Mon., 12/2 Wed., 12/4 Fri., 12/6	*(C14)* **4.4.1** Linear Dielectrics (read rest at your discretion) **6.1** Magnetization **6.2** Field of a Magnetized Object	HW10
Mon., 12/9	**6.3, 6.4** Auxiliary Field & Linear Media	HW11
Wed. 12/11	**12 noon** **Exam 3 (Ch 7, 10, 4, 6)**	

Last Time

Polarization

$\vec{P}=$ dipole moment per volume , $\vec{P}=\frac{d\vec{p}}{d\tau}$ (this is generally, a function of location)

which may be induced by an external electric field or "frozen in."

Bound Charges

$$\sigma_b=\vec{P}\cdot\hat{n} \quad \text{and} \quad \rho_b=-\vec{\nabla}\cdot\vec{P} \qquad (4.11 \text{ \& } 4.12)$$

$\hat{n}$ is a unit vector normal to the surface (pointing outward).

Potential

$$V(\vec{r})=V_{surf}(\vec{r})+V_{vol}(\vec{r})=\tfrac{1}{4\pi\varepsilon_o}\oint\frac{\sigma_b(\vec{r}')da'}{\mathscr{r}}+\tfrac{1}{4\pi\varepsilon_o}\oint\frac{\rho_b(\vec{r}')d\tau'}{\mathscr{r}}$$

$\vec{E}=-\vec{\nabla}V$

This Time

Summary

The Electric Displacement

We'll define the *electric displacement* as

$$\vec{D}\equiv\varepsilon_0\vec{E}+\vec{P}.$$

$$\vec{\nabla}\cdot\vec{D}=\rho_f.$$

$$\vec{\nabla}\times\vec{D}=\varepsilon_0\left(\vec{\nabla}\times\vec{E}\right)+\vec{\nabla}\times\vec{P}=\vec{\nabla}\times\vec{P}. \qquad (4.25)$$

Summary

Linear Dielectrics

A "Linear Dielectric" is one in which, one way or another, the Polarization is linearly proportional to the applied field. You can imagine two scenarios. One is that you have a substance full of permanent dipoles that are free to rotate, so, at a moderate temperature, they tend to be randomly oriented (giving no average polarization), but then when you apply an electric field, you start lining them up. Another scenario is that the substance is full of unpolarized objects that, when the external field is applied, become polarized – get induced. In either case, the effect needs to be small enough to be well approximated by the linear term in a Taylor Series expansion. Back when we introduced induced dipoles, we ran some numbers and saw that even for a fairly sizable external field, we'd expect the linear approximation to be quite good (specifically, we saw that a fairly large external field could off-center an atom's electron cloud by only a miniscule fraction of its radius).

For linear dielectrics, the polarization is proportional to the electric field,

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} \,, \tag{4.30}$$

where χ_e is the *electric susceptibility*. This is a good approximation for weak fields.

Now, this looks hauntingly familiar to something we'd suggested at the beginning of the chapter, and it's worth seeing how the two fit together. We started the chapter by talking about individual dipoles, and we said that, in a linear medium

$$\vec{p} = \alpha \vec{E}_{ext}$$

The induced dipole would be proportional to the electric field that induced it, where we called the proportionality constant the polarizability.

Later, we said that it would be handy to talk about a whole conglomerate, not individual dipoles, but whole distributions, and for that, it's handier talking about the *density* of dipoles,

$$\vec{P} \equiv \frac{d\vec{p}}{d\tau}$$

So, when we now return to linear media, if each individual dipole is proportional to the field it experiences, the same is true for the local density – double the field, and you double the density of dipoles,

$$\vec{P} = \frac{d\alpha}{d\tau} \vec{E}$$

So the proportionality constant for the polarization should be the density of polarizabilities. So, one could say that the *electric susceptibility* is defined by

$$\chi_e = \frac{1}{\varepsilon_o} \frac{d\alpha}{d\tau}$$

For what it's worth (which may not be much). Note a little slight of hand. When talking about the individual dipole, I stressed that it was the "external" field that mattered, i.e., the field that the dipole *experiences,* not the one that it produces. Of course, if we're focusing on one dipole in a sea of billions, then the "external" field is that due to any far away sources plus that due to all the neighboring dipoles. So when we talk about the polarization for the whole conglomerate, that "external" includes again, the fields due to both 'external' sources and sources within the conglomerate.

