

Wed., 11/20	(C 14) 4.1 Polarization	HW9
Fri., 11/22	(C 14) 4.2 Field of Polarized Object	
Mon., 11/25	(C14) 4.3 Electric Displacement	
Mon., 12/2	(C14) 4.4.1 Linear Dielectrics (read rest at your discretion)	HW10
Wed., 12/4	6.1 Magnetization	

Summary

This chapter is about how insulators respond to electric fields. These are materials that don't have any 'free' electrons to flow about in response to externally imposed electric fields. Instead, the electrons are pretty well bound to their atoms. Still, it's just an electric interaction that holds them in place; they certainly aren't *immune* to the influence of another electric field. How do they respond? Qualitatively, if the atoms are more or less locked in place, then there are two things they can do in response to the external fields – they can't *translate*, but they can *stretch* and they can *rotate*. Today, we'll start by considering what just one such object (atom, molecule, whatever) can do; next time we'll then think about what a whole chunk of them would do.

We'll start with that 'stretching.'

Induced Dipoles.

Consider a net-neutral atom or molecule. An externally imposed electric field will have the effect of drawing the + charges in one direction and the – charges in the other. Meanwhile, assuming that the charges were initially in some kind of stable configuration, then the internal forces that hold them together will increase (at least initially) as they get drawn apart, thus pulling them harder and harder back toward each other. Thus the charges will get pulled apart just to a distance at which the external and internal forces balance, establishing a new equilibrium.

$$\begin{aligned}\vec{F}_{ext} &= -\vec{F}_{int} \\ q\vec{E}_{ext} &= -\vec{F}_{int}\end{aligned}$$

How does the force grow with charge separation? Presumably we can apply a Taylor series expansion to it, and, if the separation is small enough we can keep only the 1st order term (0th-order term is 0 since we're expanding around equilibrium). Now, a nice way to quantify 'charge separation' is the dipole moment, so what we end up with is something like

$$q\vec{E}_{ext} = -\left(\left. \frac{\partial \vec{F}_{int}}{\partial p} \right|_{p=0} p + \dots \right)$$

Flipping that relation around, we expect that the induced charge separation // dipole moment will be proportional to the externally applied field.

$$\vec{p} \approx \alpha \vec{E}_{ext}$$

where the constant of proportionality, α is the atomic polarizability (clearly a material-specific constant.)

(the flip of sign is because, while the internal force is one way, the polarization is the other)

How small is small enough?

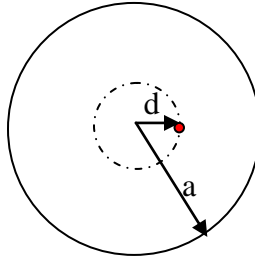
This linear relationship should only hold for relatively *small* distortions. “Small” compared to what? Let’s consider a case in which it turns out being an *exact* relation, and then we’ll be able to see what was special about that case.

Ex. 4.1: Griffiths shows that it works for the simple case of a uniformly charged spherical “electron cloud” and point charge nucleus.

$$\vec{F}_{ext \rightarrow nuc} = q\vec{E}_{ext \rightarrow nuc} = -q\vec{E}_{elect \rightarrow nuc}$$

$$\vec{E}_{ext \rightarrow nuc} = -\vec{E}_{elect \rightarrow nuc}$$

Now, the picture looks something like this:



This is a nice candidate for applying Gauss’s Law: the electric field is clearly radial and angularly symmetric.

$$\oint \vec{E}_{elect \rightarrow nuc} \cdot d\vec{a} = \frac{1}{\epsilon_o} \int \rho d\tau = \frac{1}{\epsilon_o} \int_{\frac{4}{3}\pi a^3}^{-q} d\tau$$

$$E_{elect \rightarrow nuc} 4\pi d^2 = \frac{-q}{\frac{4}{3}\pi a^3 \epsilon_o} \frac{4}{3}\pi d^3$$

$$E_{elect \rightarrow nuc} = \frac{-q}{4\pi a^3 \epsilon_o d^2} d^3 = \frac{-q}{4\pi a^3 \epsilon_o} d$$

But, our equilibrium condition is that this field is equal and opposite to the externally imposed field.

