_y about the z-axis; if it were moving in the x direction while crossing the y axis, it would have angular momentum $-yp_x$ (since that's *clock-wise*) about the z axis. Now, if the object were more generally in the x-y plane and moving with a velocity in the x-y plane, the angular momentum about the z-axis would be



$$\vec{L} = (xp_y - yp_x)e_z$$

The same reasoning can be applied for moving in the x-z plane about the y axis or moving in the y-z plane about the x axis. The fully 3-D generalization is then the familiar

$$\vec{L} = (yp_z - zp_y)e_x + (zp_x - xp_z)e_y + (xp_y - yp_x)e_z = L_x e_x + L_y e_y + L_z e_z$$

$$\vec{L} \equiv \vec{r} \times \vec{p} = \det\begin{bmatrix} e_x & e_y & e_z \\ x & y & z \\ p_x & p_y & p_z \end{bmatrix}$$

Quantum Mechanical Angular momentum Operator

Well, to get from classical to quantum mechanics, we replace each of the physical parameters with operators (that can extract the measured values for the parameters by operating upon the wavefunction.) So,

$$\hat{L}_x = (y\hat{p}_z - z\hat{p}_y) = \left(y\frac{\hbar}{i}\frac{\partial}{\partial z} - z\frac{\hbar}{i}\frac{\partial}{\partial y} \right) = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right)$$

And similarly for the others.

4.3.1 Eigenvalues

Griffiths launches right into demonstrating that the three components of angular momentum do not commute with each other. This is of conceptual significance and immediately proves of mathematical utility as he sets about determining the eigenvalues.

Meaning

As for the *conceptual* significance: if two operators *don't* commute, we know it means that they *don't* share eigenvectors, and so they *aren't* "compatible", that is, the values for both operators' corresponding measurements are not well determined at the same time. So, we can say that the wavefunction does not simultaneously have well defined x and y, y and z, or z and x components of its angular momentum.

Math

As for the *mathematics*, the argument goes something like this

$$\left[\hat{L}_y,\hat{L}_z\right]=\left[\left(z\hat{p}_x-x\hat{p}_z\right),\left(x\hat{p}_y-y\hat{p}_x\right)\right]=\left[\left(z\hat{p}_x-x\hat{p}_z\right),\left(x\hat{p}_y-y\hat{p}_x\right)\right]$$

Now, you can go about this one of two ways:

break this down into all 8 terms and work it out:

$$\left[\left(z\hat{p}_x-x\hat{p}_z\right),\left(x\hat{p}_y-y\hat{p}_x\right)\right]=\left(z\hat{p}_x-x\hat{p}_z\right)\left(x\hat{p}_y-y\hat{p}_x\right)-\left(x\hat{p}_y-y\hat{p}_x\right)\left(z\hat{p}_x-x\hat{p}_z\right)=\ldots$$

Which, in point of fact, would take less work, but not give us so much practice working with commutators as:

or rephrase this as combinations of commutators and use some of what you've already learned about commutators in general and those of *p*'s and *x*'s in particular:

$$\begin{aligned}\left[\hat{L}_y,\hat{L}_z\right]&=\left[\left(z\hat{p}_x-x\hat{p}_z\right),\left(x\hat{p}_y-y\hat{p}_x\right)\right]=\left[z\hat{p}_x,\left(x\hat{p}_y-y\hat{p}_x\right)\right]-\left[x\hat{p}_z,\left(x\hat{p}_y-y\hat{p}_x\right)\right]\\&=\left[z\hat{p}_x,x\hat{p}_y\right]-\left[z\hat{p}_x,y\hat{p}_x\right]-\left[x\hat{p}_z,x\hat{p}_y\right]+\left[x\hat{p}_z,y\hat{p}_x\right]\end{aligned}$$

1. *Conceptual*: Why does [yp_z,x_p_z]=0? Explain. Why does [yp_z,zp_x]=yp_x[p_z,z]? Why can *y* and p_x come out of the commutator?