The particular virtue of using P rather than p is

a) how it relates to charge densities

$$\sigma_b = \vec{P}\cdot\hat{n} \quad \text{and} \quad \rho_b = -\vec{\nabla}\cdot\vec{P} \qquad (4.11\ \&\ 4.12)$$

b) how it relates to D and the simplicity of doing calculations with D (if the symmetry is right.)

The electric displacement for linear dielectrics is

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P} = \varepsilon_0\vec{E} + \varepsilon_0\chi_e\vec{E}$$
$$\vec{D} = \varepsilon_0\left(1+\chi_e\right)\vec{E} = \varepsilon\vec{E}$$

where $\varepsilon \equiv \varepsilon_0\left(1+\chi_e\right)$ is the *permittivity*. Often, tables (and problems) give the *dielectric constant*, which is $\varepsilon_r = \varepsilon/\varepsilon_0 \equiv 1+\chi_e$. This gives $\vec{D} = \varepsilon_0\varepsilon_r\vec{E}$.

Or, for that matter,

$$\vec{P} = \frac{\chi_e}{\varepsilon_r}\vec{D} = \frac{\chi_e}{1+\chi_e}\vec{D}$$

We can think of the polarization in this case as sort of a feedback (iterative) process:

(1) the external electric field causes a polarization

(2) the polarized material produces an electric field in the opposite direction inside, so it reduces the electric field inside

(3) the polarization adjusts to the new electric field…

We'll use this in an alternate way to do a calculation (or two) at the end of class.

If $\vec{\nabla}\times\vec{D} = 0$

As we discussed last time, if there is enough symmetry to find the electric displacement $\vec{D}$ using Gauss's law, it is easy to find the electric field in linear dielectrics.

Instead of $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$, we have

$$\vec{\nabla}\cdot\vec{D} = \rho_f.$$

Of course, that's the only charge that would be there *if* there *weren't* any dielectric. So, if it helps to think of it this way, the D you find is essentially the E you'd find if there hadn't been any Dielectric, $\vec{E}_{vac}$. If the entire space between the free charges is filled with dielectric, the free charge doesn't change, so the electric displacement that results is $\vec{D} = \varepsilon_0\vec{E}_{vac}$ (the Gauss's laws differ by ε_0). If the dielectric is linear, then

$$\vec{E} = \frac{\vec{D}}{\varepsilon} = \frac{\varepsilon_0\vec{E}_{vac}}{\varepsilon} = \frac{\vec{E}_{vac}}{\varepsilon_r}.$$

This *doesn't* work across a boundary between vacuum and dielectric because $\vec{\nabla}\times\vec{P} \neq 0$, so $\vec{\nabla}\times\vec{D} \neq 0$.

<u>Examples/Exercises</u>:

Problem 50 / 4.18 Let's consider a simplified version of problem 4.18 just for the sake of brevity. Say we have only one dielectric material, of constant ε_r between two capacitor plates.

a) Electric Displacement

$$\oint D \cdot d\vec{a} = Q_{encl}$$

Do a Gaussian box through the top slab

$$D_{outside} A_{top} + D_{inside} A_{bottom} = Q_{encl}$$

$$0 + D_{inside} A_{bottom} = Q_{encl}$$

$$D_{inside} = \frac{Q_{encl}}{A_{bottom}} = \sigma_{free}$$

b) Electric Field

$$\vec{E} = \frac{\vec{D}}{\varepsilon} = \frac{\sigma_{free}}{\varepsilon}(-\hat{z}) = \frac{\sigma_{free}}{\varepsilon_r \varepsilon_o}(-\hat{z})$$

c) Polarization

$$\vec{P} = \vec{D} - \varepsilon_o \vec{E} = \sigma_{free}(-\hat{z}) - \frac{\sigma_{free}}{\varepsilon_r}(-\hat{z}) = -\left(1 - \frac{1}{\varepsilon_r}\right)\sigma_{free}(\hat{z})$$

d) Potential Difference

$$\Delta V = -\int_{bottom}^{top} \vec{E} \cdot d\vec{l} = -\int_{bottom}^{top} \frac{\sigma_{free}}{\varepsilon_r \varepsilon_o}(-\hat{z}) \cdot d\vec{z} = \frac{\sigma_{free}}{\varepsilon_r \varepsilon_o} a$$

e) Bound charge

$$\rho_b = -\vec{\nabla} \bullet \vec{P} = 0$$

$$\sigma_b = \vec{P} \cdot \hat{n}$$

At the top,

$$\sigma_b = \vec{P} \cdot \hat{z} = -\left(1 - \frac{1}{\varepsilon_r}\right)\sigma_{free}$$

