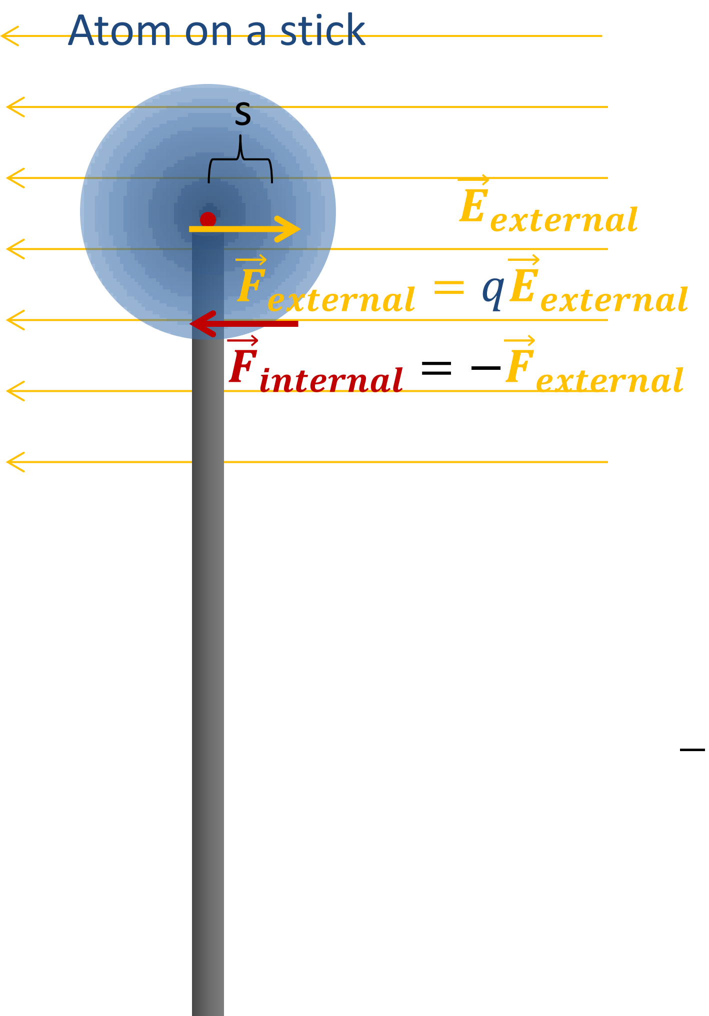
Fri.	(C 14) 4.1 Polarization	
Mon. Wed. Fri.	(C 14) 4.2 Field of Polarized Object (C14) 4.3 Electric Displacement Washington 3-2 Rep. 7pm AHoN 116 (C14) 4.4.1 Linear Dielectrics (read rest at your discretion)	 HW5

Wed. after next – Sr. Spring Registration

Atom's Response to Electric Field



For *small* stretch, first term in Taylor Series (Hook's Law)

$$F_{\text{int}} \approx -\left|\frac{\partial F_{\text{int}}}{\partial s}\bigg|_{s=0}\right| s + \ldots$$

$$F_{\text{int}} \approx -\left|\frac{\partial F_{\text{int}}}{\partial (qs)}\bigg|_{s=0}\right| (qs) + \ldots$$

$$F_{\text{int}} \approx -\left|\frac{\partial F_{\text{int}}}{\partial p}\bigg|_{s=0}\right| p + \ldots$$

Electric Dipole moment

$$-F_{ext} = F_{\text{int}} \approx -\left|\frac{\partial F_{\text{int}}}{\partial p}\bigg|_{s=0}\right| p + \ldots$$

$$-qE_{ext} = F_{\text{int}} \approx -\left|\frac{\partial F_{\text{int}}}{\partial p}\bigg|_{s=0}\right| p + \ldots$$

So, for small enough stretch, weak enough field

$p \propto -qE_{ext}$ or $\vec{p} \approx \alpha \vec{E}_{ext}$ $\alpha \equiv$ polarizability

