7	Wed. 10/15 Thurs 10/16 Fri. 10/17	3.5 **Uncertainty Principle** 3.6 **Dirac Notation (Q5.6)**	**Daily 7.W** **Weekly 7** **Daily 7.F**

Equipment

- Griffith's text
- Printout of roster with what pictures I have
- Whiteboards and pens

Check dailies

Announcements:

Note: Weekly does include 3.11 and 3.15 as advertised in related daily's but not the weekly.

Daily 7.W Wednesday 10/15 Griffiths **3.5** Uncertainty Principle

1. *Math:* Write $\begin{bmatrix} a \\ b \end{bmatrix}$ in terms of |+x> and |-x>.

"Why does the axiom on functions in Hilbert spaces need to be and axiom? Is it unprovable as a theorem?" Casey P

Exactly. He notes that it can be demonstrated / proven for some particular cases but not for others. In fact, he mentions that it's odd calling such a thing an "axiom" but it certainly isn't a provable "theorem".

3.5 The Uncertainty Principle

1. Any questions about the mathematical derivations in this section. Are there steps that didn't make sense?

Without having generally proved that it *should* be true, we've been using, checking and double checking, the momentum-position uncertainty relation, $\sigma_x \sigma_p \geq \hbar/2$, for a while now. Here, Griffiths doesn't just prove that inequality, he goes a step further and proves the *general* uncertainty principle for all operators. That nifty relationship, the commutator, plays a key role.

3.5.1 Proof of the Generalized Uncertainty Principle

Way back in Chapter 1, Griffiths had defined the "Variance" of a distribution of measurements as the average of the square of the deviation of individual measurements from the average measurement:

$$\sigma_A^2 \equiv \left\langle (\Delta a)^2 \right\rangle \text{ where } \Delta a \equiv a - \langle a \rangle \text{ (eqn's 1.10 and 1.11)}$$

So

$$\sigma_A^2 = \left\langle \left(a - \langle a \rangle\right)^2 \right\rangle$$

Now, the next step is to invoke the identification that the average of any observable, <q> is found in our model (quantum mechanics) by taking the inner implied in the notation

$$\langle q \rangle = \left\langle \hat{Q} \right\rangle = \left\langle \Psi \middle| \hat{Q}\Psi \right\rangle$$

Arguably, $\Delta a = a - \langle a \rangle$ is an observable, and the associated operator would be

$$\hat{A} - \langle a \rangle$$

(to avoid confusion, I'm using the lower case *a* to represent the *value* returned and the upper-case to represent the *operator*.)

So,

$$\sigma_A^2 = \left\langle \Psi \middle| \left(\hat{A} - \langle a \rangle\right)^2 \Psi \right\rangle = \left\langle \Psi \middle| \left(\hat{A} - \langle a \rangle\right)\left(\hat{A} - \langle a \rangle\right)\Psi \right\rangle$$

Now, in this chapter we've argued that all operators associated with observables must be hermitian, i.e., must be able to be applied to the *other* function in the inner product and yield the same result (given that at least one of the functions dies off at infinity, thus is normalizable –that was a key step in the derivation).

$$\sigma_A^2 = \left\langle \Psi \middle| \left(\hat{A} - \langle a \rangle\right)\left(\hat{A} - \langle a \rangle\right)\Psi \right\rangle = \left\langle \left(\hat{A} - \langle a \rangle\right)\Psi \middle| \left(\hat{A} - \langle a \rangle\right)\Psi \right\rangle$$

As a short-hand, Griffiths denotes the new function, $f \equiv \left(\hat{A} - \langle a \rangle\right)\Psi$ so we can write

$$\sigma_A^2 = \langle f \mid f \rangle$$

Similarly, for some other measurable, *b*, we can reason that

$$\sigma_B^2 = \left\langle \left(\hat{B} - \langle b \rangle\right)\Psi \middle| \left(\hat{B} - \langle b \rangle\right)\Psi \right\rangle = \langle g | g \rangle. \text{ Where } g \equiv \left(\hat{B} - \langle b \rangle\right)\Psi$$