$$-\vec{E}_{ext \rightarrow nuc} = \vec{E}_{elect \rightarrow nuc} = \frac{-q}{4\pi a^3 \epsilon_o} d\hat{d}$$

$$qd\hat{d} = 4\pi a^3 \epsilon_o \vec{E}_{ext \rightarrow nuc}$$

$$\vec{p} = \alpha \vec{E}_{ext \rightarrow nuc}$$

We can return to the question of ‘how small is small enough?’ The math worked out perfectly here when the “electron cloud” was a) spherically symmetric and b) radially independent (until you got outside it). So, you might expect that “small enough” would be small enough that you can ignore a real atom’s charge distribution’s dependence on *r*. Looking at the physicists favorite atom, Hydrogen, the charge density goes like

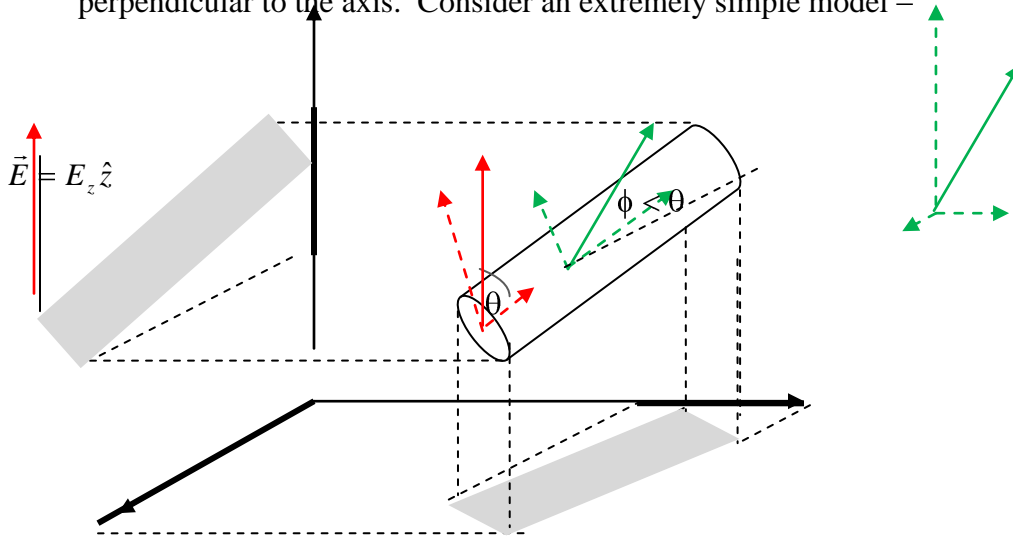
$\rho(r) = \frac{q}{\pi a^3} e^{-2r/a}$. So, the charge density drops off exponentially, with a decay constant of the Bohr radius (around 10^{-10} m.) For any other atom, we're probably talking about variations on this scale.

(best done by Gauss's Law, and a couple of steps of integration by parts)

Problem 4.1 demonstrates that, for a 5×10^5 V/m field, the separation in hydrogen is on the order of 10^{-16} m (a 10^{th} the radius of a proton). So, it seems plausible that this is small enough.

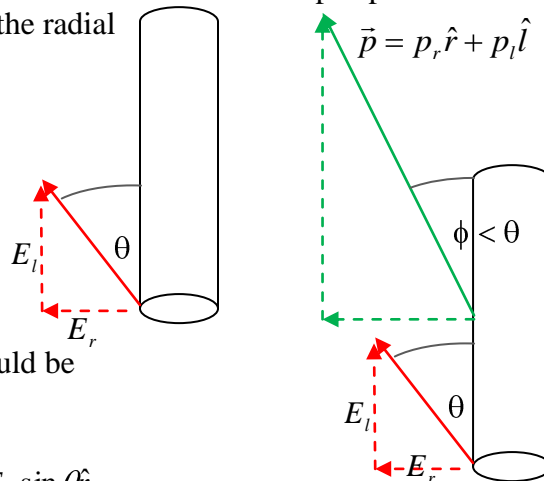
Polarizability Tensor

If we consider something a tad more complex than an atom, say, a linear molecule, then it becomes evident that the polarizability will be different in different directions. After a sphere, the next simplest geometry is a cylinder, and that's enough to see this complication. – it may be easy to push charge along the axis, but not so easy to push it perpendicular to the axis. Consider an extremely simple model –



