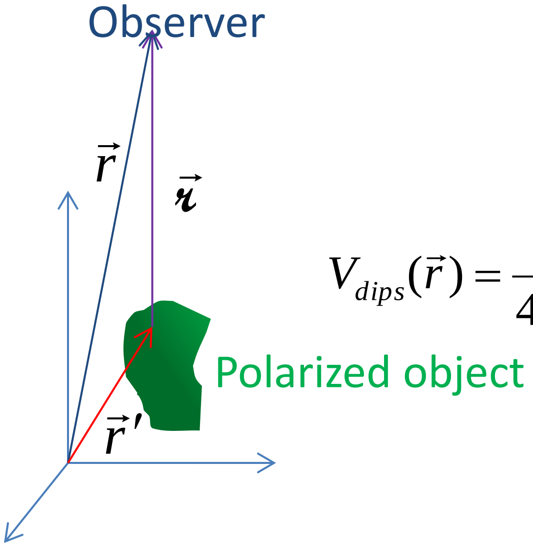
Wed.	(C14) 4.3 Electric Displacement Washington 3-2 Rep 7pm AHoN 116	
Thurs.		HW5
Fri.	(C14) 4.4.1 Linear Dielectrics (read rest at your discretion)	

From last Time: Polarization



$$\vec{P} \equiv \frac{d\vec{p}}{d\tau}$$

$$V_{dips}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_{volume} \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) \cdot \vec{P}(\vec{r}\,')\, d\tau' = \frac{1}{4\pi\varepsilon_0} \left[\int_{surface} \frac{\sigma_b}{\mathscr{r}} da' + \int_{volume} \frac{\rho_b}{\mathscr{r}} d\tau' \right]$$

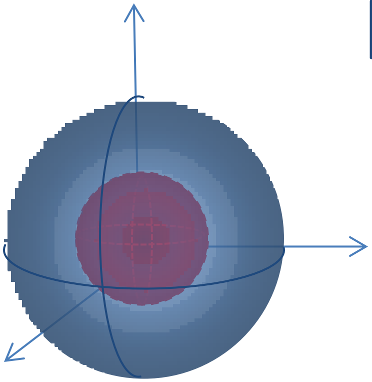
where
$$\sigma_b = \vec{P} \cdot \hat{a} \quad \text{and} \quad \rho_b = -\vec{\nabla} \cdot \vec{P}$$

$$Q_b = \int_{volume} \rho_b \, d\tau' + \oint_{surface} \sigma_b \, da$$

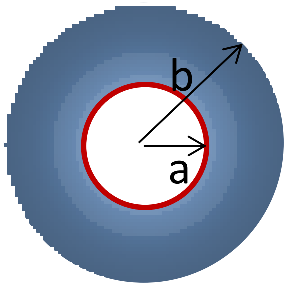
Polarization & "Bound Charge"

Exercise

$$\sigma_b = \vec{P} \cdot \hat{a} \quad \text{and} \quad \rho_b = -\vec{\nabla} \cdot \vec{P}$$



A thick spherical shell (inner radius *a* and outer radius *b*) is made of dielectric material with a "frozen-in" polarization $\vec{P}(\vec{r}) = \frac{k}{r}\hat{r}$



Locate all of the bound charge and use Gauss's law to calculate the electric field in the three regions.

Cross-sectional view

[redacted]

The Electric "Displacement"

Quite Generally, Gauss's law says

$$\vec{\nabla} \cdot \vec{E} = \tfrac{1}{\varepsilon_o} \rho_{(all)}$$

Now we also relate "bound" charge (due to variation in density of dipoles)

$$-\vec{\nabla} \cdot \vec{P} = \rho_b$$

So, if you have a region with dipoles *and* free charges,

$$\rho_{(all)} = \rho_{free} + \rho_{bound}$$

or,

$$\rho_{free} = \rho_{(all)} - \rho_{bound}$$

$$\rho_{free} = \varepsilon_o \vec{\nabla} \cdot \vec{E} - \left(-\vec{\nabla} \cdot \vec{P} \right)$$

$$\rho_{free} = \vec{\nabla} \cdot \left(\varepsilon_o \vec{E} + \vec{P} \right)$$