Then

$$\sigma_A^2 \sigma_B^2 = \langle f \mid f \rangle \langle g \mid g \rangle$$

The next step goes something like this:

$$|h\rangle \equiv |g\rangle - \frac{\langle f | g \rangle}{\langle f | f \rangle} | f \rangle \text{ and thus } \langle h | = \langle g | - \frac{\langle f | g \rangle^*}{\langle f | f \rangle^*} \langle f | = \langle g | - \frac{\langle f | g \rangle^*}{\langle f | f \rangle} \langle f |$$

Of course the inner product of something with itself is positive and real so we can drop the * for the *f*-*f* inner product.

For that matter, the *h*-*h* inner product must be positive and real too:

$$0 \le \langle h|h\rangle = \left(\langle g| - \frac{\langle f|g\rangle^*}{\langle f|f\rangle}\langle f|\right)\left(|g\rangle - \frac{\langle f|g\rangle}{\langle f|f\rangle}|f\rangle\right) = \langle g|g\rangle + \frac{\langle f|g\rangle^2}{\langle f|f\rangle^2}\langle f|f\rangle - \frac{\langle f|g\rangle}{\langle f|f\rangle}\langle g|f\rangle - \frac{\langle f|g\rangle^*}{\langle f|f\rangle}\langle f|g\rangle$$

$$0 \le \langle g|g\rangle + \frac{\langle f|g\rangle^2}{\langle f|f\rangle} - \frac{\langle f|g\rangle}{\langle f|f\rangle}\langle f|g\rangle^* - \frac{\langle f|g\rangle^*}{\langle f|f\rangle}\langle f|g\rangle$$

$$0 \le \langle g|g\rangle + \frac{\langle f|g\rangle^2}{\langle f|f\rangle} - \frac{\langle f|g\rangle^2}{\langle f|f\rangle} - \frac{\langle f|g\rangle^2}{\langle f|f\rangle}$$

$$0 \le \langle g|g\rangle - \frac{\langle f|g\rangle^2}{\langle f|f\rangle}$$

$$0 \le \langle g|g\rangle\langle f|f\rangle - \langle f|g\rangle^2$$

$$\langle f|g\rangle^2 \le \langle g|g\rangle\langle f|f\rangle$$

So,
We can now say that

$\sigma_A^2\sigma_B^2 = \langle f|f\rangle\langle g|g\rangle \ge \langle f|g\rangle^2$ (1) We'll return to this later

The next step in the derivation is to observe that
$\sigma_A^2\sigma_B^2 \ge \langle f|g\rangle^2 = \mathcal{Re}\langle f|g\rangle^2 + \mathcal{Im}\langle f|g\rangle^2 \ge \mathcal{Im}\langle f|g\rangle^2$ (2) we'll return to this later too
What *is* the imaginary part in terms of the whole? Well, if we can write it in the form of
$z = z_{re} + iz_{im}$ and similarly $z^* = z_{re} - iz_{im}$ then $z_{im} = \frac{1}{2i}\left(z - z^*\right)$
So,

$\sigma_A^2\sigma_B^2 \ge \left(\frac{\langle f|g\rangle - \langle f|g\rangle^*}{2i}\right)^2$ where $\langle f|g\rangle^* = \langle g|f\rangle$

$$\sigma_A^2\sigma_B^2 \ge \left(\frac{\langle f|g\rangle - \langle g|f\rangle}{2i}\right)^2$$

Time to go back to talking about the operators A and B.