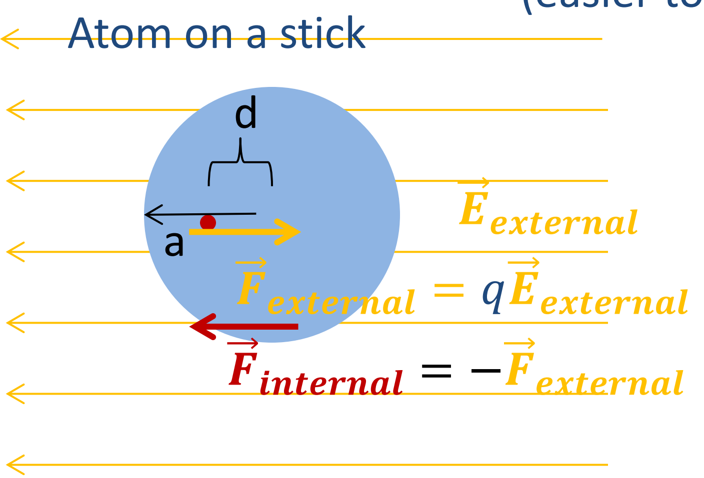
Atom's Response to Electric Field

$$\vec{p} \approx \alpha \vec{E}_{ext}$$

How Small is Small Enough?

Special case: uniform 'electron cloud' $\rho = \dfrac{q}{\frac{4}{3}\pi a^3}$

(easier to consider nucleus being displaced)



$$\vec{F}_{ext\to nuc} = q\vec{E}_{ext\to nuc} = -q\vec{E}_{elect\to nuc}$$

$$\vec{E}_{ext\to nuc} = -\vec{E}_{elect\to nuc}$$

$$\oint \vec{E}_{elect\to nuc}\cdot d\vec{a} = \frac{1}{\varepsilon_o}\int \rho\, d\tau' = \frac{1}{\varepsilon_o}\int \frac{-q}{\frac{4}{3}\pi a^3}\, d\tau$$

$$E_{elect\to nuc}\, 4\pi d^2 = \frac{-q}{\frac{4}{3}\pi a^3 \varepsilon_o}\, \tfrac{4}{3}\pi d^3$$

$$E_{elect\to nuc} = \frac{-q}{4\pi a^3 \varepsilon_o d^2}\, d^3 = \frac{-q}{4\pi a^3 \varepsilon_o}\, d$$

$$-\vec{E}_{ext\to nuc} = \vec{E}_{elect\to nuc} = \frac{-q}{4\pi a^3 \varepsilon_o}\, d\hat{d}$$

$$\underbrace{4\pi a^3 \varepsilon_o}\vec{E}_{ext\to nuc} = \underbrace{q d\hat{d}}$$

$$\alpha \vec{E}_{ext\to nuc} = \vec{p}$$

Key to this working: charge distribution is radially uniform and angularly symmetric

Atom's Response to Electric Field

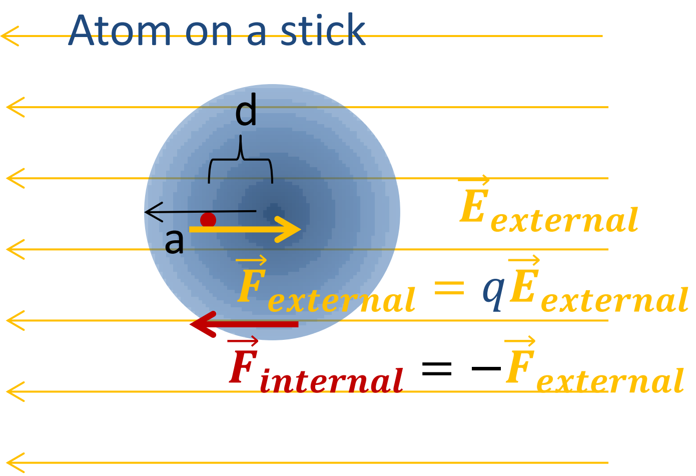
$$\vec{p} \approx \alpha \vec{E}_{ext}$$

Key to this working: charge distribution is radially uniform and angularly symmetric

How Small is Small Enough?
Real (simple): electron cloud

$$\rho(r) = \frac{q}{\pi a^3} e^{-2r/a}$$

(easier to consider nucleus being displaced)



$$\vec{F}_{ext \to nuc} = q\vec{E}_{ext \to nuc} = -q\vec{E}_{elect \to nuc}$$

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$$\oint \vec{E}_{elect \to nuc} \cdot d\vec{a} = \frac{1}{\varepsilon_o} \int \rho \, d\tau' = \frac{1}{\varepsilon_o} \int \frac{-q}{\pi a^3} e^{-2r/a} d\tau$$

$$E_{elect \to nuc} 4\pi d^2 = \frac{1}{\varepsilon_o} \frac{-q}{\pi a^3} 4\pi \int_0^d e^{-2r/a} r^2 dr$$