"Could we please go over how the canonical commutation relations can be manipulated?"Kyle B,

Yes, this is very confusing Jessica

Of course, if the derivatives aren't with respect to the position components that are inside the commutators, then those components are effectively constants and can come outside

$$\left[\hat{L}_y,\hat{L}_z\right]=z\left[\hat{p}_x,x\hat{p}_y\right]-zy\left[\hat{p}_x,\hat{p}_x\right]-x^2\left[\hat{p}_z,\hat{p}_y\right]+y\left[x\hat{p}_z,\hat{p}_x\right]$$

2. *Conceptual*: Work out the canonical commutation relations for the components of *r* and *p* (eq 4.10). Actually, you can explain instead of doing the math, if you want.
3. *Starting Weekly HW*: Griffiths problem 4.19

"I get that L and p are supposed to be shortcuts, but how do we apply the operators to formulas to use them? Especially L_+ and L_-." Anton

$$\left[\hat{p}_i,\hat{p}_j\right]=\left(\frac{\hbar}{i}\right)^2\left(\frac{\partial^2}{\partial r_i\partial r_j}-\frac{\partial^2}{\partial r_j\partial r_i}\right)=0$$

$$\left[\hat{p}_i, r_j\right] = \left(\frac{\hbar}{i}\right)\left(\left(\left(\ \right)\frac{\partial r_j}{\partial r_i} + r_j \frac{\partial(\ )}{\partial r_i}\right) - \left(r_j \frac{\partial(\ )}{\partial r_i}\right)\right) = \left(\frac{\hbar}{i}\right)\left(\left(\ \right)\frac{\partial r_j}{\partial r_i}\right) = \frac{\hbar}{i}\delta_{i,j}$$

Of course, p_x commutes with itself, so that term vanishes, and since the order of derivatives doesn't matter, p_z and p_y commute, so that term vanishes too. Finally, I'll switch the order if the first commutator and pay for that by introducing a negative sign.

$$\left[\hat{L}_y, \hat{L}_z\right] = -z\left[x\hat{p}_y, \hat{p}_x\right] + y\left[x\hat{p}_z, \hat{p}_x\right]$$

It was for this next step that you were asked to:

4. *Conceptual*: Prove the following commutator identity: [AB,C]=A[B,C]+[A,C]B.

So then this could be rephrased as

$$\left[\hat{L}_y, \hat{L}_z\right] = -zx\left[\hat{p}_y, \hat{p}_x\right] - z\left[x, \hat{p}_x\right]\hat{p}_y + yx\left[\hat{p}_z, \hat{p}_x\right] + y\left[x, \hat{p}_x\right]\hat{p}_z$$

Again, the order of derivatives doesn't matter, so two of these vanish and we're just left with the two $[x,p_x]$ ones; since those are reverse order of what we just demonstrated, I'll pick up a negative sign

$$\left[\hat{L}_y, \hat{L}_z\right] = z\frac{\hbar}{i}\hat{p}_y - y\frac{\hbar}{i}\hat{p}_z = \frac{\hbar}{i}\left(z\hat{p}_y - y\hat{p}_z\right)$$

$$\boxed{\left[\hat{L}_y, \hat{L}_z\right] = i\hbar\hat{L}_x}$$

If we just cycle the subscripts, we'll get the other two relations for free:

$\boxed{\left[\hat{L}_z, \hat{L}_x\right] = i\hbar\hat{L}_y}$ and $\boxed{\left[\hat{L}_x, \hat{L}_y\right] = i\hbar\hat{L}_z}$

Ladder Operators

Can we also talk about where eqn 4.105 comes from?" ladder operator def. Jessica

Notice for later use that making a complex linear combination of a couple of these relations gives

$$\left[\hat{L}_z, \hat{L}_x\right] \pm i\left[\hat{L}_z, \hat{L}_y\right] = i\hbar\hat{L}_y \pm i\left(-i\hbar\hat{L}_x\right)$$
$$\left[\hat{L}_z, \hat{L}_x \pm i\hat{L}_y\right] = \hbar\left(\hat{L}_x \pm i\hat{L}_y\right)$$

More compactly, defining $\boxed{\hat{L}_\pm \equiv \hat{L}_x \pm i\hat{L}_y,}$

$$\left[\hat{L}_z, \hat{L}_\pm\right] = \pm\hbar\hat{L}_\pm$$

What does it mean for a commutator to yield up one of the operators again?