$$\langle f|g\rangle - \langle g|f\rangle = \left\langle (\hat{A} - \langle a\rangle)\Psi \middle| (\hat{B} - \langle b\rangle)\Psi \right\rangle - \left\langle (\hat{B} - \langle b\rangle)\Psi \middle| (\hat{A} - \langle a\rangle)\Psi \right\rangle = \left\langle \Psi \middle| (\hat{A} - \langle a\rangle)(\hat{B} - \langle b\rangle)\Psi \right\rangle - \left\langle \Psi \middle| (\hat{B} - \langle b\rangle)(\hat{A} - \langle a\rangle)\right.$$

once again using that the operators are hermitian to move them across the inner product

$$\langle f|g\rangle - \langle g|f\rangle = \left\langle \Psi \middle| \left(\left(\hat{A}-\langle a\rangle\right)\left(\hat{B}-\langle b\rangle\right) - \left(\hat{B}-\langle b\rangle\right)\left(\hat{A}-\langle a\rangle\right)\right)\Psi \right\rangle$$

$$= \left\langle \Psi \middle| \left(\left(\hat{A}\hat{B} - \hat{A}\langle b\rangle - \langle a\rangle\hat{B} + \langle a\rangle\langle b\rangle\right) - \left(\hat{B}\hat{A} - \langle b\rangle\hat{A} - \hat{B}\langle a\rangle + \langle b\rangle\langle a\rangle\right)\right)\Psi \right\rangle$$

$$= \left\langle \Psi \middle| \left(\hat{A}\hat{B} - \hat{B}\hat{A}\right)\Psi \right\rangle = \left\langle \hat{A}\hat{B} - \hat{B}\hat{A} \right\rangle = \left\langle \left[\hat{A},\hat{B}\right] \right\rangle$$

So, at long last, we have

$$\boxed{\sigma_A^2\sigma_B^2 \ge \left(\frac{\left\langle\left[\hat{A},\hat{B}\right]\right\rangle}{2i}\right)^2}$$

Starting Weekly HW: Griffiths Problem 3.15 Show that two noncommuting operators cannot have a complete set of common eigenfunctions.

Say $f = \sum c_{n,a}\psi_{n,a} = \sum c_{m,b}\psi_{m,b}$, so it's describable in terms of both basis sets, that is, they are both complete *enough* to describe it.

$$\left(\hat{A}\hat{B} - \hat{B}\hat{A}\right)f = \sum c_{mb}\left(\hat{A}\hat{B} - \hat{B}\hat{A}\right)\psi_{m,b} = \sum c_{mb}\left(\hat{A}\hat{B}\psi_{m,b} - \hat{B}\hat{A}\psi_{m,b}\right) = \sum c_{mb}\left(\hat{A}b_m\psi_{m,b} - \hat{B}\hat{A}\psi_{m,b}\right)$$

Now, *if* the eigen states are the same for A and B, then $\psi_{n,a} = \psi_{m,b}$ and we have

$$\left(\hat{A}\hat{B} - \hat{B}\hat{A}\right)f = \sum c_{mb}\left(\hat{A}b_m\psi_{n,a} - \hat{B}\hat{A}\psi_{n,a}\right) = \sum c_{mb}\left(a_n b_m\psi_{n,a} - \hat{B}a_n\psi_{n,a}\right)$$

And switching back again, $\psi_{n,a} = \psi_{m,b}$,

$$\left(\hat{A}\hat{B} - \hat{B}\hat{A}\right)f = \sum c_{mb}\left(a_n b_m\psi_{m,b} - \hat{B}a_n\psi_{m,b}\right) = \sum c_{mb}\left(a_n b_m\psi_{m,b} - b_m a_n\psi_{m,b}\right) = 0$$

So, the commutator is 0 and thus the uncertainty principle simply says that the products of the standard deviations is equal to or greater than 0.