$$\varepsilon_o \vec{E} + \vec{P} \equiv \vec{D}$$ Electric Displacement

So Gauss's Law for *free* charge and Electric *Displacement*

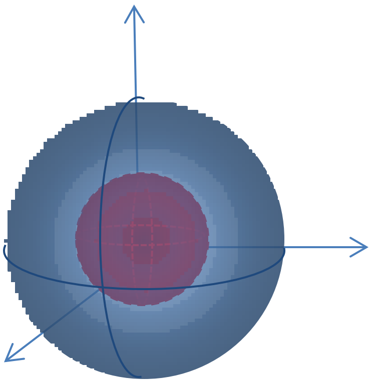
$$\rho_{free} = \vec{\nabla} \cdot \vec{D}$$

and

$$Q_{free} = \int \rho_{free} d\tau = \int \vec{D} \cdot d\vec{a}$$

Polarization & Electric Displacement

Exercise – take 2



$$Q_{free} = \int \rho_{free} d\tau = \int \vec{D} \cdot d\vec{a} \qquad \varepsilon_o \vec{E} + \vec{P} \equiv \vec{D}$$

A thick spherical shell (inner radius *a* and outer radius *b*) is made of dielectric material with a "frozen-in" polarization $\vec{P}(\vec{r}) = \frac{k}{r}\hat{r}$. There are no free charges.

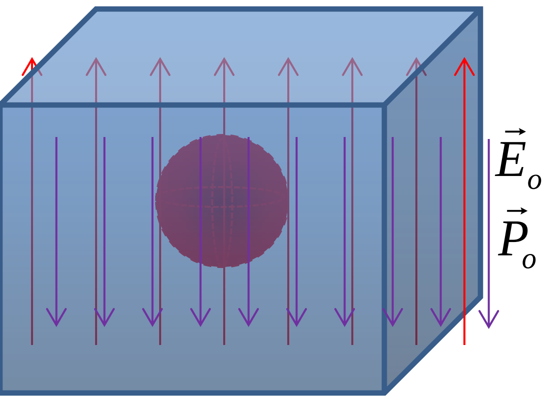
Find D from our new Gauss's Law in all three regions, and then find E from it.

Polarization & Electric Displacement

Example

$$Q_{free} = \int \rho_{free} d\tau = \int \vec{D} \cdot d\vec{a} \qquad \varepsilon_o \vec{E} + \vec{P} \equiv \vec{D}$$

Consider a huge slab of dielectric material initially with uniform field, $\vec{E}_o$ and corresponding uniform polarization and electric displacement $\vec{D}_o = \varepsilon_o \vec{E}_o + \vec{P}_o$



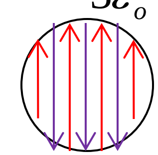
You cut a small spherical hole out of it. What is the field in its center in terms of $\vec{E}_o$ and $\vec{P}_o$?

For illustrative purposes only, take the polarization to be anti-parallel to the field, and imagine both to be in the z direction.

By Superposition Principle, cutting out a sphere is the same as inserting a sphere of opposite polarization.

Quoting Example 4.2 (which in turn builds on 3.9), the field *inside* a uniformly polarized sphere is $\vec{E}_{sphere} = -\frac{1}{3\varepsilon_o} \vec{P}_{sphere}$

So, we 'add in ' a sphere of polarization $-\vec{P}_o$



Adding field $\vec{E}_{added} = \frac{1}{3\varepsilon_o} \vec{P}_o$

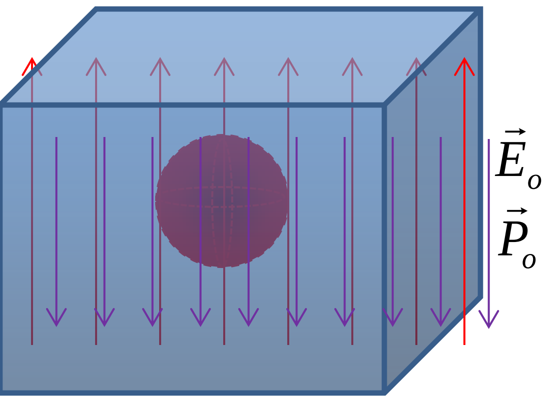
$$\vec{E}_{in.sphere} = \vec{E}_o + \tfrac{1}{3\varepsilon_o} \vec{P}_0$$

Polarization & Electric Displacement

Example

$$Q_{free} = \int \rho_{free} d\tau = \int \vec{D} \cdot d\vec{a} \qquad \varepsilon_o \vec{E} + \vec{P} \equiv \vec{D}$$