Well, let's see what it does to L_z's eigenvectors:
Saw we have an eigenvector of L_z, such that

$$\hat{L}_z f_z = \lambda_z f_z$$

$$\left[\hat{L}_z, \hat{L}_\pm\right] f_z = \pm\hbar\hat{L}_\pm f_z$$
$$\hat{L}_z\hat{L}_\pm f_z - \hat{L}_\pm\hat{L}_z f_z = \pm\hbar\hat{L}_\pm f_z$$
$$\hat{L}_z\hat{L}_\pm f_z - \hat{L}_\pm\lambda_z f_z = \pm\hbar\hat{L}_\pm f_z$$
$$\hat{L}_z\hat{L}_\pm f_z - \lambda_z\hat{L}_\pm f_z = \pm\hbar\hat{L}_\pm f_z$$
$$\hat{L}_z\hat{L}_\pm f_z = (\lambda_z \pm \hbar)\hat{L}_\pm f_z$$

For making the implications obvious, let's define $g_z \equiv \hat{L}_\pm f_z$,

Then the final line reads $\hat{L}_z f_z = (\lambda_z \pm \hbar) g_z$

Apparently, what this funny commutator relationship means is that

If $\hat{L}_z f_z = \lambda_z f_z$ (*f_z is an eigenfunctionof L_z*)

Then $\boxed{\hat{L}_z(\hat{L}_\pm f_z) = (\lambda_z \pm \hbar)(\hat{L}_\pm f_z)}$ ($g_z \equiv \hat{L}_\pm f_z$ *is too.*)

That one of these operators returns another eigenvector with the eigenvaulue *raised* by $\hbar$ while the other returns an eigenvector with the eigenvalue *lowered* by as much makes these "ladder operators" in the same family as a_+ and a_- for the harmonic oscillator.

We're most definitely coming back to this in a moment, but first, to round out the relations between angular momentum measurable.

Magnitude of Angular Momentum

Now, aside from finding the components of the angular momentum, we'll be interested in finding its magnitude. Of course, angular momentum is a vector whose magnitude is related to its components through

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

So, the operator that, when it acts upon the wavefunction, predicts the magnitude-squared of the angular momentum should relate to the three component operators similarly:

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$$

Finding how *it* commutes with the component operator s is now trivial:

Math

$$\left[\hat{L}^2, \hat{L}_x\right] = \left[\hat{L}_x^2, \hat{L}_x\right] + \left[\hat{L}_y^2, \hat{L}_x\right] + \left[\hat{L}_z^2, \hat{L}_x\right]$$
$$= \left(\hat{L}_x\left[\hat{L}_x, \hat{L}_x\right] + \left[\hat{L}_x, \hat{L}_x\right]\hat{L}_x\right) + \left(\hat{L}_y\left[\hat{L}_y, \hat{L}_x\right] + \left[\hat{L}_y, \hat{L}_x\right]\hat{L}_y\right) + \left(\hat{L}_z\left[\hat{L}_z, \hat{L}_x\right] + \left[\hat{L}_z, \hat{L}_x\right]\hat{L}_z\right)$$
$$= 0 + \left(\hat{L}_y\left(-i\hbar\hat{L}_z\right) + \left(-i\hbar\hat{L}_z\right)\hat{L}_y\right) + \left(\hat{L}_z\left(i\hbar\hat{L}_y\right) + \left(i\hbar\hat{L}_y\right)\hat{L}_z\right) = -2i\hbar\hat{L}_z\hat{L}_y + 2i\hbar\hat{L}_z\hat{L}_y = 0$$

Ditto for the other two components, so they all commute.

Concept

Interestingly, while you *can't* know any two components at the same time, you *can* know a component and the magnitude at the same time, leaving the other two components undetermined.

Homework

In your homework, you'll demonstrate that these commute with the Hamiltonian too, so the angular momentum eigenvectors can be energy eigenvectors too. Note: for that you may need to assume (as I did) that $V(r)$ can be represented as a power series, thus commuting with r^n means commuting with $V(r)$.