However, if they are merely complete, but not the same eigenfunctions, then it's the case that $\psi_{m,b} = \sum_n \langle\psi_{n,a}|\psi_{m,b}\rangle\psi_{n,a} = \sum_n d_{n,m}\psi_{n,a}$ and similarly $\psi_{n,a} = \sum_m \langle\psi_{m,b}|\psi_{n,a}\rangle\psi_{m,b} = \sum_n d^*_{n,m}\psi_{m,b}$

So

$$\left(\hat{A}\hat{B} - \hat{B}\hat{A}\right)f = \sum_m c_{mb}\left(\hat{A}\hat{B} - \hat{B}\hat{A}\right)\psi_{m,b} = \sum_m c_{mb}\left(\hat{A}\hat{B}\psi_{m,b} - \hat{B}\hat{A}\psi_{m,b}\right) = \sum_m c_{mb}\left(\hat{A}b_m\psi_{m,b} - \hat{B}\hat{A}\sum_n d_{n,m}\psi_{n,a}\right)$$

$$= \sum_m c_{mb}\left(\hat{A}\left(\sum_n d_{n,m}\psi_{n,a}\right)b_m - \hat{B}\sum_n d_{n,m}a_n\psi_{n,a}\right) = \sum_m c_{mb}\left(\left(\sum_n d_{n,m}a_n\psi_{n,a}\right)b_m - \hat{B}\sum_n d_{n,m}a_n\psi_{n,a}\right)$$

$$= \sum_m c_{mb}\left(\left(\sum_n d_{n,m}a_n\psi_{n,a}\right)b_m - \hat{B}\sum_n d_{n,m}a_n\left(\sum_{m'} d^*_{n,m'}\psi_{m',b}\right)\right)$$

$$= \sum_m c_{mb}\left(\left(\sum_n d_{n,m}a_n\psi_{n,a}\right)b_m - \hat{B}\sum_n d_{n,m}a_n\left(\sum_{m'} d^*_{n,m'}\psi_{m',b}\right)\right)$$

$$=\sum_m c_{mb}\left(\left(\sum_n d_{n,m}a_n\psi_{n,a}\right)b_m-\sum_n d_{n,m}a_n\left(\sum_{m'} d^*_{n,m'}b_{m'}\psi_{m',b}\right)\right)=\sum_m c_{mb}\sum_n d_{n,m}a_n\left(b_m\psi_{n,a}-\sum_{m'} d^*_{n,m'}b_{m'}\psi_{m',b}\right)$$

And then getting it all back in the same basis set

$$=\sum_m c_{mb}\sum_n d_{n,m}a_n\left(b_m\sum_{m'} d^*_{n,m'}\psi_{m',b}-\sum_{m'} d^*_{n,m'}b_{m'}\psi_{m',b}\right)=\sum_m\sum_n\sum_{m'} c_{mb}d^*_{n,m'}d_{n,m}a_n\psi_{m',b}\left(b_m-b_{m'}\right)$$

Which won't go to 0 unless $d_{n,m}=\delta_{n,m}$, that is, the two basis sets are the same.

Could we talk about how we know if two observables are compatible or incompatible? What does it mean to have shared eigenfunctions?" Spencer

I agree, can we go over this? Jessica

Does shared eigenfunctions literally mean they have the exact same eigenfunctions (and eigenvalues)? Gigja

Does the complete set of eigenfunctions have to be simultaneous for observables to be compatible? Kyle B

The laboratory explanation makes sense to me, in that the outcome of the second measurement can't contradict the first for them to be compatible, but how to determine it mathematically is not clear. Bradley W

Now, you may be wondering "why go with the imaginary in the first place?" We could have gone after the real part instead:

$$\sigma_A^2\sigma_B^2\ge\langle f|g\rangle^2=\mathcal{Re}\langle f|g\rangle^2+\mathcal{Im}\langle f|g\rangle^2\ge\mathcal{Re}\langle f|g\rangle^2$$

$z_{re}=\frac{1}{2}\left(z+z^*\right)$ which would have led to $\sigma_A^2\sigma_B^2\ge\left(\frac{\langle f|g\rangle+\langle g|f\rangle}{2}\right)^2$

