Mon., 11/25	*(C14)* **4.3** Electric Displacement	
Mon., 12/2 Wed., 12/4	*(C14)* **4.4.1** Linear Dielectrics (read rest at your discretion) **6.1** Magnetization	HW10

Last Time

Polarization

$$\vec{P} = \text{dipole moment per volume} \ , \ \vec{P} = \frac{d\vec{p}}{d\tau} \text{ (this is generally, a function of location)}$$

which may be induced by an external electric field or "frozen in."

Bound Charges

$$\sigma_b = \vec{P}\cdot\hat{n} \quad \text{and} \quad \rho_b = -\vec{\nabla}\cdot\vec{P} \qquad (4.11\ \&\ 4.12)$$

$\hat{n}$ is a unit vector normal to the surface (pointing outward).

Potential

$$V(\vec{r}) = V_{surf}(\vec{r}) + V_{vol}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\oint \frac{\sigma_b(\vec{r}')da'}{\mathscr{r}} + \frac{1}{4\pi\varepsilon_o}\oint \frac{\rho_b(\vec{r}')d\tau'}{\mathscr{r}}$$

$$\vec{E} = -\vec{\nabla}V$$

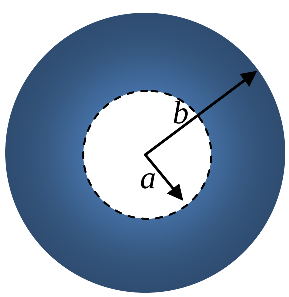
This Time

Example: In the spirit of last time,

Problem 4.15

A thick spherical shell (inner radius *a* and outer radius *b*) is made of dielectric material with a "frozen-in" polarization

$$\vec{P}(\vec{r}) = \frac{k}{r}\hat{r}\,.$$



(There is no free charge in the problem.)

a. Locate all of the bound charge and use Gauss's law to calculate the electric field.

The bound volume charge is

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 P_r\right) = -\frac{1}{r^2}\frac{\partial}{\partial r}(kr) = -\frac{k}{r^2}\,.$$

On the outer surface, $r = b$ and $\hat{n} = \hat{r}$, so the bound surface charge there is

$$\sigma_b = \vec{P}\cdot\hat{n} = \left(\frac{k}{b}\hat{r}\right)\cdot\hat{r} = \frac{k}{b}.$$

On the inner surface, $r = a$ and $\hat{n} = -\hat{r}$, so the bound surface charge there is

$$\sigma_b = \vec{P}\cdot\hat{n} = \left(\frac{k}{a}\hat{r}\right)\cdot(-\hat{r}) = -\frac{k}{a}.$$

The problem is spherically symmetric, so $\vec{E} = E(r)\hat{r}$ and a sphere of radius r is a good choice for a Gaussian surface. Regardless of the radius, the electric flux is $\oint \vec{E}\cdot d\vec{a} = E\cdot 4\pi r^2$. The charge enclosed is $Q_{enc} = 0$ for $r < a$ and $r > b$ (net bound charge had *better* be zero) and for $a < r < b$ it is

$$Q_{enc} = \left(-\frac{k}{a}\right)(4\pi a^2) + \int_a^r \left(-\frac{k}{r^2}\right)(4\pi r^2 dr) = -4\pi ka - 4\pi k(r-a) = -4\pi kr.$$

Using Gauss's law, $\oint \vec{E}\cdot d\vec{a} = Q_{enc}/\varepsilon_0$, the electric field is

$$\vec{E} = \begin{cases} 0 & r < a,\ r > b, \\ -(k/\varepsilon_0 r)\hat{r} & a < r < b. \end{cases}$$

<u>Summary</u>

Electric Field inside a Dielectric (4.2.3)

The *microscopic* electric field calculated using the positions of all of the charges in a polarized dielectric is impossible to calculate and not useful unless you're looking at microscopic-scaled interactions within that medium.