Assuming d<<a

$$e^{-2r/a} \approx 1 - \tfrac{2r}{a}$$

$$E_{elect \to nuc} 4\pi d^2 \approx \frac{1}{\varepsilon_o} \frac{-q}{\pi a^3} 4\pi \int_0^d \left(1 - \tfrac{2r}{a}\right) r^2 dr$$

$$E_{elect \to nuc} 4\pi d^2 \approx \frac{1}{\varepsilon_o} \frac{-q}{\pi a^3} 4\pi \left(\tfrac{1}{3} d^3 - \tfrac{1}{2} \tfrac{d^4}{a}\right)$$

$$E_{elect \to nuc} \approx \frac{1}{\varepsilon_o} \frac{-q}{\pi a^3} d \left(\tfrac{1}{3} - \tfrac{1}{2} \tfrac{d}{a}\right)$$

$$\varepsilon_o \pi a^3 E_{elect \to nuc} \approx -qd\left(\tfrac{1}{3} - \tfrac{1}{2} \tfrac{d}{a}\right) = -p\left(\tfrac{1}{3} - \tfrac{1}{2} \tfrac{d}{a}\right)$$

Indeed, constant polarizability only for radially constant charge distribution

But if d<<a, good enough approximation

Atom's Response to Electric Field

$$\vec{p} \approx \overleftrightarrow{\alpha}\vec{E}_{ext}$$

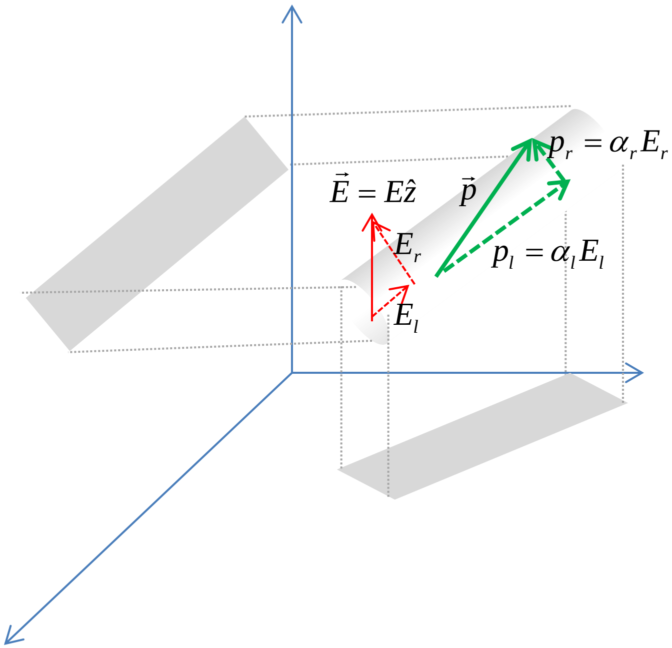
Polarizability Tensor

Mass on un-even springs analogy

Atom's Response to Electric Field

$$\vec{p} \approx \overleftrightarrow{\alpha}\vec{E}_{ext}$$

Polarizability Tensor



Imagine cylindrical 'molecule' which is easier to polarize along length than radially: $\alpha_l > \alpha_r$.

So when field is applied in z-direction,
Get poloarization at an angle

$$\begin{bmatrix} p_r \\ p_l \end{bmatrix} \approx \begin{bmatrix} \alpha_r & 0 \\ 0 & \alpha_l \end{bmatrix} \begin{bmatrix} E_r \\ E_l \end{bmatrix}$$

More generally expressed in terms of Cartesian coordinates that may not be aligned with p or E,