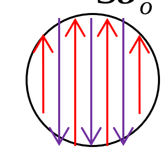
Consider a huge slab of dielectric material initially with uniform field, $\vec{E}_o$ and corresponding uniform polarization and electric displacement $\vec{D}_o = \varepsilon_o \vec{E}_o + \vec{P}_o$



You cut a small spherical hole out of it. What is the field in its center in terms of $\vec{E}_o$ and $\vec{P}_o$?

$$\vec{E}_{sphere} = -\frac{1}{3\varepsilon_o} \vec{P}_{sphere}$$

$$\vec{E}_{in.sphere} = \vec{E}_o + \tfrac{1}{3\varepsilon_o} \vec{P}_0$$



What is the electric displacement in its center in terms of $\vec{D}_o$ and $\vec{P}_o$?

$$\vec{D}_{sphere} = \varepsilon_o \vec{E}_{shere} + \vec{P}_{sphere}$$

There is no material in the sphere, so

$$\vec{D}_{sphere} = \varepsilon_o \left(\vec{E}_o + \tfrac{1}{3\varepsilon_o} \vec{P}_0 \right) \quad \text{where} \quad \vec{E}_o = \tfrac{1}{\varepsilon_o} \left(\vec{D}_o - \vec{P}_o \right)$$

so

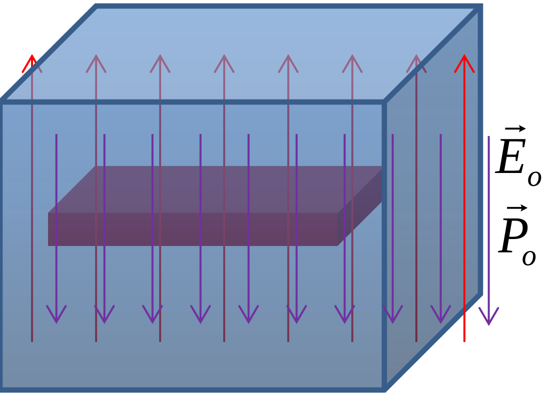
$$\vec{D}_{sphere} = \varepsilon_o \left(\tfrac{1}{\varepsilon_o} \left(\vec{D}_o - \vec{P}_o \right) + \tfrac{1}{3\varepsilon_o} \vec{P}_0 \right) = \left(\vec{D}_o - \tfrac{2}{3} \vec{P}_0 \right)$$

Polarization & Electric Displacement

Exercise

$$Q_{free} = \int \rho_{free} d\tau = \int \vec{D} \cdot d\vec{a} \qquad \varepsilon_o \vec{E} + \vec{P} \equiv \vec{D} \qquad \sigma_b = \vec{P} \cdot \hat{a} \quad \text{and} \quad \rho_b = -\vec{\nabla} \cdot \vec{P}$$

Consider a huge slab of dielectric material initially with uniform field, $\vec{E}_o$ and corresponding uniform polarization and electric displacement $\vec{D}_o = \varepsilon_o \vec{E}_o + \vec{P}_o$.



You cut out a wafer-shaped cavity perpendicular to $\vec{P}_o$.

What is the field in its center in terms of $\vec{E}_o$ and $\vec{P}_o$?

Hint: Think of *inserting* the appropriate waver-sized capacitor.

What is the electric displacement in its center in terms of $\vec{D}_o$ and $\vec{P}_o$?

The Electric "Displacement"

So Gauss's Law for *free* charge and Electric *Displacement*

$$\rho_{free} = \vec{\nabla} \cdot \vec{D} \quad \text{and} \quad Q_{free} = \int \rho_{free} d\tau = \int \vec{D} \cdot d\vec{a} \qquad \text{where} \quad \vec{D} \equiv \varepsilon_o \vec{E} + \vec{P}$$

Of practical use – with polarizable materials, you might directly control ρ_{free}, but ρ_{bound} unavoidably changes in response (reminiscent of the free energies in thermo)

Warning: while $\vec{\nabla} \times \vec{E} = 0$ in electrostatics, and so $\vec{E} = \frac{1}{4\pi\varepsilon_o} \int \frac{\rho}{\mathscr{r}^2} \hat{\mathscr{r}} d\tau'$