Say the charge more easily moves axially than radially, $\alpha_l > \alpha_r$. then applying a field in only the z direction would be viewed from the perspective of the rod as applying some in the axial and some in the radial

$$\begin{aligned} \vec{E} &= E_z \hat{z} = E_r \hat{r} + E_l \hat{l} \\ &= E_z \sin \theta \hat{r} + E_z \cos \theta \hat{l} \end{aligned}$$



Then the response would be

$$\begin{aligned} \vec{p} &= \alpha_r E_r \hat{r} + \alpha_l E_l \hat{l} \\ \vec{p} &= \alpha_r E_z \cos \theta \hat{l} + \alpha_l E_z \sin \theta \hat{r} \end{aligned}$$

What this illustrates is that, for a fairly simple system, with the field *not* parallel to one of the object's natural axes, you're going to get polarization along multiple natural axes; and

if you want to resolve that into arbitrary x,y,z axes, you're generally going to see polarization on all three (even if you choose one of those axes to be the direction of the original field).

We could rephrase the result in this case as, (for which we could determine the alphas in terms of angles and the radial and axial alphas)

$$\begin{aligned} p_x &= \alpha_{xz} E_z \\ p_y &= \alpha_{yz} E_z \\ p_z &= \alpha_{zz} E_z \end{aligned}$$

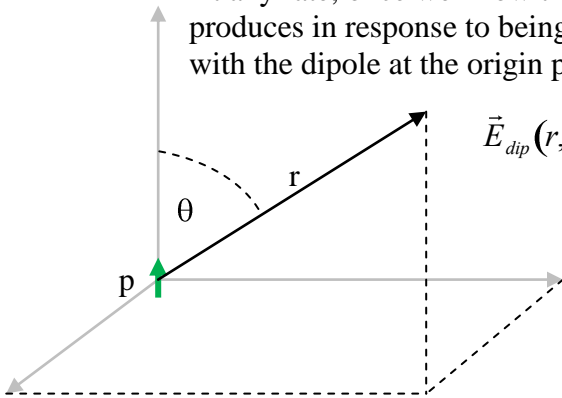
More generally, if we were so unwise as to choose a coordinate system that aligned with *neither* the object's natural axes *nor* the field, then

$$\vec{p} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \alpha_{xx} E_x + \alpha_{xy} E_y + \alpha_{xz} E_z \\ \alpha_{yx} E_x + \alpha_{yy} E_y + \alpha_{yz} E_z \\ \alpha_{zx} E_x + \alpha_{zy} E_y + \alpha_{zz} E_z \end{pmatrix} = \tilde{\alpha} \vec{E}$$

So polarizability is generally a matrix, i.e., a tensor, though it's always possible to choose your axes more sensibly (so there are no cross terms).

Field due to Dipole

At any rate, once we know the dipole moment, we know the electric field that *it* produces in response to being subjected to the external field. In spherical coordinates with the dipole at the origin pointing in the z direction:



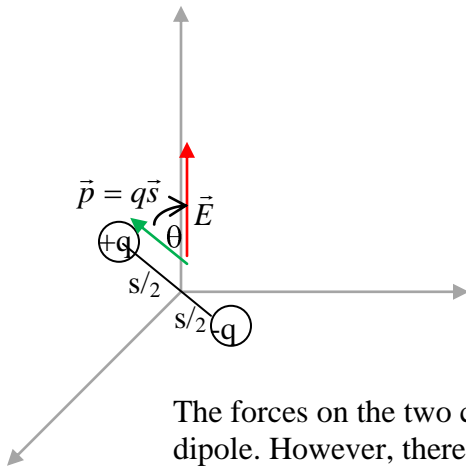
$$\vec{E}_{dip}(r, \theta) = \frac{P}{4\pi\epsilon_0 r^3} (2\cos\theta \hat{r} + \sin\theta \hat{\theta}) \quad (3.103)$$

Polar Molecules

Some molecules, such as water (H₂O), have a charge separation / dipole moment even if there is no external electric field. It doesn't have to be induced, but is due to the higher probability of finding the electrons in certain locations (in H₂O, they spend more time by the oxygen atom). Then a more significant effect than further polarizing the molecule is aligning it.