Finding Eigenvalues

Yes I'd also like this. But can we go through the derivation of 4.118 as well? Jessica

I would also like this. Gigja

1. *Conceptual*: For each equation from 4.103 through 4.118, write down whether that equation is: A result of something earlier (if so, state what), or an assumption, or a guess he's trying to prove. Which of these equations proves that $L_\pm$ are ladder operators? Example: 4.103 is a direct result of the definition of the *r* and *p* operators.

Now we've learned all we need to about these operators to set about finding the eigenvalues. Now, since $\left[\hat{L}^2, \hat{L}_z\right] = 0$, we can define a basis set of vectors that are eigenvectors of both, *f*, that is

$\hat{L}^2 f = \lambda_2 f$ and $\hat{L}_z f = \lambda_z f$

But we've also found that $\hat{L}_z\left(\hat{L}_\pm f\right) = \left(\lambda_z \pm \hbar\right)\left(\hat{L}_\pm f\right)$.

So if we could only figure out what the *maximum* or *minimum* possible eigenvalue is, we could then bootstrap/recur our way up to find the eigenvalue for any subsequent eigenfunction of L_z and L^2.

Now, L_z is a component of a vector, with + and – possibilities, so there's no *minimum* that jumps to mind; however, L^2 is the magnitude of a vector, clearly it's eigenvalues (corresponding to measurable values) can't be negative. So we may be able to get a foothold by considering its possible eigenvalues.

So, how are $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$ and $\hat{L}_\pm \equiv \hat{L}_x \pm i\hat{L}_y$ related?

Looks like multiplying L_+ by its complex conjugate, i.e., L_-, should return the first two terms if it were made of mere numbers; so let's see what it gets us:

$$\hat{L}_\pm \hat{L}_\mp = \left(\hat{L}_x \pm i\hat{L}_y\right)\left(\hat{L}_x \mp i\hat{L}_y\right) = \hat{L}_x^2 + \hat{L}_y^2 \pm i\left(\hat{L}_y\hat{L}_x - \hat{L}_y\hat{L}_x\right) = \hat{L}_x^2 + \hat{L}_y^2 \pm i\left[\hat{L}_y, \hat{L}_x\right] = \hat{L}_x^2 + \hat{L}_y^2 \mp ii\hbar\hat{L}_z$$

$$\hat{L}_\pm \hat{L}_\mp = \hat{L}_x^2 + \hat{L}_y^2 \pm \hbar\hat{L}_z \Rightarrow \hat{L}_x^2 + \hat{L}_y^2 = \hat{L}_\pm\hat{L}_\mp \mp \hbar\hat{L}_z$$

So,

$$\hat{L}^2 = \hat{L}_\pm\hat{L}_\mp \mp \hbar\hat{L}_z + \hat{L}_z^2$$

Now, let's imagine two states with the *same magnitude* of angular momentum, λ_2 but one that is aligned with the z direction as much as possible, so it's the f_{top} state, and one that's aligned with the $-z$ direction as much as possible, so it's the f_{bottom} state.

That is, $\hat{L}_- f_{bottom} = 0$ and

$$\hat{L}^2 f_b = \hat{L}_+ \hat{L}_- f_b - \hbar \hat{L}_z f_b + \hat{L}_z^2 f_b$$
$$\lambda_2 f_b = \hat{L}_\pm 0 - \hbar \lambda_{z.b} f_b + \lambda_{z.b}^2 f_b$$
$$\lambda_2 = \lambda_{z.b}(\lambda_{z.b} - \hbar)$$

And $\hat{L}_+ f_{top} = 0$ and

$$\hat{L}^2 f_t = \hat{L}_- \hat{L}_+ f_t + \hbar \hat{L}_z f_t + \hat{L}_z^2 f_t$$
$$\lambda_2 f_t = \hat{L}_- 0 + \hbar \lambda_{z.t} f_t + \lambda_{z.t}^2 f_t$$
$$\lambda_2 = \lambda_{z.t}(\lambda_{z.t} + \hbar)$$