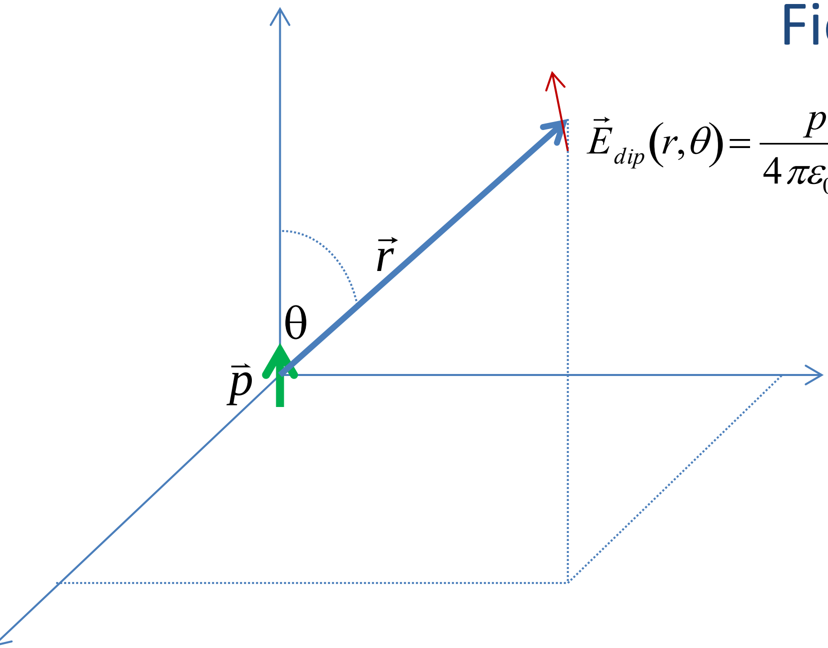
$$p_x = \alpha_{xx}E_x + \alpha_{xy}E_y + \alpha_{xz}E_z$$
$$p_y = \alpha_{yx}E_x + \alpha_{yy}E_y + \alpha_{yz}E_z$$
$$p_z = \alpha_{zx}E_x + \alpha_{zy}E_y + \alpha_{zz}E_z$$

$$\vec{p} = \begin{bmatrix} \alpha_{xx} & \alpha_{xy} & \alpha_{xz} \\ \alpha_{yx} & \alpha_{yy} & \alpha_{yz} \\ \alpha_{zx} & \alpha_{zy} & \alpha_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

$$\vec{p} = \overleftrightarrow{\alpha}\vec{E}$$

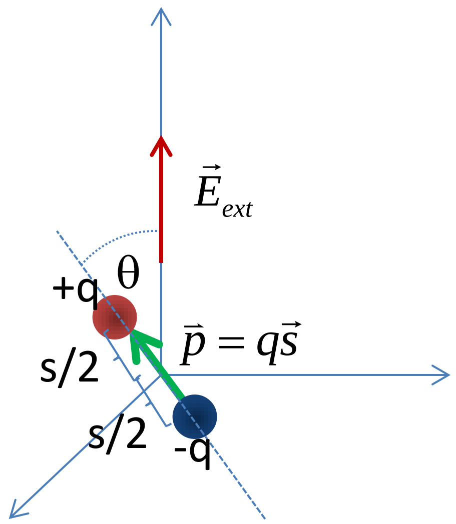
Atom’s Response to Electric Field

Field *of* a dipole



$$\vec{E}_{dip}(r,\theta) = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

Atom's Response to Electric Field



Response of a dipole to fields

Torque in uniform field

$$\vec{N} = \sum \vec{r}_i \times \vec{F}_i$$

$$\vec{N} = \left[\left(\frac{\vec{s}}{2}\right) \times \left(q\vec{E}\right) + \left(-\frac{\vec{s}}{2}\right) \times \left(-q\vec{E}\right)\right]$$

$$\vec{N} = q\vec{s} \times \vec{E}$$

$$\vec{N} = \vec{p} \times \vec{E}$$

$$N = pE\sin\theta$$

Change in energy with Rotation

$$\Delta U(\vec{r}) = -\int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{\ell}$$

$$\Delta U_{rot} = -2\int_{\vec{r}_i}^{\vec{r}_f} q\vec{E} \cdot \tfrac{s}{2}\, d\vec{\theta} = -\int_{\vec{r}_i}^{\vec{r}_f} qEs\sin\theta d\theta = -qsE\left(\cos\theta_f - \cos\theta_i\right)$$

$$\Delta U_{rot} = -\Delta\left(\vec{p} \cdot \vec{E}\right)\Big|_{p,E}$$

Change in energy with vibration

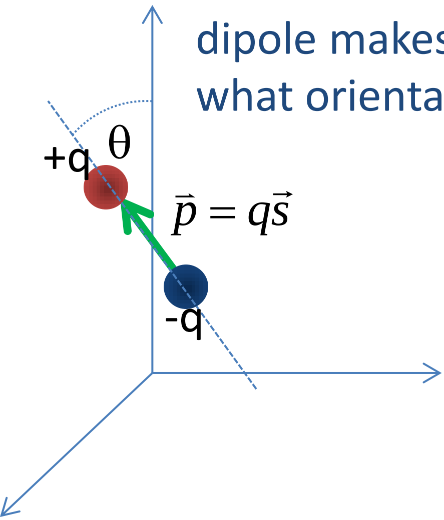
$$\Delta U_{stretch} = -2\int_{s_i}^{s_f} q\vec{E} \cdot \tfrac{d\vec{s}}{2} = -2\int_{s_i}^{s_f} q\vec{E} \cdot d\vec{s} = -\Delta\left(\vec{p} \cdot \vec{E}\right)\Big|_{E,\theta}$$