At the bottom

$$\sigma_b = \vec{P} \cdot -\hat{z} = \left(1 - \frac{1}{\varepsilon_r}\right)\sigma_{free}$$

f) E from charge distribution

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{encl}}{\varepsilon_o} = \frac{Q_{f.encl} + Q_{b.encl}}{\varepsilon_o}$$

$$E_{inside} A_{bottom} = \frac{Q_{f.encl} + Q_{b.encl}}{\varepsilon_o}$$

$$E_{inside} = \frac{Q_{f.encl}/A + Q_{b.encl}/A}{\varepsilon_o} = \frac{\sigma_f + \sigma_b}{\varepsilon_o} = \frac{\sigma_f - \left(1 - \frac{1}{\varepsilon_r}\right)\sigma_f}{\varepsilon_o} = \frac{\sigma_f}{\varepsilon_r \varepsilon_o}$$

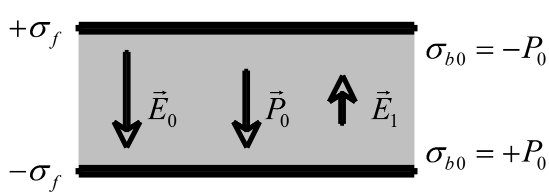
And geometry again tells us that the field should point down.

Example 4.6 – Dielectric Material in a Parallel-Plate Capacitor (Alternate Method!)

Suppose a material with a dielectric constant ε_r is placed inside a parallel-plate capacitor. How does this affect the electric field inside?

Think of the polarizing of the material as an iterative process. This will seem like an unnecessarily complex approach for this problem, but we'll see a pay-off on the next problem

First, the vacuum electric field $\vec{E}_0 = \vec{E}_{vac}$ (make it downward) produces a polarization of $\vec{P}_0 = \varepsilon_0 \chi_e \vec{E}_0$ in the same direction. The polarization $\vec{P}_0$ produces an additional electric field in the opposite direction because of bound surface charges ($\sigma_b = \vec{P} \cdot \hat{n}$) of $-P_0$ on the top and $+P_0$ on the bottom. The additional electric field is $\vec{E}_1 = -\vec{P}_0 / \varepsilon_0 = -\chi_e \vec{E}_0$. The diagram below shows these steps.



The additional electric field $\vec{E}_1$ produces a polarization of $\vec{P}_1 = \varepsilon_0 \chi_e \vec{E}_1 = -\varepsilon_0 \chi_e^2 \vec{E}_0$ in the upward direction. This additional polarization produces bound charges $|\sigma_{b1}| = P_1$ with the positive on the bottom and negative on top. This results in an additional downward electric field of $\vec{E}_2 = -\vec{P}_1 / \varepsilon_0 = (-\chi_e)^2 \vec{E}_0$.

Continue this iterative process to find that the nth term in the electric field is $\vec{E}_n = (-\chi_e)^n \vec{E}_0$. (Note that this expression even works for the zero term.) The total electric inside the dielectric is

$$\vec{E}_{inside} = \vec{E}_0 + \vec{E}_1 + \vec{E}_2 + \ldots = \sum_{n=0}^{\infty} \vec{E}_n = \sum_{n=0}^{\infty} (-\chi_e)^n \vec{E}_0 .$$

If $\chi_e < 1$, this geometric series converges to

$$\vec{E}_{inside} = \left(\frac{1}{1+\chi_e} \right) \vec{E}_0 = \frac{1}{\varepsilon_r} \vec{E}_0 .$$

The field inside the dielectric is uniform and smaller than the external electric field because $\varepsilon_r > 1$.

Problem 4.23 – Dielectric Sphere in a Uniform Electric Field (Alternate Method!)