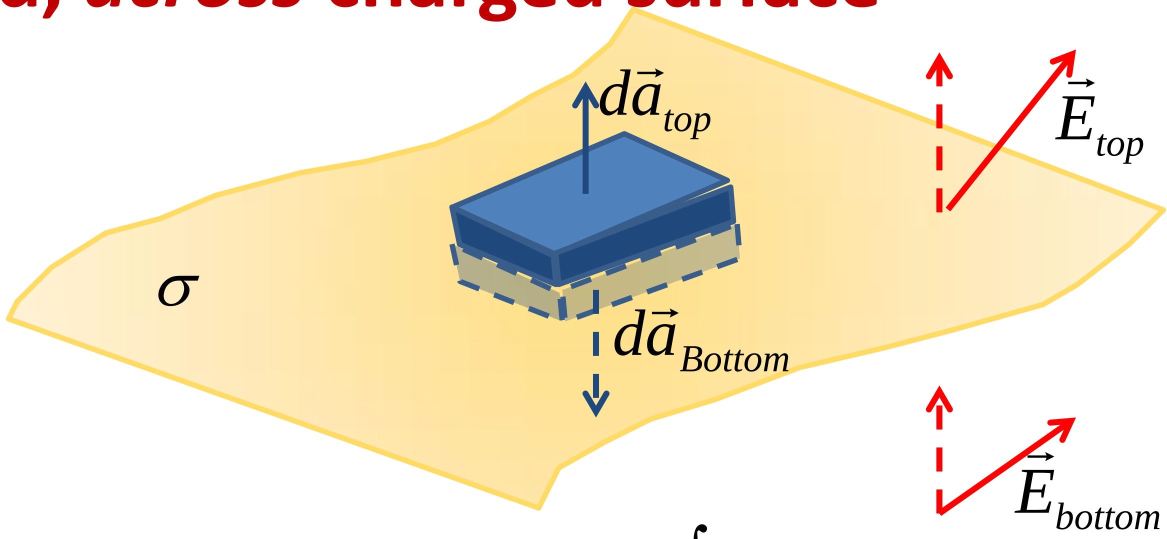
For electric displacement $\vec{\nabla} \times \vec{D} = \vec{\nabla} \times \varepsilon_o \vec{E} + \underbrace{\vec{\nabla} \times \vec{P}}$

and so $\vec{D} \neq \frac{1}{4\pi\varepsilon_o} \int \frac{\rho_{free}}{\mathscr{r}^2} \hat{\mathscr{r}} d\tau'$

Not necessarily 0

Which means it can't necessarily be expressed as gradient of a scalar field

Boundary Conditions
Electric field, *across* charged surface



$$\vec{\nabla}\cdot\vec{E} = \frac{\rho_{encl}}{\varepsilon_o}$$

$$\oint \vec{E}\cdot d\vec{a} = \frac{Q_{encl}}{\varepsilon_o}$$

$$\int \vec{E}_{top}\cdot d\vec{a}_{top} + \int \vec{E}_{bottom}\cdot d\vec{a}_{bottom} + \int \vec{E}_{sides}\cdot d\vec{a}_{sides} = \frac{Q_{encl}}{\varepsilon_o} = \frac{\int \sigma da_{surface}}{\varepsilon_o}$$