Which eventually gets us to

$$\left\langle\Psi\middle|\left(\left(\hat{A}\hat{B}-\hat{A}\langle b\rangle-\langle a\rangle\hat{B}+\langle a\rangle\langle b\rangle\right)+\left(\hat{B}\hat{A}-\langle b\rangle\hat{A}-\hat{B}\langle a\rangle+\langle b\rangle\langle a\rangle\right)\right)\middle|\Psi\right\rangle$$

$$=\left\langle\Psi\middle|\left(\hat{A}\hat{B}+\hat{B}\hat{A}+2\left(\langle a\rangle\langle b\rangle-\hat{A}\langle b\rangle-\langle a\rangle\hat{B}\right)\right)\middle|\Psi\right\rangle$$

$$=\left\langle\hat{A}\hat{B}+\hat{B}\hat{A}\right\rangle+2\left(\langle a\rangle\langle b\rangle-\langle a\rangle\langle b\rangle-\langle a\rangle\langle b\rangle\right)$$

$$=\left\langle\hat{A}\hat{B}+\hat{B}\hat{A}\right\rangle$$

So it must also be true that $\sigma_A^2\sigma_B^2\ge\left(\frac{\left\langle\hat{A}\hat{B}+\hat{B}\hat{A}\right\rangle}{2}\right)^2$

I don't a priori see a reason one of these conditions would be more restrictive than the other.

Let's give this a go for the position and momentum operators:

$$\hat{A} \to \hat{x} = x \qquad \sigma_x^2 \sigma_p^2 \ge \left(\frac{\langle [\hat{x}\hat{p}, \hat{p}\hat{x}] \rangle}{2i} \right)^2$$

$$\hat{B} \to \hat{p} = \frac{\hbar}{i}\frac{d}{dx}$$

$$\langle [\hat{x}, \hat{p}] \rangle = \left\langle \Psi \middle| x \frac{\hbar}{i}\frac{d}{dx}\Psi - \frac{\hbar}{i}\frac{d}{dx}(x\Psi) \right\rangle$$

$$\langle [\hat{x}, \hat{p}] \rangle = \left\langle \Psi \middle| x \frac{\hbar}{i}\frac{d}{dx}\Psi - x\frac{\hbar}{i}\frac{d}{dx}(\Psi) - \Psi\frac{\hbar}{i}\frac{d}{dx}(x) \right\rangle$$

$$\langle [\hat{x}, \hat{p}] \rangle = \left\langle \Psi \middle| -\Psi\frac{\hbar}{i} \right\rangle = -\frac{\hbar}{i}\langle \Psi | \Psi \rangle = -\frac{\hbar}{i}$$

$$\sigma_x^2 \sigma_p^2 \ge \left(\frac{-\hbar / i}{2i} \right)^2 = \left(\frac{\hbar}{2} \right)^2$$

$$\sigma_x \sigma_p \ge \frac{\hbar}{2}$$

The *general uncertainty principle's* derivation is quite general. It only supposes that there *are* two measurable. As you'll connect in the homework, if they completely share a basis set of eigen states, then the commutator for the observables will be 0, and they two measurements are compatible – being in a state for which one measurement is definite, the other measurement is also definite. Conversely, if the two observables do *not* completely share a basis set, then measuring one observable can leave you in a state for which the other observable's value is *not* determined, and there will be an uncertainty.

Math: Construct the uncertainty relation for Energy and position.

2. *Starting Weekly HW:* Consider three observables, A, B, and C. We know that [B, C] = A and [A, C] = B. Show that $\sigma_{AB}\sigma_C \ge \frac{1}{2i}\langle A^2 + B^2 \rangle$.