Torque on a Polar Molecule in an External Electric Field

Suppose a polar molecule is in a uniform, external electric field \vec{E} . To understand what happens, consider a physical dipole made of charges $+q$ and $-q$ with a separation \vec{s} (pointing from the negative charge to the positive one).



The forces on the two charges are equal and opposite, so there is no *net* force on dipole. However, there is a net torque about the center of the dipole (sorry, Griffiths uses tau for volume, so we need a new letter, N):

$$\vec{N} = \sum \vec{r}_i \times \vec{F}_i = \left[\left(\frac{\vec{s}}{2} \right) \times (q\vec{E}) + \left(-\frac{\vec{s}}{2} \right) \times (-q\vec{E}) \right] = q\vec{s} \times \vec{E} = \vec{p} \times \vec{E} .$$

The size of the torque is

$$N = pE \sin \theta ,$$

where θ is the angle from the dipole moment to the field.

The right-hand-rule tells us the direction assigned to the torque (in this case it's into the board / -x)

Potential Energy of a Dipole in an External Electric Field

For an object moving with a (conservative) force on it, the potential energy (relative to a reference point) is

$$U(\vec{r}) = -W_{ref \rightarrow \vec{r}} = - \int_{ref}^{\vec{r}} \vec{F} \cdot d\vec{\ell} .$$

Let's look at the change in potential energy associated with rotating the two charges

$$U_{rot} \curvearrowright = -2 \int_{ref}^{\vec{r}} q\vec{E} \cdot \frac{\vec{s}}{2} d\theta = - \int_{ref}^{\vec{r}} qE s \sin \theta d\theta = -qsE \cos \theta \Big|_{ref}^{\theta} = -\vec{p} \cdot \vec{E} \Big|_{ref}^{\theta}$$

Note: this was derived under very specific conditions: E is uniform and we're rotating (not stretching or translating).

Similarly, if you imagine stretching/compressing the dipole, you get

$$U_{stretch} \curvearrowright = -2 \int_{s_i}^{s_f} q\vec{E} \cdot \frac{d\vec{s}}{2} = -2 \int_{s_i}^{s_f} q\vec{E} \cdot d\vec{s} = -\vec{p} \cdot \vec{E} \Big|_{p_i}^{p_f}$$

So rotating and vibrating the dipole would do work

Force on a Dipole in an External Electric Field

If the external electric field is not uniform, there will be a force on a dipole because the forces on the two charges (think of the physical dipole) are not the same size. The size of the force depends on how quickly the electric field changes with position.

The net force exerted on the dipole is then

$$\vec{F} = \vec{F}_+ + \vec{F}_- = +q\vec{E}(\vec{r}_+) - q\vec{E}(\vec{r}_-) = q(\vec{E}(\vec{r}_+) - \vec{E}(\vec{r}_+ - \vec{d}))$$

$$\vec{F} = qd \frac{\vec{E}(\vec{r}_+) - \vec{E}(\vec{r}_+ - \vec{d})}{d}$$

For the sake of simplicity, let's say that the dipole points in the z-direction, and we take this to the limit for which d is infinitesimal (making an ideal dipole).

$$\vec{F} = p_z \frac{\partial \vec{E}}{\partial z}$$

Alternatively, if the dipole pointed in the x or y, we'd get $\vec{F} = p_x \frac{\partial \vec{E}}{\partial x}$ or $\vec{F} = p_y \frac{\partial \vec{E}}{\partial y}$

So, more generally,

$$\vec{F} = p_z \frac{\partial \vec{E}}{\partial z} + p_x \frac{\partial \vec{E}}{\partial x} + p_y \frac{\partial \vec{E}}{\partial y} = \vec{p} \cdot \vec{\nabla} \vec{E}$$

Note: some sources say

$$\vec{F} = -\vec{\nabla} U = \vec{\nabla} (\vec{p} \cdot \vec{E})$$

But that is wrong for two reasons: 1st: the dipole moment must be held constant during the derivative, so it's slopy to not note that restriction. 2nd:

$$\vec{\nabla} (\vec{p} \cdot \vec{E}) \Big|_{\vec{p}} = \vec{p} \times (\vec{\nabla} \times \vec{E}) + \vec{p} \cdot \vec{\nabla} \vec{E} \neq \vec{p} \cdot \vec{\nabla} \vec{E}$$

So unless the electric field is curl-less, i.e., unless we're talking electro-statics, there's an additional term mistakenly predicted by this expression.