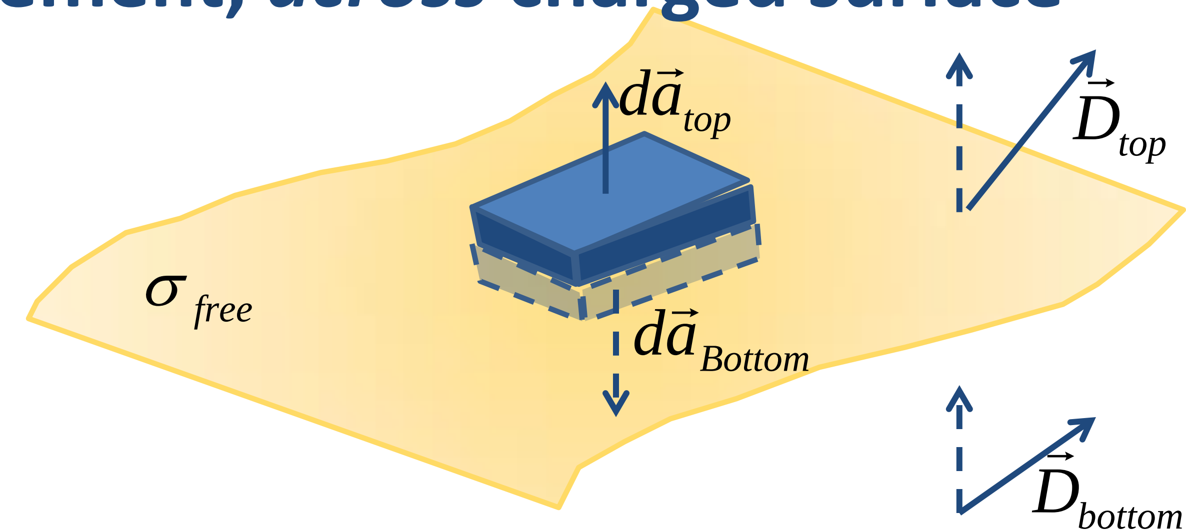
Send side height / area to 0

$$\int \vec{E}_{top}\cdot d\vec{a}_{top} + \int \vec{E}_{bottom}\cdot d\vec{a}_{bottom} = \frac{\int \sigma da_{surface}}{\varepsilon_o}$$

$$E_{\perp top}A + E_{\perp bottom}A(-1) = \frac{\sigma A}{\varepsilon_o}$$

$$E_{\perp top} - E_{\perp bottom} = \frac{\sigma}{\varepsilon_o}$$

Boundary Conditions
Electric Displacement, *across* charged surface



$$\vec{\nabla}\cdot\vec{D} = \rho_{free.encl}$$

$$\oint \vec{D}\cdot d\vec{a} = Q_{free.encl}$$

$$\int \vec{D}_{top}\cdot d\vec{a}_{top} + \int \vec{D}_{bottom}\cdot d\vec{a}_{bottom} + \int \vec{D}_{sides}\cdot d\vec{a}_{sides} = Q_{free.encl} \quad = \int \sigma_{free}\, da_{surface}$$

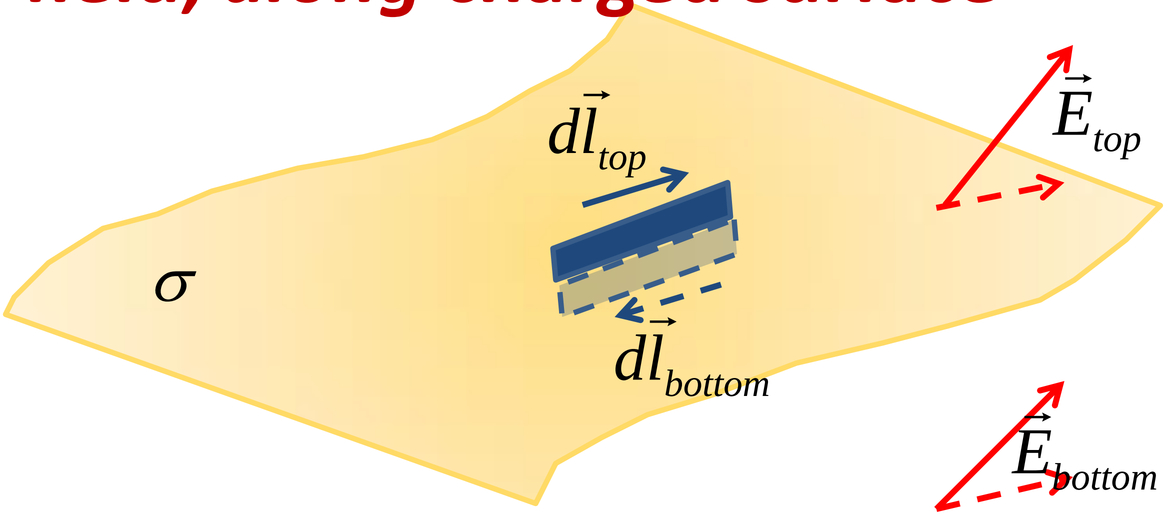
Send side height / area to 0

$$\int \vec{D}_{top}\cdot d\vec{a}_{top} + \int \vec{D}_{bottom}\cdot d\vec{a}_{bottom} = \int \sigma_{free}\, da_{surface}$$

$$D_{\perp top}A + D_{\perp bottom}A(-1) = \sigma_{free}A$$

$$D_{\perp top} - D_{\perp bottom} = \sigma_{free}$$

Boundary Conditions
(static) Electric field, *along* charged surface



$$\vec{\nabla} \times \vec{E} = 0$$

$$\oint \vec{E} \cdot d\vec{l} = 0$$

$$\int \vec{E}_{top} \cdot d\vec{l}_{top} + \int \vec{E}_{bottom} \cdot d\vec{l}_{bottom} + \int \vec{E}_{sides} \cdot d\vec{l}_{sides} = 0$$