3.5.2 The Minimum-Uncertainty Wave Packet

Looking back over our derivation, we have

$$\sigma_A^2\sigma_B^2 = \langle f|f\rangle\langle g|g\rangle \ge \langle f|g\rangle^2 = \mathcal{Re}\langle f|g\rangle^2 + \mathcal{Im}\langle f|g\rangle^2 \ge \mathcal{Im}\langle f|g\rangle^2 = \left(\frac{\langle f|g\rangle - \langle g|f\rangle}{2i}\right)^2 = \left(\frac{\langle[\hat{A},\hat{B}]\rangle}{2i}\right)^2$$

where $f \equiv (\hat{A}-\langle a\rangle)\Psi$ and $g \equiv (\hat{B}-\langle b\rangle)\Psi$.

The minimum uncertainty is when these limits are equalities:

$$\langle f|f\rangle\langle g|g\rangle = \langle f|g\rangle\langle g|f\rangle$$

This works if $g = cf$:

$$\langle f|f\rangle\langle cf|cf\rangle = \langle f|cf\rangle\langle cf|f\rangle$$

$$\langle f|f\rangle c^* c\langle f|f\rangle = \langle f|f\rangle cc^*\langle f|f\rangle$$

$$\mathcal{Re}\langle f|g\rangle^2 + \mathcal{Im}\langle f|g\rangle^2 = \mathcal{Im}\langle f|g\rangle^2$$

This works if $\mathcal{Re}\langle f|g\rangle^2 = 0$, so

$\langle f|g\rangle$ is purely imaginary.

Putting these two together,
$\langle f|g\rangle = \langle f|cf\rangle = c\langle f|f\rangle = c$ where c is imaginary. Making that explicit:
$c = id$ where d is real.

So, we have that $g = idf$
Or plugging back in in terms of the operators and average values,

$$(\hat{B}-\langle b\rangle)\Psi = id(\hat{A}-\langle a\rangle)\Psi$$

Momentum & Position

$$\hat{A} \to \hat{x} = x$$

$$\hat{B} \to \hat{p} = \frac{\hbar}{i}\frac{d}{dx}$$

$$\left(\frac{\hbar}{i}\frac{d}{dx} - \langle p\rangle\right)\Psi = id(x-\langle x\rangle)\Psi$$

This differential equation is solved by $\Psi(x) = Ae^{-d(x-\langle x\rangle)^2}e^{i\langle p\rangle/\hbar}$ a Gaussian.

3.5.3 The Energy-Time Uncertainty Principle

1. *Starting Weekly HW*: Show that the expectation value of any observable in a stationary state does not change with time, provided the time rate of change of the operator for the observable is zero.

Since time isn't a 'measurable' in the sense that it's a property of a particle; as Griffiths puts it, time is the *independent* variable upon which the system's position, momentum, and energy may be *dependent.* Say we have an observation that depends on time.

$$\frac{d}{dt}\langle q\rangle = \frac{d}{dt}\langle\Psi|\hat{Q}\Psi\rangle = \left\langle\frac{\partial}{\partial t}\Psi\middle|\hat{Q}\Psi\right\rangle + \left\langle\Psi\middle|\frac{\partial}{\partial t}\left(\hat{Q}\Psi\right)\right\rangle$$

If the operator itself has time dependence, $\hat{Q}(x,p,t)$, then the second term splits into two

$$\frac{d}{dt}\langle q\rangle = \left\langle\frac{\partial}{\partial t}\Psi\middle|\hat{Q}\Psi\right\rangle + \left\langle\Psi\middle|\frac{\partial\hat{Q}}{\partial t}(\Psi)\right\rangle + \left\langle\Psi\middle|\hat{Q}\frac{\partial}{\partial t}\Psi\right\rangle$$

We can call on Schrodinger's equation to substitute $i\hbar\frac{\partial}{\partial t}\Psi = \hat{H}\Psi$

$$\frac{d}{dt}\langle q\rangle = \left\langle\frac{1}{i\hbar}\hat{H}\Psi\middle|\hat{Q}\Psi\right\rangle + \left\langle\Psi\middle|\frac{\partial\hat{Q}}{\partial t}(\Psi)\right\rangle + \left\langle\Psi\middle|\hat{Q}\frac{1}{i\hbar}\hat{H}\right\rangle$$