We'll concentrate on the *macroscopic* electric field which is the average over a region large enough to contain thousands of atoms. This eliminates all of the microscopic fluctuations.

The Electric Displacement

We'll define the *electric displacement* as

$$\vec{D} \equiv \varepsilon_0\vec{E} + \vec{P}.$$

This is a useful definition because we can write a "Gauss's law" for $\vec{D}$ that only depends on the free charge (<u>not</u> the bound charge):

$$\vec{\nabla}\cdot\vec{D} = \varepsilon_0(\vec{\nabla}\cdot\vec{E}) + (\vec{\nabla}\cdot\vec{P}) = \rho - \rho_b = \rho_f,$$

$$\vec{\nabla}\cdot\vec{D} = \rho_f.$$

The integral form of the law (use the Divergence Theorem) is

$$\oint \vec{D}\cdot d\vec{a} = Q_{f,enc}.$$

$\vec{D}$ is Not Exactly Analogous to $\vec{E}$

The free charge density ρ_f does <u>not</u> completely determine $\vec{D}$ in the same way that the total charge density ρ determines the *electrostatic* field $\vec{E}$!

This is because it takes both curl and divergence (Helmholtz Theorem) to completely define a vector field. In electrostatics, the curl of E is 0, so all (non-trivial) information is wrapped up in E's divergence, which is determined by ρ.

$$\vec{\nabla}\times\vec{D}=\varepsilon_0\left(\vec{\nabla}\times\vec{E}\right)+\vec{\nabla}\times\vec{P}=\vec{\nabla}\times\vec{P}. \qquad (4.25)$$

Since $\vec{\nabla}\times\vec{D}\neq 0$, it *can't* be represented by a gradient of a scalar, i.e., it generally can't have a scalar potential associated with it.

Recall all the work we had to go through to get the "generalized" Coulomb's Law in Chapter 10, when E was *not* curl-less. Without going through all that work we can pretty safely say that, our *not-curl-less* D would not be solved by a simple Coulomb's Law:

$$\vec{D}\neq\frac{1}{4\pi}\int\rho_f\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^2}\right)d\tau.$$

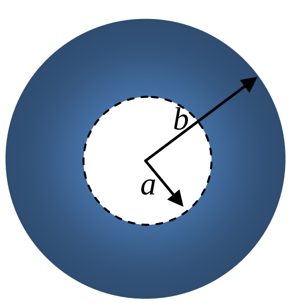
However, if there is enough symmetry to use Gauss's law for the electric displacement, then $\vec{\nabla}\times\vec{P}=0$, thus $\vec{\nabla}\times\vec{D}=0$. Then knowing the divergence is sufficient for finding D. Otherwise, you'll have to use a more indirect rout (like finding E and P individually)

For those reading ahead to the next section, a new relationship between D and E is introduced; it's worth commenting on what it does and does not imply. In a *linear* medium, the polarization, and therefore D is proportional to E: $\vec{D}=\varepsilon_x\vec{E}$. Two warnings about that: first, this relation *only holds for linear media* (that is, when the polarization is *induced* and the response is small enough as to be approximated as linear) and it's piece-wise defined – that is, if you have two different media right next to each other (say plastic and air) the constant of proportionality changes discontinuously. Thus D doesn't simply mimic E across the boundary.

Example: Return to Problem 4.15

A thick spherical shell (inner radius *a* and outer radius *b*) is made of dielectric material with a "frozen-in" polarization

$$\vec{P}(\vec{r})=\frac{k}{r}\hat{r}.$$



(There is no free charge in the problem.) Find the electric field everywhere

b. Use the new version of Gauss's law to find $\vec{D}$, then get the electric field from the definition of $\vec{D}$.

The problem is spherically symmetric, so we can use Gauss's law, $\oint \vec{D} \cdot d\vec{a} = Q_{f,enc}$. Since $Q_{f,enc} = 0$ for any sphere, $\vec{D} = 0$ everywhere.