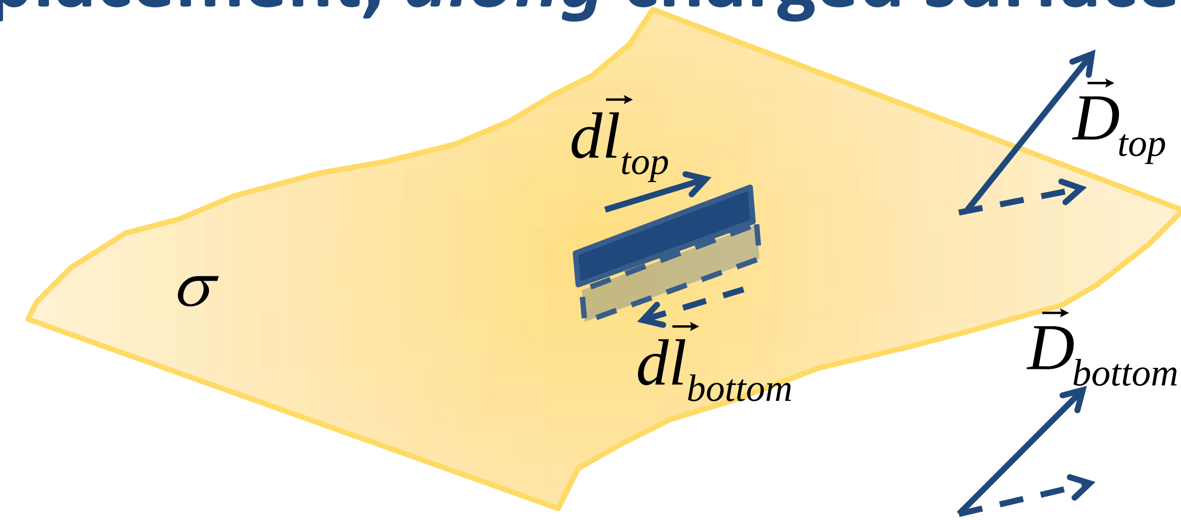
Send side height to 0

$$\int \vec{E}_{top} \cdot d\vec{l}_{top} + \int \vec{E}_{bottom} \cdot d\vec{l}_{bottom} = 0$$

$$E_{\parallel top} L + E_{\parallel bottom} L(-1) = 0$$

$$E_{\parallel top} - E_{\parallel bottom} = 0$$

Boundary Conditions
(static) Electric displacement, *along* charged surface



$$\vec{\nabla}\times\vec{D}=\vec{\nabla}\times\vec{P}$$

$$\oint\vec{D}\cdot d\vec{l}=\oint\vec{P}\cdot d\vec{l}$$

$$\int\vec{D}_{top}\cdot d\vec{l}_{top}+\int\vec{D}_{bottom}\cdot d\vec{l}_{bottom}+\int\vec{D}_{sides}\cdot d\vec{l}_{sides}=\int\vec{P}_{top}\cdot d\vec{l}_{top}+\int\vec{P}_{bottom}\cdot d\vec{l}_{bottom}+\int\vec{P}_{sides}\cdot d\vec{l}_{sides}$$

