Wed. 9/11 Thurs 9/12 Fri. 9/13	*(C 21.1-.5,.8)* **2.2.3** Using Gauss **T2** Numerical Quadrature (continued), 1.3 Integral Calc (recommended) *(C21.1-.5,.8)* **2.2.3-.2.4** Using Gauss	HW1
Mon. 9/16 Wed. 9/18 Thurs 9/19	*(C 16)* **1.6, 2.3.1 -.3.3** Electric Potential *(C 16)* **2.3.4-.3.5** Electric Potential	HW2

Last time we "met" Gauss's Law, and saw its two forms: integral and differential.

- **Integral**: the flux or "flow" of electric field through a closed surface depends on the sources enclosed (like water from a showerhead & through a mesh baggy).
- $$\oint_S \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{\text{enc}}$$
- **Differential**: the Divergance is something like the rate at which the enclosed volume empties or fills
- $$\lim_{Vol\to 0}\frac{\oint_S \vec{E}\cdot d\vec{a}}{Vol} = \lim_{Vol\to 0}\frac{1}{\varepsilon_0}\frac{Q_{\text{enc}}}{Vol}$$
$$Div\vec{E} = \frac{1}{\varepsilon_0}\rho$$
$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho$$

Integral Form

The following is always true, but <u>not</u> always useful!

$$\oint_S \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{\text{enc}}$$

To be useful for finding the electric field, a charge distribution must be symmetric. There are three kinds of symmetry that are helpful:

(1) Spherical: $\rho(r)$, not a function of θ or ϕ $\rightarrow$ $\vec{E} = E(r)\hat{r}$

(2) Cylindrical: $\rho(s)$, not a function of z or ϕ $\rightarrow$ $\vec{E} = E(s)\hat{s}$

(3) Planar: $\rho(|z|)$, not a function of x or y (for symmetry about xy plane) $\rightarrow$ $\vec{E} = \pm E(z)\hat{z}$ (+ for $z > 0$, – for $z < 0$)

In these cases, it is possible to choose a *Gaussian surface* so that the electric field is either perpendicular or parallel to each part of the surface. The surface should be chosen so that where the field is perpendicular, it is has a constant magnitude.

When Gauss's Law is useful, the integral on the left-hand side of the equation should be easy!

Tips for Using Gauss's Law

(1) Use a symmetry argument to determine the *direction* of the electric field.

(2) Figure out what shape of Gaussian surface can you draw so that each part is either perpendicular or parallel to the electric field. (Draw it!)

(3) Apply Gauss's Law to the chosen surface to determine the *magnitude* of the electric field. (Some dimensions won't appear in the answer.)

Examples

Example 1: Spherical shell with radius *R* and uniformly distributed charge *Q*

By symmetry, the electric field must point radially and its magnitude can only depend on the distance from the center of the shell. A sphere of radius *r* is a good Gaussian surface since it will be perpendicular to the electric field everywhere. The electric flux is:

$$\sum \vec{E} \cdot \hat{n}\, \Delta A = E\left(4\pi r^2\right)$$

$r < R$: no charge inside $\left(\sum q_{\text{inside}} = 0\right)$, so $E = 0$

$r > R$: all charge is inside $\left(\sum q_{\text{inside}} = Q\right)$, so

$$\sum \vec{E} \cdot \hat{n}\, \Delta A = E\left(4\pi r^2\right) = \frac{\sum q_{\text{inside}}}{\varepsilon_0} = \frac{Q}{\varepsilon_0}$$

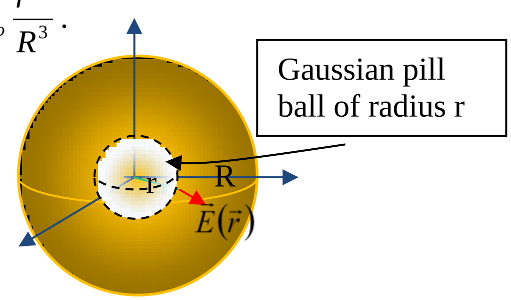
$$E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

We've been using these results for quite a while now (the second part was just stated)!

Note: Gauss's law only helps get the magnitude of the electric field, not the direction.

When we're using Gauss's Law, it's generally because the integral on the *Left* is easy, but if the charge enclosed is phrased not simply as Q, but as a charge *density,* then you may have a corresponding integral on the *right* to do.

Example 2: Sphere of radius R with varying charge density $\rho(r') = \rho_o \frac{r'^3}{R^3}$.



What is the field at a point a distance *r_1< R?*

What is the field at a point a distance *r_2> R?*

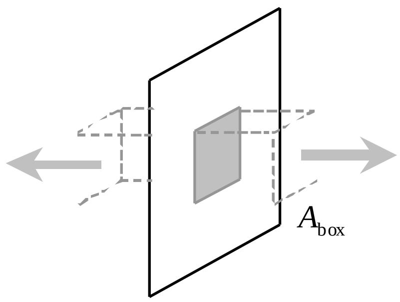
$$\oint_S \vec{E}(\vec{r}) \cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{\text{enc}} = \int_{0<r'<r'_{max}} \rho(\vec{r}')\, d\tau'$$

$\int_{r'=0}^{r'_{max}} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \left(\rho_o \frac{r'^3}{R^3}\right) r'^2 dr' \sin\theta d\theta d\phi$ (limit of r is the radius of the Gaussian pill box for a location inside the sphere, and it's R for a location outside.)

Example 3: Large uniformly charged plate with charge per area of *Q*/*A*

By symmetry, the electric field near the center of the plate must point perpendicularly away from the plate. Its magnitude could depend on the distance from the plate.

A box that extends on each side of the plate is a good Gaussian surface since the electric field will be parallel or perpendicular to each side.