Now here's a trickier problem. A sphere made of linear dielectric material is placed in an otherwise uniform electric field $\vec{E}_0$. Find the electric field inside the sphere in terms of the material's dielectric constant ε_r.

To get started, we'll have to use the result of Example 4.2: the electric field inside a sphere with uniform polarization $\vec{P}$ is $\vec{E} = -\vec{P}/3\varepsilon_0$.

The external electric field $\vec{E}_0$ produces a polarization of $\vec{P}_0 = \varepsilon_0 \chi_e \vec{E}_0$. The polarization $\vec{P}_0$ produces an additional electric field in the opposite direction, $\vec{E}_1 = -\vec{P}_0/3\varepsilon_0 = (-\chi_e/3)\vec{E}_0$. The additional electric field $\vec{E}_1$ causes the polarization to adjust by $\vec{P}_1 = \varepsilon_0 \chi_e \vec{E}_1$. The adjustment to the polarization $\vec{P}_1$ creates an extra electric field in the opposite direction, $\vec{E}_2 = -\vec{P}_1/3\varepsilon_0 = (-\chi_e/3)^2 \vec{E}_0$. The n[th] contribution to the electric field is $\vec{E}_n = (-\chi_e/3)^n \vec{E}_0$. The total electric field inside the linear dielectric is

$$\vec{E}_{inside} = \vec{E}_0 + \vec{E}_1 + \vec{E}_2 + \ldots = \sum_{n=0}^{\infty} \vec{E}_n = \sum_{n=0}^{\infty} \left(-\frac{\chi_e}{3} \right)^n \vec{E}_0 .$$

If $\chi_e < 3$ (it is curious that there is an extra condition on the solution using this method!), this geometric series converges to

$$\vec{E}_{inside} = \left(\frac{1}{1+\chi_e/3} \right) \vec{E}_0 = \left(\frac{3}{3+\chi_e} \right) \vec{E}_0 = \left[\frac{3}{2+(1+\chi_e)} \right] \vec{E}_0 = \left(\frac{3}{2+\varepsilon_r} \right) \vec{E}_0 .$$

Which is what Griffith's obtains by a much more complicated method.

The field inside the dielectric is uniform and smaller than the external electric field because $\varepsilon_r > 1$. Note that in this case, $\vec{E} \neq \vec{E}_{vac}/\varepsilon_r$ because the space between the free charges (a large capacitor) is not completely filled with dielectric material.

Example 4.6 – Capacitor Filled with a Dielectric

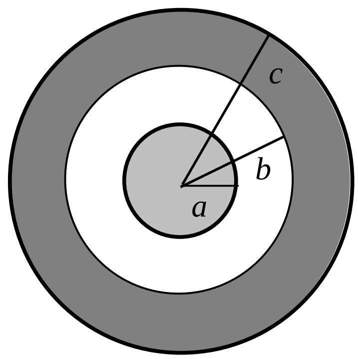
Suppose a material with a dielectric constant ε_r is placed inside a parallel-plate capacitor. How does this affect the capacitance?

For a given charge, the electric field inside is reduced by a factor of $1/\varepsilon_r$, so the potential difference is also reduced by the same factor. The capacitance is $C = Q/V$, so it is increased by a factor of ε_r. If we call the capacitance when filled with vacuum is C_{vac}, then the capacitance when filled with a linear dielectric is

$$C = \varepsilon_r C_{vac}.$$

Problem 4.21

A coaxial cable consists of a copper wire of radius *a* surrounded by a concentric copper tube of inner radius *c*. The space between is partially filled (from *b* to *c*) with material of dielectric constant ε_r as shown below. Find the capacitance per length of the cable.



Let *Q* be the charge on a length *L* of the inner conductor. By symmetry, we know that $\vec{D} = D(s)\hat{s}$, so use a cylinder of radius *s* and length *L* as a Gaussian surface.