Equating the two expressions, we find that

$$\lambda_{z.t}(\lambda_{z.t} + \hbar) = \lambda_{z.b}(\lambda_{z.b} - \hbar)$$

And thus

$$\lambda_{z.b} = -\lambda_{z.t} \quad (1)$$

(or $\lambda_{z.b} = (\lambda_{z.t} + \hbar)$ which is counter to how we set up this problem – that the top state had the most positive possible eigenvalue)

Then again, since the raising operator tells us that the eigenvalues are separated by integer values of $\hbar$, so

$\lambda_{z.t} = \lambda_{z.b} + N\hbar$ (2) where N is some integer,

Then putting (1) into (2), we learn that

$$\lambda_{z.t} = -\lambda_{z.t} + N\hbar$$
$$\lambda_{z.t} = \frac{1}{2} N\hbar$$

So it must be that $\lambda_{z.t}$ is an integer or half integer multiple of $\hbar$.

Call it l.

$\hat{L}^2 f_l = \lambda_2 f_l = l(l+1)\hbar^2 f_l$ for l integer or half integer

Could we go over the paragraph above equation 4.118?" Spencer

But this same wavefunction, when acted upon by $\hat{L}_z$ has an eigenvalue λ_z that runs from $\lambda_{z.b} = -l\hbar$ up to $\lambda_{z.t} = l\hbar$ in integer steps. So superscripting the function to indicate *that* quantization,

$\hat{L}_z f_l^m = m\hbar f_l^m$ where $m = -l, \ldots l,$ *and* $\hat{L}^2 f_l^m = l(l+1)\hbar^2 f_l^m$ for l integer or half integer

4.3.2 Eigenfunctions

It'll be convenient to re-write the angular momentum operators that we've derived in spherical coordinates, then we'll be able recognize it lurking in the Hamiltonian that we've already solved in this chapter.

$$\vec{L} \equiv \vec{r} \times \vec{p}$$

where $\vec{p} = \frac{\hbar}{i}\left(\frac{\partial}{\partial x}e_x + \frac{\partial}{\partial y}e_y + \frac{\partial}{\partial z}e_z\right)$ in Cartesian. Of course, that's the gradient:

$$\vec{p} = \frac{\hbar}{i}\vec{\nabla}$$

Now, we could carefully project each of these infinitesimal changes and the unit vectors into spherical coordinates, but in general, the gradient tells us how a function changes for infinitesimal steps in each of the ortho-normal coordinate directions, so

$$\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{r} + \frac{\partial f}{r\partial\theta}\hat{\theta} + \frac{\partial f}{r\sin\theta\partial\phi}\hat{\phi}$$

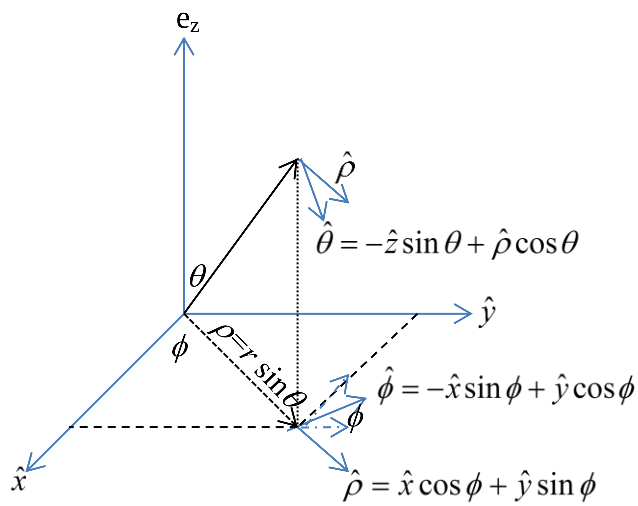
Then

$$\vec{p} = \frac{\hbar}{i}\left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$

And then

$$\vec{L} \equiv r\hat{r} \times \frac{\hbar}{i}\left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right) = \frac{\hbar}{i}\det\begin{bmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ r & 0 & 0 \\ \frac{\partial}{\partial r} & \frac{1}{r}\frac{\partial}{\partial\theta} & \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \end{bmatrix} = \frac{\hbar}{i}\left(-\hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} + \hat{\phi}\frac{\partial}{\partial\theta}\right)$$

