Fri. 9/20	**2.4.3-4.4** Work & Energy in Electrostatics	
Mon. 9/23 Wed. 9/25 Thurs 9/26	**2.5** Conductors **3.1-.2** Laplace & Images ***Summer Science Research Poster Session**: Hedco7pm~9pm*	 HW3

Materials

Announcements

Research Presentations tomorrow 4pm. Dinner at Julie & Tyler's afterwards – would like a guesstimate at how much food we'll need.

Last Time

Finished with boundary conditions

Work & Energy with fields

The work of moving one charge through an existing potential terrain

$$W_{\to q\, a\to b} = q\left(V(b) - V(a)\right)$$

The work of moving all the charges into place (from infinity)

$$W = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{j<i}\frac{q_i q_j}{r_{ij}}$$

This Time

Work to build a Continuous Charge Distribution

We want to rephrase the double sum we developed last time,

$$W_{assemble} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{j<i}\frac{q_i q_j}{r_{ij}}$$

in a way that we can easily write an integral. The work to assemble a discrete charge distribution is

$$W_{assemble} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{j<i}\frac{q_i q_j}{r_{ij}} = \frac{1}{2}\left[\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{i\neq j}\frac{q_i q_j}{r_{ij}}\right] = \frac{1}{2}\sum_{i=1}^{n} q_i\left(\frac{1}{4\pi\varepsilon_0}\sum_{i\neq j}\frac{q_j}{r_{ij}}\right) = \frac{1}{2}\sum_{i=1}^{n} q_i V(P_i).$$

The ½ in the second expression is because of double counting. The second sum can be rewritten to give the potential at the location of charge q_i due to all of the other charges – called $V(P_i)$.

- **No Self-Energy.** This is an important qualification, so I'll repeat it: we're defining $V(P_i) \equiv \frac{1}{4\pi\varepsilon_0}\sum_{i\neq j}\frac{q_j}{r_{ij}}$ that's the potential at the location of charge *i* due to all charges *but* charge *i* (relative to what it would be if everyone else were at infinity.) That means that

we don't have include a term for *assembling* the point charge itself (which is embarrassingly infinite: $\frac{1}{4\pi\varepsilon_o}\frac{q_i^2}{r_{ii}} = \frac{1}{4\pi\varepsilon_o}\frac{q_i^2}{0} = \infty$).

If we have a continuous, or piece-wise continuous, charge density, then

$$W_{assemble} = \lim_{dq\to 0}\frac{1}{2}\sum_{i=1}^{n} dq_i V(\vec{r}_i) = \lim_{d\tau\to 0}\frac{1}{2}\sum_{i=1}^{n}\rho d\tau V(\vec{r}_i)$$

$$W_{assemble} = \frac{1}{2}\int \rho V\, d\tau \qquad [2.43]$$

Almost Self-Energy. With this representation, it isn't hard to inadvertently *include* the self-voltage terms in the sum – you probably aren't defining the voltage to *exclude* that associated with the charge right where you're evaluating it. On the upside, we usually use this integral form only when ρ is everywhere finite, and therefore the charge evaluated at any specific location is vanishingly small:

$$q(r') = \lim_{d\tau\to 0}\rho(r')d\tau' \to 0$$

Thus the self-energy contribution is coincidentally absent from the sum.

Inapplicable for Point Charges. This condition is decidedly *not* met for point charges. The charge density is a Dirac delta, so

$$q(r') = \lim_{d\tau\to 0}\rho_{point}(r')d\tau' = q \neq 0$$

Which means that we *would* have a Self-Energy term. While you may not be tempted to accidentally use integral 2.43 for point charges, it is more tempting to use one we'll derive from it which inherits this limitation.

Charging Capacitor. While it's important to build the general tools, it's also valuable to look at a concrete and specific example. We'll take a moment to remember how we found the work done to assemble a charged capacitor. That will hopefully serve two purposes – connect the more general arguments of this chapter with what you've previously learned, and give you something concrete to focus on when we address a couple of bigger questions about energy and fields.

Say you want to built up a capacitor, perhaps you bring the charges in from infinity, or, more easily, you just move them between the initially-neutral plates of the capacitor. For an ideal capacitor, the electric field is uniform between the two plates

$$E = \frac{Q}{\varepsilon_o A} \text{ (thank you Gauss's Law)}$$

Alternatively, then the voltage difference between the two plates is

$\Delta V = Es = \frac{Qs}{\varepsilon_o A}$ (just worried about magnitude, s is separation of the plates)

So the work of moving an additional little bit of charge, dQ_j, from one plate to the other is simply

$$dW_j = dQ_j E_j s = \frac{Q_j s}{\varepsilon_o A} dQ_j$$

Of course, in the course of moving the morsel, dQ_j, of charge between plates, you've increased the charge on the plates by the same amount, $Q_j \to Q_j + dQ_j$.