Regardless of the radius, the flux of $\vec{D}$ is $\oint \vec{D}\cdot d\vec{a} = D\cdot 2\pi sL$. The free charge enclosed is

$$Q_{f,enc} = \begin{cases} 0 & s < a, \\ Q & a < s < c. \end{cases}$$

Apply Gauss's law, $\oint \vec{D}\cdot d\vec{a} = Q_{f,enc}$, to get $\vec{D} = (Q/2\pi sL)\hat{s}$ for $a < s < c$. The electric field is $\vec{E} = \vec{D}/\varepsilon_0\varepsilon_r$, so

$$\vec{E} = \begin{cases} (Q/2\pi\varepsilon_0 sL)\hat{s} & a < s < b, \\ (Q/2\pi\varepsilon_0\varepsilon_r sL)\hat{s} & b < s < c. \end{cases}$$

The potential difference between the metal parts is

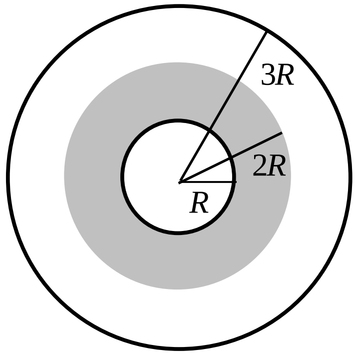
$$V = -\int_c^a \vec{E}\cdot d\vec{\ell} = -\int_c^b \frac{Q}{2\pi\varepsilon_0\varepsilon_r sL}\,ds - \int_b^a \frac{Q}{2\pi\varepsilon_0 sL}\,ds = \frac{Q}{2\pi\varepsilon_0 L}\left[\ln\left(\frac{b}{a}\right) + \frac{1}{\varepsilon_r}\ln\left(\frac{c}{b}\right)\right].$$

The capacitance per length is

$$\frac{C}{L} = \frac{Q}{LV} = \frac{2\pi\varepsilon_0}{\left[\ln(b/a) + (1/\varepsilon_r)\ln(c/b)\right]}.$$

Spherical Capacitor Partially Filled with Dielectric – have students try this

There are two metal spherical shells with radii R and $3R$. There is material with a dielectric constant $\varepsilon_r = 3/2$ between radii R and $2R$. What is the capacitance?



Suppose that there is a positive (free) charge Q on the inner metal shell and a negative charge $-Q$ on the outer metal shell. By symmetry, we know that $\vec{D} = D(r)\hat{r}$, so use a sphere of radius r as a Gaussian surface. Regardless of the radius, the flux of $\vec{D}$ is $\oint \vec{D}\cdot d\vec{a} = D\cdot 4\pi r^2$. For $R < r < 3R$, the free charge enclosed is $Q_{f,enc} = Q$. Apply Gauss's law, $\oint \vec{D}\cdot d\vec{a} = Q_{f,enc}$, to get $\vec{D} = \left(Q/4\pi r^2\right)\hat{r}$ between the metal shells. The electric field is $\vec{E} = \vec{D}/\varepsilon_0\varepsilon_r$, so

$$\vec{E} = \begin{cases} \left(Q/4\pi\varepsilon_0\varepsilon_r r^2\right)\hat{r} = \left(Q/6\pi\varepsilon_0 r^2\right)\hat{r} & R < r < 2R, \\ \left(Q/4\pi\varepsilon_0 r^2\right)\hat{r} & 2R < r < 3R. \end{cases}$$

The potential difference between the shells is

$$V = -\int_{3R}^{R} \vec{E}\cdot d\vec{\ell} = -\int_{3R}^{2R} \frac{Q}{4\pi\varepsilon_0 r^2}\,dr - \int_{2R}^{R} \frac{Q}{6\pi\varepsilon_0\varepsilon_r r^2}\,dr = \frac{Q}{\pi\varepsilon_0}\left\{\left[\frac{1}{4r}\right]_{3R}^{2R} + \left[\frac{1}{6r}\right]_{2R}^{R}\right\}$$

$$= \frac{Q}{\pi\varepsilon_0 R}\left\{\left[\frac{1}{8} - \frac{1}{12}\right] + \left[\frac{1}{6} - \frac{1}{12}\right]\right\} = \frac{Q}{8\pi\varepsilon_0 R}.$$

The capacitance is

$$C = Q/V = 8\pi\varepsilon_0 R.$$

Note about Example 4.8. In this example, a point charge is positioned above a linear dielectric. Naturally, this induces a polarization, but since

$$\vec{P} = \frac{\chi_e}{\varepsilon_r}\vec{D} = \frac{\chi_e}{1+\chi_e}\vec{D}$$

That means

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\frac{\chi_e}{1+\chi_e}\vec{\nabla}\cdot\vec{D} = -\frac{\chi_e}{1+\chi_e}\rho_f = 0$$

Since there is no free charge in the dielectric. Thus all the charge is on its surface – a great case for a boundary value problem. I.e., if you find a solution that is consistent with the boundary, you've found *the* solution.