1. If we're dealing with a polar molecule and assume that the dipole moment is unchanged by the external field, then we don't need to worry about the derivative of \vec{p} , so we get the result in the book:

$$\vec{F} = (\vec{p} \cdot \vec{\nabla}) \vec{E}$$

The x component of this equation is (depends on changes in E_x):

$$F_x = p_x \frac{\partial E_x}{\partial x} + p_y \frac{\partial E_x}{\partial y} + p_z \frac{\partial E_x}{\partial z}$$

2. If we're dealing with an induced dipole which is proportional to the applied field ($\vec{p} = \alpha \vec{E}$), the force is

$$\vec{F} = \vec{p} \cdot \vec{\nabla} \vec{E} = \alpha \vec{E} \cdot \vec{\nabla} \vec{E}$$

In the simple case that the polarizability is uniform (so the tensor reduces to a constant), through product rule 4,

$\vec{F} = \nabla(\vec{E} \cdot \vec{E}) = \alpha(\nabla E^2 - \vec{E} \times \nabla \times \vec{E})$ where the second term is 0 unless there's a time-varying source (time varying current density & accompanying time-varying magnetic field.)

Note that this result differs by a factor of $\frac{1}{2}$ from what you'd get using $\vec{F} = -\nabla U = \nabla(\vec{p} \cdot \vec{E})$. Since there are two ways to find the force on an induced dipole – by finding that directly on the dipole or finding that which the dipole exerts on the source and applying Newton's 3rd Law, you can verify that you do need the factor of $\frac{1}{2}$.

There will be a force directed toward a region with a higher electric field. One application of this is “optical tweezers” where a neutral object will be attracted to the focus of a laser beam because the (average) electric field there is maximum (the average force is $\langle \vec{F} \rangle = \frac{1}{2} \alpha \nabla \langle E^2 \rangle$). If the object reflects any light, there will also be a force in the direction of the laser beam.

Polarization

Suppose that there are a lot of little dipoles pointing in the same direction. We can define the polarization (a vector):

$$\vec{P} = \text{dipole moment per volume} ,$$

which can be induced by an external electric field or “frozen in.”

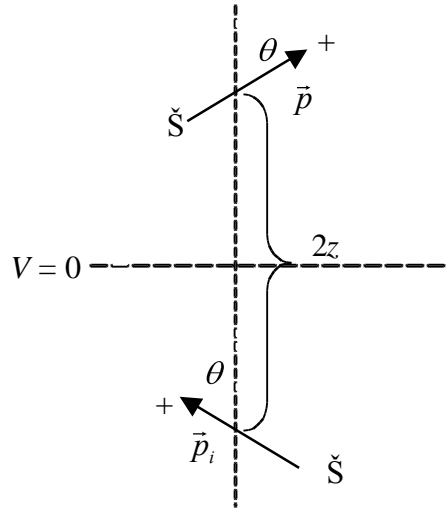
Examples/Exercises:

Problem 4.6

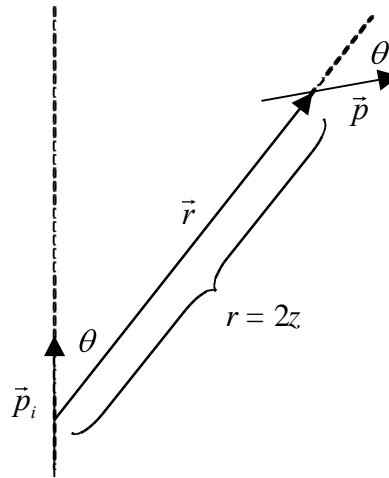
A dipole \vec{p} is a distance z above an infinite grounded conducting plane (see diagram below). The dipole makes an angle θ with the perpendicular to the plane. Find the torque on \vec{p} . In what orientation is it stable?