Atom's Response to Electric Field

Response of a dipole to fields

$$\vec{N} = \vec{p} \times \vec{E} \qquad N = pE\sin\theta$$

Exercise: A dipole is a distance *z* above an infinite grounded conducting plane. The dipole makes an angle θ with the perpendicular to the plane. Find the torque on it. In what orientation is it stable?



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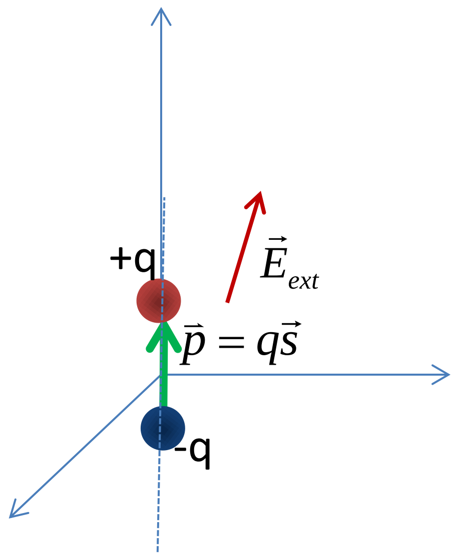
Atom's Response to Electric Field



Response of a dipole to fields

Force in uniform field

$$\vec{F}_{net} = \sum \vec{F}_i$$

$$\vec{F}_{net} = q\vec{E} - q\vec{E} = 0$$

Force in a *non*-uniform field

$$\vec{F}_{net} = q\vec{E}_+ - q\vec{E}_-$$

$$\vec{F}_{net} = q\left(\vec{E}\left(\vec{r}_+'\right) - \vec{E}\left(\vec{r}_-'\right)\right)$$

$$\vec{F}_{net} = q\left(\vec{E}\left(\vec{r}_+'\right) - \vec{E}\left(\vec{r}_+' - \vec{s}\right)\right)$$

First, imagine dipole along z-axis,

$$\vec{F}_{net} = qs\left(\frac{\vec{E}\left(\vec{r}_+'\right) - \vec{E}\left(\vec{r}_+' - \vec{s}\right)}{s}\right) = p_z\left(\frac{\partial \vec{E}}{\partial z}\right)$$

Alternatively, if dipole points along y-axis $p_y\left(\frac{\partial \vec{E}}{\partial y}\right)$ or dipole points along y-axis $p_x\left(\frac{\partial \vec{E}}{\partial x}\right)$

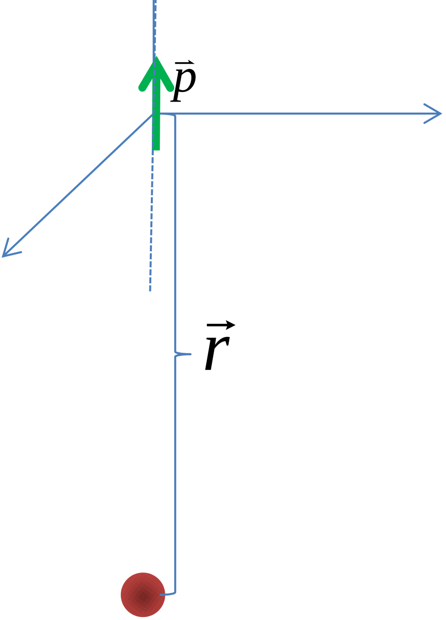
generally

$$\vec{F}_{net} = p_x\left(\frac{\partial \vec{E}}{\partial x}\right) + p_y\left(\frac{\partial \vec{E}}{\partial y}\right) + p_z\left(\frac{\partial \vec{E}}{\partial z}\right) = \left(\vec{p} \cdot \vec{\nabla}\right)\vec{E}$$