If the sides perpendicular to the plate each have an area A_{box}, the electric flux is:

$$\sum \vec{E}\cdot\hat{n}\,\Delta A = 2EA_{\text{box}}$$

The amount of charge inside the Gaussian surface is $\sum q_{\text{inside}} = \frac{Q}{A}A_{\text{box}}$, so

$$\sum \vec{E}\cdot\hat{n}\,\Delta A = 2EA_{\text{box}} = \frac{\sum q_{\text{inside}}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\frac{Q}{A}A_{\text{box}}$$

$$E = \frac{Q/A}{2\varepsilon_0}$$

Example 4. Capacitor. After this one, they asked for capacitor – I started trying to do one large Gaussian Box that pierced both plates, but then I realized that while Qencl was 0, that could simply be because the field in the bottom was equal to that out the top (not requiring that they were 0.) Ultimately, just reasoned by saying, now that we've got the field for one, we can add it to the field for another.

Exercises

A. Uniformly-Charged Rod

A thin rod of length L has a positive charge Q distributed uniformly along its length.

- **Field Geometry**. Use a *symmetry* argument to determine the *direction* of the electric field near the center of the rod.

- **Choosing the Gaussian bubble.** What shape of Gaussian surface can you draw so that each part is either perpendicular or parallel to the electric field?

- **Doing the Math.** Use Gauss's law to find the magnitude of the electric field at a radial distance r from the rod near its center.

B. Solid Uniformly-Charged Sphere

A solid sphere of radius R has a positive charge Q distributed uniformly throughout its volume.

- **Field Geometry.** Use a symmetry argument to determine the direction of the electric field.

- **Choosing a Gaussian Bubble.** What shape of Gaussian surface can you draw so that each part is either perpendicular or parallel to the electric field?

- **Doing the Math.** Use Gauss's law to find the magnitude of the electric field at a distance r from the center of the sphere for:

  $r > R$

  $r < R$

More with the Numerical Quatrature.

Problem 2.11 – Uniformly charged spherical shell

Problem (like part of 2.16) – Solid cylinder with uniform charge density

Problem 2.17 – Sheet of thickness 2*d* with uniform charge density

Curl of E

Well, Del can 'multiply' a vector in two ways. Guass's Law comes from dotting Del into E. What about crossing it into E?

Do for Point Charge and then Superposition-Principle up to many charges.

Griffiths references Stoke's theorem and then is all done. Go ahead and read section 1.3, in particular 1.3.5 on Stoke's theorem. It should make conceptual sense but it may feel a little unproven. What I want to do today is very straightforward math. It won't be particularly conceptually enlightening, but the math is clear and simple. Together, this and Griffiths argument should make it pretty clear that there *is no curl* for E of a stationary point charge.

Stationary Point Charge. It is relatively straightforward to calculate the curl of the electric field due to a single point charge. If a charge q is located at $\vec{r}' = x'\hat{x} + y'\hat{y} + z'\hat{z}$, then the electric field at $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ is

$$\vec{E}_1(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\frac{\vec{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^3} = \frac{q}{4\pi\varepsilon_0}\frac{\left[(x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}\right]}{\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{3/2}}$$

so

$$\vec{\nabla}\times\vec{E}_1 = \left(\frac{q}{4\pi\varepsilon_0}\right)\vec{\nabla}\times\left(\frac{\vec{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^3}\right)$$

$$= \left(\frac{q}{4\pi\varepsilon_0}\right)\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \dfrac{(x-x')}{\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{3/2}} & \dfrac{(y-y')}{[\ ]^{3/2}} & \dfrac{(z-z')}{[\ ]^{3/2}} \end{vmatrix}$$

Look at just the x component of the determinant, which is

$$\frac{\partial}{\partial y}\left\{\frac{(z-z')}{[\ ]^{3/2}}\right\}-\frac{\partial}{\partial z}\left\{\frac{(y-y')}{[\ ]^{3/2}}\right\}=(z-z')\frac{\left(-\frac{3}{2}\right)2(y-y')}{[\ ]^{5/2}}-(y-y')\frac{\left(-\frac{3}{2}\right)2(z-z')}{[\ ]^{5/2}}=0.$$

The other components (y and z) are also zero, so $\vec{\nabla}\times\vec{E}_1=0$.

Supper Position Principle.

Any *static* charge distribution can be built up of stationary point charges. If the curl is 0 for the field due to one, it's zero for the field due to all.

$$\vec{E}=\vec{E}_1+\vec{E}_2+\ldots,\ \text{so}$$

$$\vec{\nabla}\times\vec{E}=\vec{\nabla}\times\left(\vec{E}_1+\vec{E}_2+\ldots\right)=\left(\vec{\nabla}\times\vec{E}_1\right)+\left(\vec{\nabla}\times\vec{E}_2\right)+\ldots=0.$$

If an electric field doesn't satisfy this condition, it couldn't be produced by *static* charges!

The *Fundamental Theorem of Curls* (*Stoke's Theorem*) says

$$\int_S\left(\vec{\nabla}\times\vec{v}\right)\cdot d\vec{a}=\oint_P\vec{v}\cdot d\vec{\ell},$$

where S is a surface and P is a closed path that bounds S. This holds for any surface and $\vec{\nabla}\times\vec{E}=0$, which imply that

$$\oint\vec{E}\cdot d\vec{\ell}=0,$$

for *any* closed loop. The differential and integral forms of this statement are true for any static arrangement of charges, but <u>not</u> for situations where the *magnetic* field is changing (Faraday's Law). This property is important for the definition of the electric potential.

<u>Preview</u>

For next time, you'll read about *electric potential* (V). The definition is based on $\oint\vec{E}\cdot d\vec{\ell}=0$ for electrostatic fields. Also review how to perform *line integrals* (1.3.3).