From the definition, $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$, we know that $\vec{E} = -\vec{P}/\varepsilon_0$. There is only a polarization inside the dielectric material, so

$$\vec{E} = \begin{cases} 0 & r < a,\ r > b, \\ -\left(k/\varepsilon_0 r\right)\hat{r} & a < r < b. \end{cases}$$

Problem 4.16

Suppose the electric field inside a large piece of dielectric is $\vec{E}_0$, so that the electric displacement is $\vec{D}_o = \varepsilon_o \vec{E}_o + \vec{P}_o$.

a. A spherical cavity is carved out.

  a. Find the electric field at the center of the cavity in terms of $\vec{E}_0$ and $\vec{P}$.

    i. The hint points out that carving out a cavity is like superimposing an object of opposite polarization (so the net polarization is 0). So, imagine introducing an oppositely polarized sphere. Example 4.2 found that the electric field inside the sphere *due to the sphere alone* would be

$$\vec{E}_{sphere} = -\frac{1}{3\varepsilon_o}\vec{P}_{sphere}$$

Where the polarization of the sphere is opposite that of the medium, so $\vec{E}_{sphere} = \frac{1}{3\varepsilon_o}\vec{P}_o$.

Then the field at it's center is the sum,

$$\vec{E} = \vec{E}_o + \vec{E}_{sphere} = \vec{E}_o + \frac{1}{3\varepsilon_o}\vec{P}_o$$

  b. Find the electric displacement at the center of the cavity in terms of $\vec{D}_0$ and $\vec{P}_o$.

  Now, inside the sphere, the actual value of polarization is 0, so there,

$$\vec{D} = \varepsilon_o \vec{E}$$

$$\vec{D} = \varepsilon_o \left(\vec{E}_o + \vec{E}_{sphere} \right) = \varepsilon_o \left(\vec{E}_o + \frac{1}{3\varepsilon_o} \vec{P}_o \right) = \varepsilon_o \vec{E}_o + \frac{1}{3} \vec{P}_o =$$

$$\vec{D} = \vec{D}_o - \frac{2}{3} \vec{P}_o$$

b. Repeat for a thin wafer-shaped cavity perpendicular to $\vec{P}$.

If the polarization $\vec{P}$ points upward, then the bound surface charge on the bottom of the cavity ($\hat{n}$ is upward) is $\sigma_b = -P$ and on the top surface of the cavity ($\hat{n}$ is downward) is $\sigma_b = +P$. The electric field due to the bound charges is P/ε_0 in the upward direction. The electric field in the cavity is $\vec{E} = \vec{E}_0 + \vec{P}/\varepsilon_0$.

There is no polarization inside the cavity, so

$$\vec{D} = \varepsilon_0 \vec{E} = \varepsilon_0 \left(\vec{E}_0 + \vec{P}/\varepsilon_0 \right) = \varepsilon_0 \vec{E}_0 + \vec{P} = \vec{D}_0 .$$

c. A long needle-shaped cavity running parallel to $\vec{P}$ is hollowed out of the material. Find the electric field at the center of the cavity in terms of $\vec{E}_0$ and $\vec{P}$. Find the electric displacement at the center of the cavity in terms of $\vec{D}_0$ and $\vec{P}$.

Since the polarization is uniform, there was no bound volume charge removed. There are bound surface charges on the ends of the needle because $\vec{P}$ is perpendicular. These charges are small and far from the center, so $\vec{E} \approx \vec{E}_0$.

There is no polarization inside the cavity, so $\vec{D} = \varepsilon_0 \vec{E} \approx \varepsilon_0 \vec{E}_0 = \vec{D}_0 - \vec{P}$.

Boundary Conditions

Suppose there is a free surface charge σ_f on an interface between two materials.