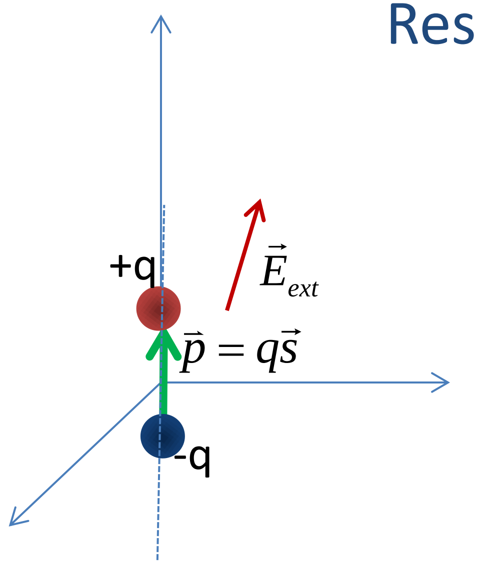
Atom’s Response to Electric Field

$$\vec{F}_{net} = \left(\vec{p} \cdot \vec{\nabla}\right)\vec{E}$$

Exercise: What is the force on a dipole due to a point charge ‘below’ it?



Atom’s Response to Electric Field



Response of an *induced* dipole to fields

$$\vec{F}_{net} = \left(\vec{p}\cdot\vec{\nabla}\right)\vec{E} \quad \text{and} \quad \vec{p} = \vec{\vec{\alpha}}\vec{E}$$

so

$$\vec{F}_{net} = \left(\vec{\vec{\alpha}}\vec{E}\cdot\vec{\nabla}\right)\vec{E}$$

Product rule 4 (if polarizability is uniform / scalar)

$$\vec{F}_{net} = \alpha\left(\tfrac{1}{2}\vec{\nabla}\left(E^2\right) - \vec{E}\times\left(\vec{\nabla}\times\vec{E}\right)\right)$$

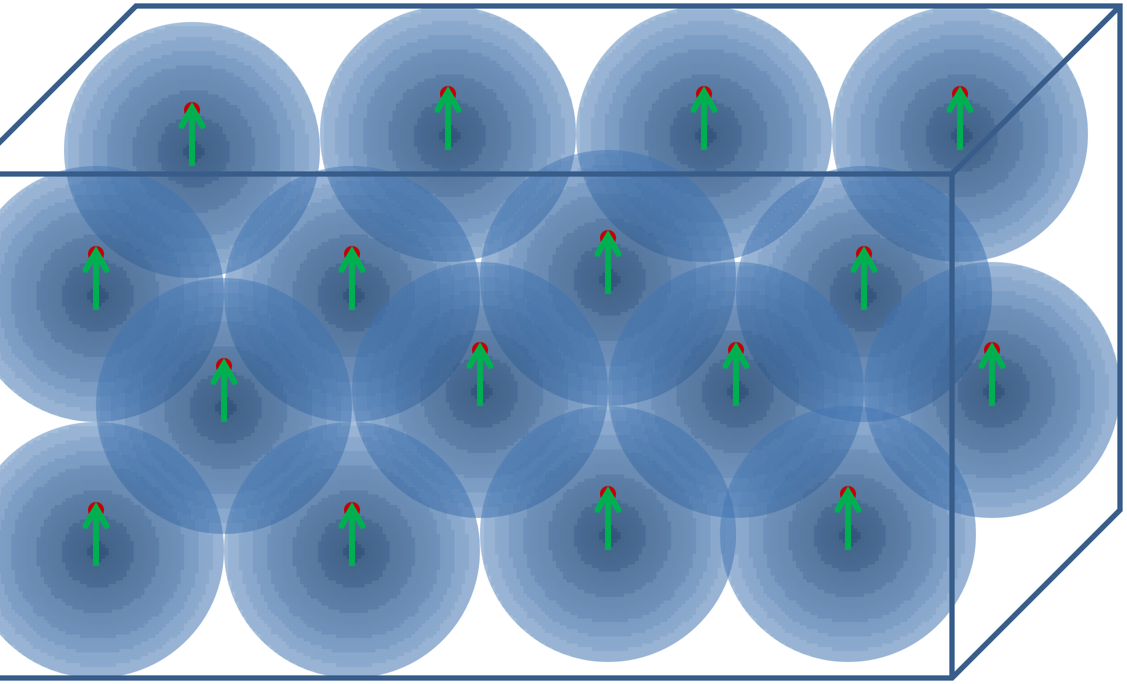
Only if time-varying current and/or charge distribution

Note: force in direction of increasing field strength - “Optical tweezers”

Bulk material's Response to Electric Field

Polarization

$\mathbf{P} \equiv$ *dipole moment per unit volume*



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