So the total work done by doing this over and over again in order to charge up the plates from neutral is

$$W_{create} = \sum dW_j = dQ_j E_j s = \sum \frac{Q_j s}{\varepsilon_o A} dQ_j \Rightarrow \int \frac{Q\, s}{\varepsilon_o A} dQ = \tfrac{1}{2} \frac{s}{\varepsilon_o A} Q^2$$

By using the relationship between the field and the charge or the potential and the charge, you can express this as either

$$W_{create} = \tfrac{1}{2} Q \Delta V = \tfrac{\varepsilon_o}{2} E^2 As = \tfrac{\varepsilon_o}{2} E^2 Vol$$

What this section of the chapter is about is addressing this same question – how much work is done in establishing a given charge distribution, but by building more general tools that can handle something more complex than the simple / uniform field capacitor. Not surprisingly, the results are quite similar.

Back to our general expression

$W = \frac{1}{2} \int \rho(r') V(r')\, d\tau'$ (work done in bringing charges in from infinity to form given charge distribution)

Using a product rule and the divergence theorem, this can be rewritten as

$$W = \frac{\varepsilon_0}{2} \int_{\text{all space}} E^2\, d\tau. \qquad [2.45]$$

First use the differential form of Gauss's Law, $\vec{\nabla} \cdot \vec{E} = \rho/\varepsilon_0$, to write

$$W = \frac{\varepsilon_0}{2} \int \left(\vec{\nabla} \cdot \vec{E}\right) V\, d\tau.$$

The product rule:

$$\vec{\nabla} \cdot \left(\vec{E} V\right) = \left(\vec{\nabla} \cdot \vec{E}\right) V + \vec{E} \cdot \left(\vec{\nabla} V\right)$$

and the relation $\vec{\nabla} V = -\vec{E}$ to rewrite the integrand as

$$W = \frac{\varepsilon_0}{2} \left[\int \vec{\nabla} \cdot \left(\vec{E} V\right) d\tau + \int E^2 d\tau \right].$$

Apply the divergence theorem, $\int_V (\vec{\nabla}\cdot\vec{A})\,d\tau = \oint_S \vec{A}\cdot d\vec{a}$, to the first integral to get

$$W = \frac{\varepsilon_0}{2}\left[\int_{Surface} V\vec{E}\cdot d\vec{a} + \int_{Volume} E^2 d\tau\right].$$

Originally, the integral was just over the region were the charge resided, but we can expand it to regions where $\rho = 0$ without changing the answer. If we let the volume increase to all of space, the first integral goes to zero because *V* and *E* get smaller far away from the charge. This gives us the result in Eq. 2.45.

$$W = \frac{\varepsilon_0}{2}\int_{all.space} E^2 d\tau$$

Flaw of applying to Point Charge

Recall what I said about the problem with *integrating* to find the energy associated with a distribution of point charges. Their infinite charge densities preserve the self-energy contribution in the continuous sum, and thus it does not equate with the finite sum that we explicitly constrained to *not* include that term.

The book says that, all the same,

$W = \frac{\varepsilon_0}{2}\int_{all.space} E^2 d\tau$ can be taken to answer a *different* question when considering point charges: how much work would it take to assemble the *point* charges and then configure them. That seems a little dubious since it yields infinity. Then again, this whole 'assembling charges from infinity' is a little artificial. A more realistic question would be – how much work does it take to *change* the charge distribution from one realistic one to another. Either form of the sum would work for that.

daftThe book points out that there's a problem with applying this, or *any* integral representation to a system for which the charge density is infinite at any locations. The trouble is most easily seen back where we As I've tried to set up, then

$$q(r') = \lim_{d\tau\to 0} \rho(r')d\tau' \to 0$$

(No) Supper Position Principle: Note that the superposition principle does <u>not</u> hold for electrostatic energy.

- **Mathematically,** that's because it depends on the square of the field strength at each location, so you get 'cross terms' when you add a new collection of field sources.
- **Physically**, the origin of these cross terms is that your *new* collection and your *old* collection of charges interact *with each other*.
    - **Example.** Say you build a spherical shell of charge, and put it at the origin. That took yhey much work. Then you decide to build a second one near by.

Sure, it takes a certain amount of energy to build the sphere all by itslef, but it *also* takes some additional work because you're doing it near that first charged sphere. Now, you assemble a second sThus, there's not just more energy associated with building

Where's the Energy?

This actually touches on a few misconceptions as well as something more general about the nature of potential energy, so it's worth digging in a little bit.

Griffiths tries to address the common question of 'Where's the energy stored, in the charge distribution or in the field?' While there's nothing wrong with that question, in and of itself, people often appeal to three mistaken ideas when trying to answer it: two about energy and one about fields. I'll correct them before we proceed.

- First, *change* in potential energy is meaningful, a single value of potential energy all by itself is not. If you assign meaning to potential energy (not its change) then you eventually encounter troubling / embarrassing infinities.
- Second, energy isn't a *thing* like nickels, beans, or rice, rather it is a *property* of a system – a measure of its motion or capacity for motion.
- Third and similarly, electric field isn't a fundamentally separate *thing* either, it too is a *property* of the charge. The field at a particular location is a measure of the source charge's potential for influence at that location. (Warning: many physicists will beg to differ, but that's often based on their practical modeling experience in which they use knowledge of fields at boundaries to generate knowledge of fields elsewhere, and they've blurred the lines between knowledge of a thing and the thing itself.) If we must visualize the relationship between charge and field, visualize a marble (charge) glued to the center of a sheet (field.)