Applying Gauss's law, $\oint \vec{D} \cdot d\vec{a} = Q_{f,enc}$, for a thin pillbox that crosses the interface gives

$$D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f .$$

Equation 4.25 (above) can be written in integral form (using the Curl theorem) as $\oint \vec{D} \cdot d\vec{\ell} = \oint \vec{P} \cdot d\vec{\ell}$. Applying this to a thin rectangular loop that crosses the interface gives

$$\vec{D}^{\parallel}_{above} - \vec{D}^{\parallel}_{below} = \vec{P}^{\parallel}_{above} - \vec{P}^{\parallel}_{below} .$$

In contrast, the boundary conditions that we got for E were

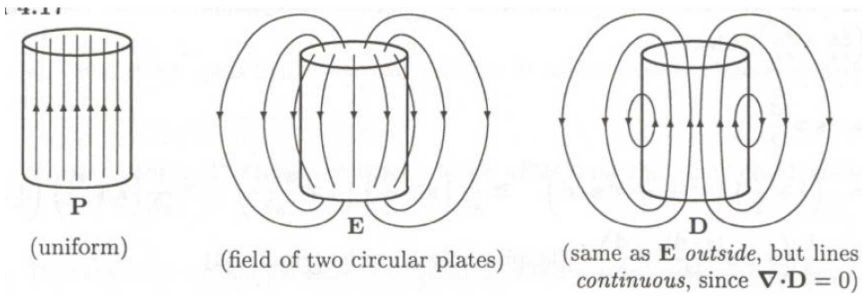
$$E_{above}^{\perp} - E_{below}^{\perp} = \frac{1}{\varepsilon_o}\sigma$$

$$E_{above}^{||} - E_{below}^{||} = 0$$

(for electro *statics* where the curl of E is 0)

Prob. 4.17 For the bar electrete of Prob. 4.1, make three careful sketches

For example, a cylinder with a "frozen-in" uniform polarization $\vec{P}$ has non-zero electric displacement $\vec{D}$, even though there is no free charge (see Prob. 4.17). The electric field is like that of a capacitor because the bound surface charge is on the ends.



Note that $\vec{D}$ "curls" around in the third diagram above since there is no free charge and $\vec{D}$ doesn't diverge, $\vec{\nabla}\cdot\vec{D} = \rho_f = 0$. In general, the curl of $\vec{D}$ is <u>not</u> zero:

$$\vec{\nabla}\times\vec{D} = \varepsilon_0\left(\vec{\nabla}\times\vec{E}\right) + \vec{\nabla}\times\vec{P} = \vec{\nabla}\times\vec{P}. \quad (4.25)$$

Preview

On Monday, we'll discuss linear dielectrics where the polarization is proportional to the total electric field.

"We've seen a couple of sections before about boundary conditions, but I was wondering if you could at least mention in a bit more detail than Griffith's did the importance/usefulness of these BCs that he mentions in section 4.3.3." Casey McGrath

"In section 4.3.2 he talks about not being able to find a vector field with only the divergence but I didn't really understand his reasoning for that." Jessica
It was one of those vector calc theorems. To fully specify a vector field you need to know its divergence AND curl. Once you have both of those things you have the field, but only one of them is not enough. Freeman,
Divergence is found by taking derivatives in each direction and summing them. This means that all the perpendicular terms are lost. Like Freeman said, those would be preserved in the vector curl, so both could give you the field together. Anton

"Can we talk a little more about why rho in Gausses law blows up at the surface of the dielectric?" Rachael Hach

"Could we possibly touch on why it's important to recognize that there is no "Coulomb's law" for D and go a little more in depth as to the parallels between E and D?" Ben Kid

"Can we talk about equation 4.25? I had trouble understanding it." Connor W,

"Can we go over example 4.4?" Casey P,

"Can we go over Griffiths' reasoning immediately following Example 4.4?" Spencer

"Can we talk about ways to find D if symmetry cannot be used? (This was mentioned at the end of the paragraph at the top of p. 185)" Sam

"I'm a little confused about rho f. The chapter says it is 'everything else' but then we do calculations and add the equation for the bound charge, creating a sum. This makes mathematical sense but I get lost with physical intuition from that point on" Anton