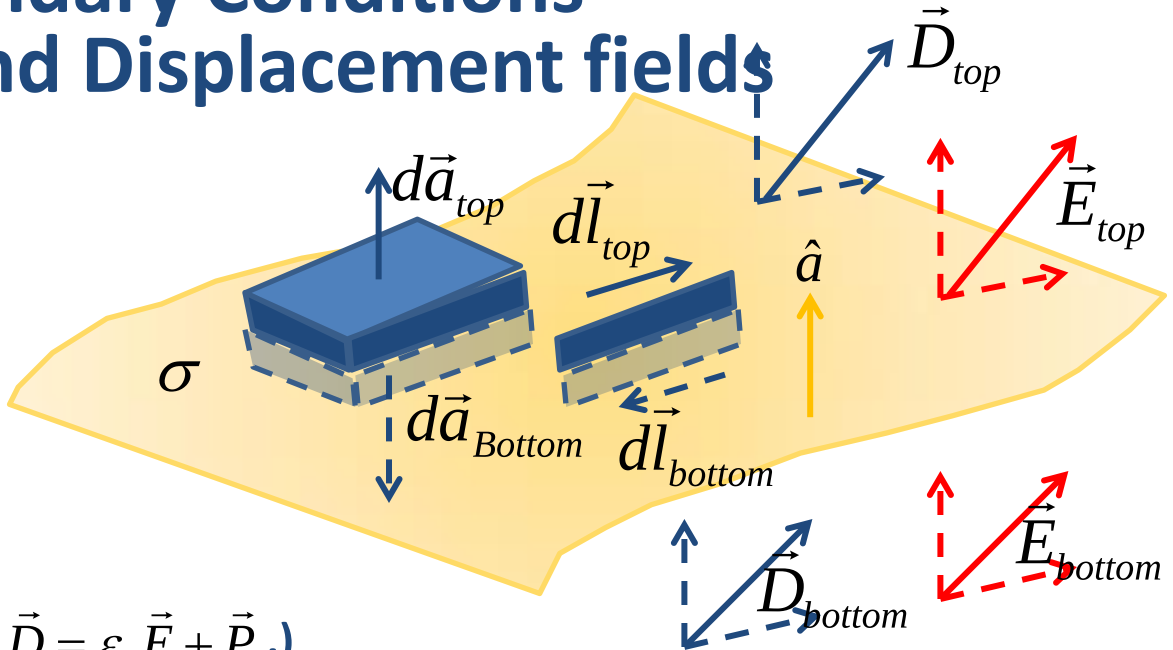
Send side height to 0

$$\int\vec{D}_{top}\cdot d\vec{l}_{top}+\int\vec{D}_{bottom}\cdot d\vec{l}_{bottom}=\int\vec{P}_{top}\cdot d\vec{l}_{top}+\int\vec{P}_{bottom}\cdot d\vec{l}_{bottom}$$

$$D_{\|top}L+D_{\|bottom}L(-1)=P_{\|top}L+P_{\|bottom}L(-1)$$

$$D_{\|top}-D_{\|bottom}=P_{\|top}-P_{\|bottom}$$

Boundary Conditions
Electric and Displacement fields



Along

$$E_{\parallel top} - E_{\parallel bottom} = 0$$

$$D_{\parallel top} - D_{\parallel bottom} = P_{\parallel top} - P_{\parallel bottom}$$

(could have guessed as much from $\vec{D} = \varepsilon_o \vec{E} + \vec{P}$ **.)**

Across

$$E_{\perp top} - E_{\perp bottom} = \frac{\sigma}{\varepsilon_o} \quad \text{or} \quad \vec{E}_{top} \cdot \hat{a} - \vec{E}_{bottom} \cdot \hat{a} = \frac{\sigma}{\varepsilon_o}$$

$$D_{\perp top} - D_{\perp bottom} = \sigma_{free} \quad \quad \vec{D}_{\perp top} \cdot \hat{a} - \vec{D}_{\perp bottom} \cdot \hat{a} = \sigma_{free}$$

(could have guessed as much from $\vec{D} = \varepsilon_o \vec{E} + \vec{P}$ **and** $\sigma - \sigma_{free} = \sigma_b = \vec{P} \cdot \hat{a}$ **.)**

Boundary Conditions
Electric and Displacement fields

Along

$$E_{\parallel top} - E_{\parallel bottom} = 0$$

$$D_{\parallel top} - D_{\parallel bottom} = P_{\parallel top} - P_{\parallel bottom}$$

$$\vec{D} = \varepsilon_o \vec{E} + \vec{P}$$

$$\sigma - \sigma_{free} = \sigma_b = \vec{P} \cdot \hat{a}$$

Across

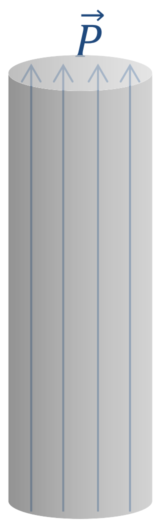
$$E_{\perp top} - E_{\perp bottom} = \frac{\sigma}{\varepsilon_o}$$