Now, let's consider our scenario for a moment, we've rearranged some charges to build a capacitor – we haven't changed their states of motion, just their configuration – so the work we've done changed potential energy, not kinetic energy. So our question is really "What changes when we change a system's potential energy?" The energy is 'invested' in that change. The answer is 'the relative positions of the system's parts in space.' We haven't changed the *individual* particles in any significant way, but their *collective configuration through space*. That's where the potential energy changed then – through the space of their changed configuration – that's exactly where you've changed the associated fields. So, the best answer is that the change in charges' potential energy can be associated with the change in the charges' fields.

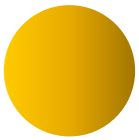
Example/Exercise:

Problem 2.32 – Work of assembling Uniformly Charged Solid Sphere

Suppose the sphere has a radius R and a total charge q (positive).

To use Eq. 2.45, we need to know the electric field everywhere. The simplest method is to use Gauss's Law,

$$\oint_S \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{\text{enc}}.$$



By symmetry, the electric field must point radially outward. For a spherical Gaussian surface of radius r, the electric flux is $E\left(4\pi r^2\right)$. The amount of charge enclosed depends on the size of r:

$$Q_{enc} = \begin{cases} \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3} q = \frac{qr^3}{R^3} & r < R \\ q & r > R \end{cases}$$

For $r < R$, the fraction is the ratio of the volume enclosed to the total volume.

Applying Gauss's Law gives

$$\vec{E} = \begin{cases} \frac{1}{4\pi\varepsilon_0}\frac{qr}{R^3}\hat{r} & r < R \\ \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} & r > R \end{cases}$$

The square of the electric field will be approximately constant in a thin spherical shell between r and $r + dr$. The energy for such a shell is

$$dW = \frac{\varepsilon_0}{2} E^2(r)\left[4\pi r^2 dr\right].$$

The integral over all of space must be broken into two parts because of the different regions for E:

$$W = \frac{\varepsilon_0}{2}\left\{ \int_0^R \left(\frac{1}{4\pi\varepsilon_0}\frac{qr}{R^3}\right)^2 \left[4\pi r^2 dr\right] + \int_R^\infty \left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\right)^2 \left[4\pi r^2 dr\right] \right\}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q^2}{2}\left\{ \frac{1}{R^6}\int_0^R r^4 dr + \int_R^\infty r^{-2} dr \right\} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{2}\left\{ \frac{1}{R^6}\left[\frac{r^5}{5}\right]_0^R + \left[\frac{-1}{r}\right]_R^\infty \right\}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q^2}{2}\left\{ \frac{1}{5R} + \frac{1}{R} \right\} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{2}\left\{ \frac{6}{5R} \right\} = \frac{1}{4\pi\varepsilon_0}\frac{3q^2}{5R}$$

What other distributions do we want to setup to analyze?

Preview

For Monday, you'll read about conductors (Section 2.5)

"Why are we really so interested in calculating the work it takes to create a charge distribution in the first place? I can understanding wanting to know the potential and the electric field, but I still don't have such a clear picture for work."
Casey McGrath
Will we use this later for other things?

Also I'd appreciate talking more conceptually about these odd "pseudo-flaws" that don't really seem to be flaws after all mentioned in section 2.4.4
Casey McGrath
I feel a little bit the same way sometimes but when he starts talking about energy, then that gets me thinking a little more in the right direction. I have no idea if that's what was on your mind but i can't say I didn't try!
Rachael Hach

"I'm a little confused about Griffith's logic on why we can integrate over all space for eqn 2.45, can we talk about this?"
Jessica
I was a little shaky on that bit too. Is the only advantage of integrating over all space getting rid of the surface integral?
Ben Kid

"So I realized just now that I have two HW packets, so if you are missing yours let me know. heheheh..."
Casey P, AHoN swag 4 liphe

"Could we talk about the variables we integrate over? I've been getting a little confused lately by all the r's and things like 's' substituted for them. Sometimes I'm not sure why I am integrating with one or another."
Anton

"We talked in Thermo. a little bit about the energy a system has to offer minus the energy it took to put it there. Now it's being mentioned in Griffiths. Was another name for that on the Hemholtz free energy? Do you think you could maybe clarify w..."
Rachael Hach
why discussing the energy it takes to create a system is important?
Rachael Hach

"can we see examples where equations 2.45 and 2.42 are used. Also would it be possible to use 2.45 for a point charge and 2.42 for a volume distribution?"
Sam

"How does Equation 2.45 take into account the energy in creating a charge? Does it have to do with the fact that we're integrating over an electric field rather than a charge distribution?"
Spencer