Thus, Griffiths does some work to find the electric field on the surface of the dielectric. Then, once he has that he notices that this field expression looks just like the field due to a pair of point charges, the original 'object' charge, q_{obj} a distance *d* above the surface, and a second 'image' charge, $q_{image} = -q_{obj}\frac{\chi_e}{\chi_e + 2}$ a distance *d* below the surface. This should be true off the plane too.

Now here's the new bit. While, from above, it looks like the image charge is location *d* below the surface, from below, it looks like the image charge is location *d above* the surface. Why should that be? Think of it this way, the charge is *really on* the surface, and it's distributed in such a way that, when viewed from above, it appears like an image charge distance *d* on the other side. This is analogous to having a 3-D rendered image on a sheet of glass – say you've got your red & blue glasses on and the image is drawn in red and blue so it looks like there's a ball distance *d* behind the glass. Now, if you go around to the other side and look (you'll also need to flip your glasses), what should it look like? A ball a distance *d* behind the glass the other way!

Think about the limiting case of a conductor (instead of a dielectric). In that case, the image charge is equal and opposite to the object charge. Now, if you go down into the conductor and look up, what do you see, the (equal and opposite) image charge sitting right on top of the object charge – thus 0 field (as we must have in a conductor.)

Preview

On Wednesday, we'll start talking about magnetic fields in materials (Ch. 6). We begin by talking about interactions of magnetic dipoles.

"I am still unsure when exactly we can use the equations for D and E on pg 189 (up until 4.35)?" Jessica

"In the first paragraph of 4.4.1, what constitutes "too strong?" Griffiths mentions this as the limit of being able to use the equations here, but doesn't give a good sense as to what systems (or specific examples) would break the theory."Casey McGrath
This was my question also.Freeman
Also, based off of equation 4.30, is it true that you can draw the conclusion using equations 4.1 and the equation hidden in the paragraph before equation 4.9 to state that epsilon_0 * Chi_e = alpha / dtau' ? Or is this mislead? One thing I thought of is that E in equation 4.30 is in reference to the entire electric field, whereas in 4.1 it may be just due to the one point charge. Basically I was just trying to draw a parallel between this new relation and the one given in equation 4.1. Casey McGrath

"Can we talk about the differences and similarities between eqns 4.35 and 4.19 which are both equations for the field inside a dielectric?" Sam

"Can we talk about why different materials would have higher or lower dielectric constants?" Casey P,

"Can we do another example of dielectric materials so we can see more uses of D and P?" Davies

"I'm just a little curious about the electric susceptibility term. If the permissivity of free space is e0, then X0 must be 0. Is there any conceptuality behind this or is it really just to serve as a term hanging of e to non-dimensionalize then term?" Rae

"I thought that footnote 11 was very interesting (pg 189). How can "nothing" be polarized? Does it have to do with the fact that QED predicts particle-antiparticle creation/annihilation out of the vacuum randomly, so in a sense even a perfect void could suddenly contain charge? And frankly, in the case that we are looking at space near a black hole, since this phenomena gives rise to Hawking radiation, perhaps the void near a black hole (imagining we have no dust particles or anything) could carry charge, which could give it polarization? Casey McGrath

"Besides the numerical representations of susceptibility and permittivity what is their conceptual meaning." Antwain

"In reference to Example 4.5 on pg 187, he gives the whole, "to get V, we need E, but to get E we need the bound charge, and to get that we need P, but we can't get P unless we know E....". So my question is, using all of these arguments derived. in this section, what makes it different than what we did in the last section? Is it that in those problems, when we calculated E we were told some P distribution, but now we are imagining we don't just happen to know P, and so we need a new way of getting to E with what we do know? Casey McGrath
How does looking at materials that obey Equation 4.30 lead us to anything different than what we looked at in previous chapters? Spencer

"Griffiths mentions not wanting to suggest that the vacuum is like another linear dialectric material. Can we talk about why this is and how permittivity is different than susceptibility?"
Ben