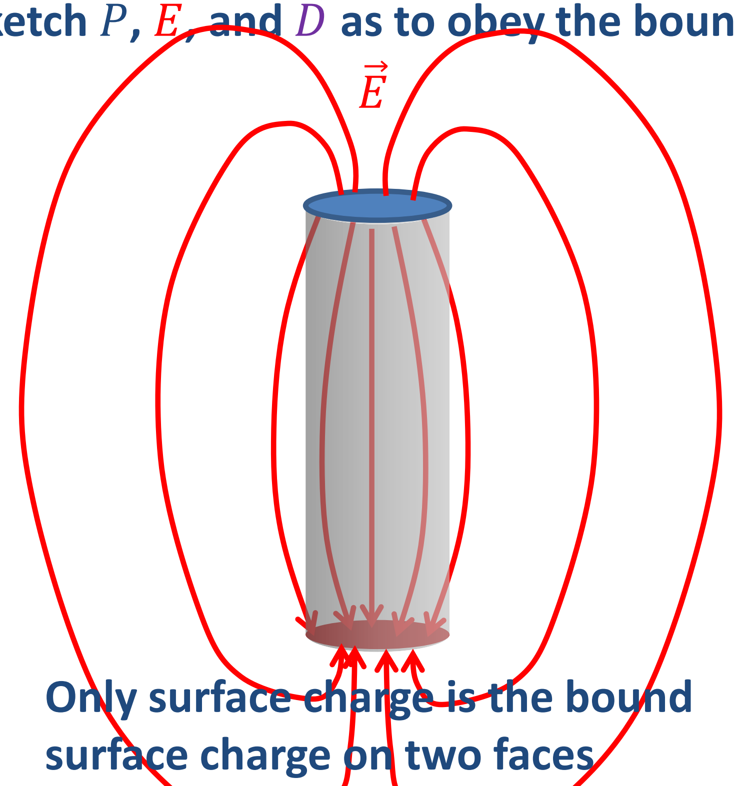
$$D_{\perp top} - D_{\perp bottom} = \sigma_{free}$$

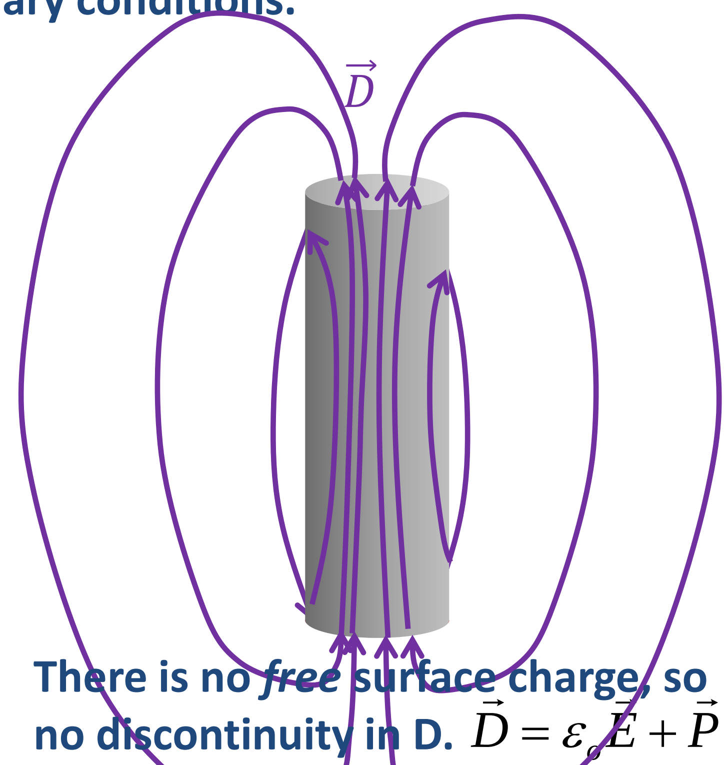
Exercise

Bar Electret (like an electric bar magnet): uniform P along axis

Sketch $\vec{P}$, $\vec{E}$, and $\vec{D}$ as to obey the boundary conditions.







Only surface charge is the bound surface charge on two faces

There is no *free* surface charge, so no discontinuity in D. $\vec{D} = \varepsilon_o \vec{E} + \vec{P}$

Mon.	(C14) 4.3 Electric Displacement	
Mon. Wed.	(C14) 4.4.1 Linear Dielectrics (read rest at your discretion) 6.1 Magnetization	 HW10