Using that the Hamiltonian (energy operator) must be hermitian to yield real measurable energies,

$$\frac{d}{dt}\langle q\rangle = \frac{1}{-i\hbar}\langle\Psi|\hat{H}\hat{Q}\Psi\rangle + \left\langle\Psi\middle|\frac{\partial\hat{Q}}{\partial t}(\Psi)\right\rangle + \frac{1}{i\hbar}\langle\Psi|\hat{Q}\hat{H}\rangle = \frac{i}{\hbar}\left\langle\Psi\middle|\left(\hat{H}\hat{Q}-\hat{Q}\hat{H}\right)\Psi\right\rangle + \left\langle\Psi\middle|\frac{\partial\hat{Q}}{\partial t}(\Psi)\right\rangle$$

$$\boxed{\frac{d}{dt}\langle q\rangle = \frac{i}{\hbar}\left\langle\left[\hat{H},\hat{Q}\right]\right\rangle + \left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle}$$

Invoking the general uncertainty principle,

$$\frac{\hbar}{i}\left(\frac{d}{dt}\langle q\rangle - \left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle\right) = \left\langle\left[\hat{H},\hat{Q}\right]\right\rangle$$

$$\sigma_H^2\sigma_Q^2 \ge \left(\frac{\left\langle\left[\hat{H},\hat{Q}\right]\right\rangle}{2i}\right)^2 = \left(\frac{\hbar}{i2i}\left(\frac{d}{dt}\langle q\rangle - \left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle\right)\right)^2 = \left(\frac{\hbar}{2}\left(\left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle - \frac{d\langle q\rangle}{dt}\right)\right)^2$$

$$\sigma_H\sigma_Q \geq \left(\frac{\left\langle\left[\hat{H},\hat{Q}\right]\right\rangle}{2i}\right)^2 = \left(\frac{\hbar}{i2i}\left(\frac{d}{dt}\langle q\rangle - \left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle\right)\right)^2 = \frac{\hbar}{2}\left|\left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle - \frac{d\langle q\rangle}{dt}\right|$$

|

$$\sigma_H \frac{\sigma_Q}{\left|\left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle - \frac{d\langle q\rangle}{dt}\right|} \geq \frac{\hbar}{2}$$

The 'uncertainty' in t is then defined as $\Delta t \equiv \frac{\sigma_Q}{\left|\left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle - \frac{d\langle q\rangle}{dt}\right|}$ and we have $\sigma_H \Delta t \geq \frac{\hbar}{2}$

In the case that the operator has not time dependence, and it's just that the observable's average is changing,

$\Delta t = \frac{\sigma_Q}{\left|\frac{d\langle q\rangle}{dt}\right|}$ Or $\Delta t\left|\frac{d\langle q\rangle}{dt}\right| = \sigma_Q$ so the time we're talking about is the time it takes for the average measured value to drift by one standard deviation (if the rate of change were constant.)

Example: mixed state for the simple harmonic oscillator, time for the average position to change.

He warns that the energy-time uncertainty principle is 'robust' enough that people can use it with pretty misguided concepts in mind and still get good answers, like the notion that conservation of energy can be violated. A spread of energy states go into constructing the system, that doesn't mean that energy itself varies.

"Can we go over what "substantial change in a system refers to in regards to delta t?"

Mark T,

Often the uncertainty relations are used for 'rule of thumb', 'back of the envelope', or 'ballpark' kinds of calculations. So folks get pretty vague when using it. Concretely, the defining relationship is

$$\Delta t\left|\frac{d\langle q\rangle}{dt}\right| = \sigma_Q$$

But if we're just spitballing, the time it takes for a noticeable change (since the standard deviation is our rough measure for the width of a distribution of measurement values.)