QuickTime™ and a
TIFF (Uncompressed) decompressor
are needed to see this picture.

First think about a physical dipole to see how the “image dipole” will be oriented (see the diagram below). The magnitude of the image dipole is the same as the real one.



Redraw the diagram with image dipole \vec{p}_i at the origin along the (new) z axis.



The electric field due to the image charge (really the surface charge) at the location of the real dipole is

$$\vec{E}_i = \frac{P}{4\pi\epsilon_0(2z)^3} (2\cos\theta\hat{r} + \sin\theta\hat{\theta}).$$

The real dipole can also be written in spherical coordinates as

$$\vec{p} = p\cos\theta\hat{r} + p\sin\theta\hat{\theta}.$$

The cross product of a unit vector with itself is zero and $\hat{r} \times \hat{\theta} = \hat{\phi}$ (add a minus if they're in the opposite order). The torque on the real dipole is

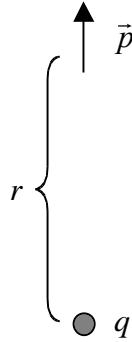
$$\begin{aligned} \vec{N} &= \vec{p} \times \vec{E}_i = \frac{P^2}{4\pi\epsilon_0(2z)^3} (\cos\theta\hat{r} + \sin\theta\hat{\theta}) \times (2\cos\theta\hat{r} + \sin\theta\hat{\theta}) \\ &= \frac{p^2}{4\pi\epsilon_0(2z)^3} [\cos\theta\sin\theta\hat{\phi} + 2\sin\theta\cos\theta(-\hat{\phi})] \\ &= -\frac{p^2\sin\theta\cos\theta}{4\pi\epsilon_0(2z)^3} \hat{\phi} \end{aligned}$$

which is out of the page. When θ is small, it will tend to rotate back toward $\theta = 0$.

The torque is zero when θ is 0° , 90° , or 180° . If the angle is between 0° and 90° , the torque will be counterclockwise. If the angle is between 90° and 180° , the torque will be clockwise. The angle of 90° is not stable, but 0° and 180° are stable.

Force of a Charge on a Dipole (Problem 4.9 simplified)

Suppose a dipole \vec{p} points away from a positive point charge q a distance r away. What is the force on the dipole?



The electric field due to the point charge at the location of the dipole is

$$\vec{E} = \frac{q}{4\pi\epsilon_0 r^2} \hat{r}.$$

In spherical coordinates, the dipole moment is $\vec{p} = p \hat{r}$.

It is easiest to find the force using the more general form of the equation (but the dipole is a constant size, so it has no derivative):

$$\vec{F} = \vec{\nabla}(\vec{p} \cdot \vec{E}) = \vec{\nabla} \left(\frac{qp}{4\pi\epsilon_0 r^2} \right) = \frac{\partial}{\partial r} \left(\frac{qp}{4\pi\epsilon_0 r^2} \right) \hat{r} = -\frac{2qp}{4\pi\epsilon_0 r^3} \hat{r}.$$

We don't need to consider other derivatives because it only depends on r . Note that the force is attractive. We could also find the force on the charge due to the electric field of the dipole and use Newton's third law.

Preview

For Friday, you'll read about the electric field of a polarized object.

"can we go over how we found eqn 4.5, the formula for the force on a dipole in a nonuniform field?" [Jessica](#)

I was also a bit unclear, namely two steps before the final equation. [Casey McGrath](#)

"Why is the atomic polarization (α) important?" [Casey P,](#)

"Why can we model an atom as a spherical electron cloud with a nucleus that is offset from the center? What about the structure of the atom would cause it to be in equilibrium in this configuration?" [Ben Kid](#)

"Would the dipole torque end up causing the molecule to oscillate back and forth in the field? I'm thinking this might be similar to a spring-like system, so there would still be some angular momentum left over when it is pointing with the field." [Freeman,](#)

"How would we go about calculating the force on a dipole if we cannot assume that the dipole is very short?" [Spencer](#)

"Can we really go over what Equation 4.3 means?" [Casey McGrath](#)

"How would the atoms "freeze in" polarization. It's hard to understand how they would stay polarized after the E-field is removed." [Connor W,](#)

"Can we go over the torque arguments and derivation of eqn 4.5?" [Sam](#)