We'll want to break this down into x,y, and z components so we can use the relations that we've already developed. Looking at the figure, we can translate the unit vectors into Cartesian ones:



$$\vec{L} = \frac{\hbar}{i}\left(-\left(-\hat{z}\sin\theta + \left(\hat{x}\cos\phi + \hat{y}\sin\phi\right)\cos\theta\right)\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} + \left(-\hat{x}\sin\phi + \hat{y}\cos\phi\right)\frac{\partial}{\partial\theta}\right)$$

$$\vec{L} = \frac{\hbar}{i}\left(\left(-\hat{x}\cos\phi + \hat{y}\sin\phi\right)\cot\theta\frac{\partial}{\partial\phi} + \left(-\hat{x}\sin\phi + \hat{y}\cos\phi\right)\frac{\partial}{\partial\theta}\right) + \frac{\hbar}{i}\hat{z}\frac{\partial}{\partial\phi}$$

$$\vec{L} = \hat{x}\frac{\hbar}{i}\left(-\cos\phi\cot\theta\frac{\partial}{\partial\phi} - \sin\phi\frac{\partial}{\partial\theta}\right) + \hat{y}\frac{\hbar}{i}\left(\cos\phi\frac{\partial}{\partial\theta} + \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right) + \hat{z}\frac{\hbar}{i}\frac{\partial}{\partial\phi}$$

So we have

$$L_x = \frac{\hbar}{i}\left(-\cos\phi\cot\theta\frac{\partial}{\partial\phi} - \sin\phi\frac{\partial}{\partial\theta}\right)$$

$$L_y = \frac{\hbar}{i}\left(\cos\phi\frac{\partial}{\partial\theta} + \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right)$$

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$$

And therefore

$$L_\pm = L_x \pm iL_y$$

$$L_\pm = \frac{\hbar}{i}\left(-\cos\phi\cot\theta\frac{\partial}{\partial\phi} - \sin\phi\frac{\partial}{\partial\theta}\right) \pm i\frac{\hbar}{i}\left(\cos\phi\frac{\partial}{\partial\theta} + \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right)$$

$$L_\pm = \hbar\left(\left(i\cos\phi \pm \sin\phi\right)\cot\theta\frac{\partial}{\partial\phi} + \left(i\sin\phi \pm \cos\phi\right)\frac{\partial}{\partial\theta}\right)$$

$$L_\pm = \hbar\left(ie^{\mp i\phi}\cot\theta\frac{\partial}{\partial\phi} \pm e^{\mp i\phi}\frac{\partial}{\partial\theta}\right) = \hbar e^{\mp i\phi}\left(i\cot\theta\frac{\partial}{\partial\phi} \pm \frac{\partial}{\partial\theta}\right) = \pm\hbar e^{\mp i\phi}\left(\frac{\partial}{\partial\theta} \mp i\cot\theta\frac{\partial}{\partial\phi}\right)$$

Then we can say that

$\hat{L}^2 = \hat{L}_\pm\hat{L}_\mp \mp \hbar\hat{L}_z + \hat{L}_z^2$ gives us

$$\hat{L}^2 = \hbar e^{-i\phi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\phi}\right)\left(\hbar e^{+i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\right) - \hbar\frac{\hbar}{i}\frac{\partial}{\partial\phi} + \left(\frac{\hbar}{i}\right)^2\frac{\partial^2}{\partial\phi^2}$$

1. *Math*: Griffiths 4.21: show that the above becomes the below.

$$\hat{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

From having found the eigenvalues of this, we can say that this operating upon an eigenvector we should have

$$\hat{L}^2 f_l^m = l(l+1)\hbar^2 f_l^m = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]f_l^m$$

$$l(l+1)f_l^m = -\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]f_l^m$$

Of course, we've seen and solved this very equation before. We know that the results are $f_l^m = Y_l^m$

2. *Conceptual*: Griffiths 4.22 (a) only
3. *Starting Weekly HW*: Griffiths: 4.23