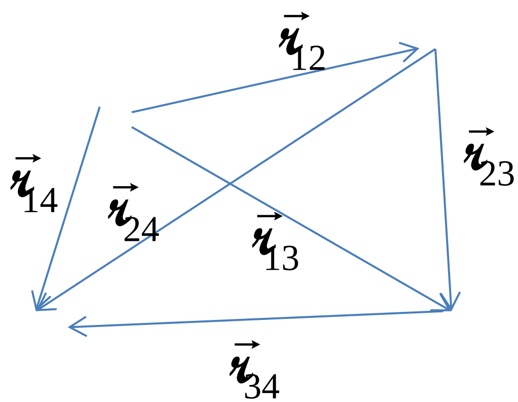
Fri.	2.4.3-4.4 Work & Energy in Electrostatics	
Mon. Wed. Thurs	2.5 Conductors 3.1-.2 Laplace & Images Summer Science Research Poster Session: Hedco7pm~9pm	HW3

Work to construct charge distribution

Source charges sensing charge, Q

$$W(a \rightarrow b) = \int_a^b \vec{F}_{you} \cdot d\vec{l} = Q\left(-\int_a^b \vec{E} \cdot d\vec{l} \right) = Q\Delta V$$



$$W = \sum_{i=2}^{n} \sum_{j=1}^{i-1} \frac{1}{4\pi\varepsilon_o} \frac{q_i q_j}{\left| \mathscr{r}_{ij} \right|} = \sum_{i} \sum_{j<i} \frac{1}{4\pi\varepsilon_o} \frac{q_i q_j}{\left| \mathscr{r}_{ij} \right|}$$

Work to construct a *discrete* charge distribution

$$W_{assemble} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{j<i}\frac{q_i q_j}{\mathscr{r}_{ij}} = \frac{1}{2}\left[\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{i\neq j}\frac{q_i q_j}{\mathscr{r}_{ij}}\right] = \frac{1}{2}\sum_{i=1}^{n} q_i\left(\frac{1}{4\pi\varepsilon_0}\sum_{i\neq j}\frac{q_j}{\mathscr{r}_{ij}}\right)$$

If we sum twice over *all* charges (but self interaction)

Avoiding *self* energy

$$V(P_i) \equiv \frac{1}{4\pi\varepsilon_0}\sum_{i\neq j}\frac{q_j}{\mathscr{r}_{ij}}$$

Potential at location of q_i due to all other charges *but* q_i

Work to construct a *continuous* charge distribution

$$W_{assemble} = \lim_{dq\to 0}\frac{1}{2}\sum_{i=1}^{n} dq_i V(P_i) = \lim_{d\tau\to 0}\frac{1}{2}\sum_{i=1}^{n}\rho\, d\tau' V(P_i) = \frac{1}{2}\int \rho V\, d\tau'$$

where

$$V = \frac{1}{4\pi\varepsilon_o}\int \frac{\rho}{\mathscr{r}}\, d\tau'$$

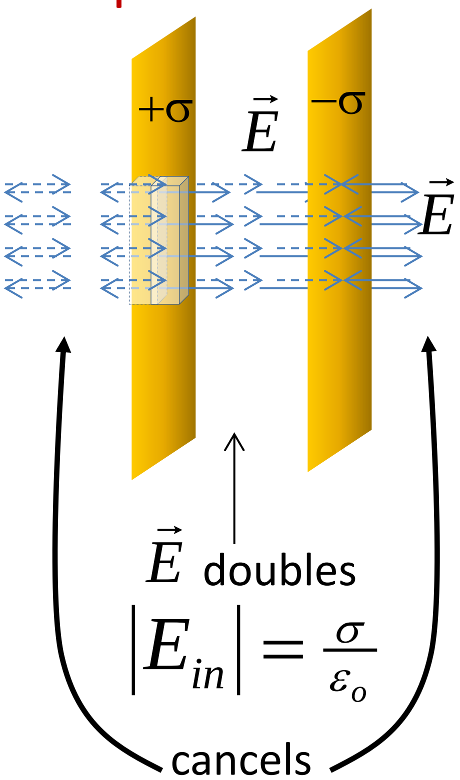
Self energy issues vanish for *continuous* distribution since

$$dq(r') = \lim_{d\tau\to 0}\rho(r')d\tau' \to 0$$

Concrete example, universal results – Charging a Capacitor

First, E-field expression

One plate   Another plate



$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{encl}}{\varepsilon_o}$$

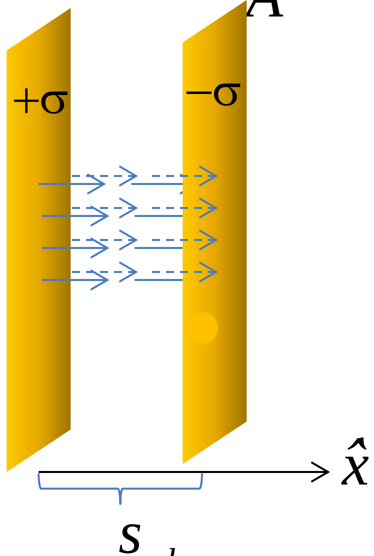
$$\int |E_r| da_r + \int |E_l| da_l = \frac{1}{\varepsilon_o} \int \sigma da$$

$$|E_r| + |E_l| = \frac{\sigma}{\varepsilon_o}$$

$$|E| = |E_r| = |E_l| = \frac{\sigma}{2\varepsilon_o}$$

Concrete example, universal results – Charging a Capacitor

$$\vec{E} = \tfrac{\sigma}{\varepsilon_o}\hat{x} = \tfrac{1}{\varepsilon_o}\frac{Q}{A}\hat{x}$$



$$\Delta V_{r\to l} = -\int_r^l \vec{E}\cdot d\vec{l} = \tfrac{1}{\varepsilon_o}\frac{Q}{A}s$$

Work to move one morsel of charge, dQ

$$dW = dQ(\Delta V) = \left(\frac{Q}{\varepsilon_o A}s\right)dQ$$

$$W = \sum dW = \sum \frac{Qs}{\varepsilon_o A}dQ \Rightarrow \int \frac{Qs}{\varepsilon_o A}dQ = \tfrac{1}{2}\frac{s}{\varepsilon_o A}Q^2$$

Or using what we'd just derived:

$$W_{assemble} = \frac{1}{2}\int \sigma V\, da' = \frac{1}{2}\int \sigma\left(\tfrac{1}{\varepsilon_o}\sigma s\right) da'$$

$$W_{assemble} = \frac{1}{2}\tfrac{1}{\varepsilon_o}\sigma^2 sA = \frac{1}{2}\tfrac{1}{\varepsilon_o}\left(\frac{Q}{A}\right)^2 sA$$

Rephrased in terms of field

$$W_{create} = \tfrac{1}{2}Q(\Delta V) = \tfrac{\varepsilon_o}{2}E^2 As = \tfrac{\varepsilon_o}{2}E^2 Vol$$

Same result, more mathematically

$$W=\frac{1}{2}\int \rho V\, d\tau$$

$$\rho=\varepsilon_0\vec{\nabla}\cdot\vec{E}$$ Gauss's Law

$$W=\frac{\varepsilon_0}{2}\int\left(\vec{\nabla}\cdot\vec{E}\right)V\, d\tau$$

$$\left(\vec{\nabla}\cdot\vec{E}\right)V=\vec{\nabla}\cdot\left(\vec{E}V\right)-\vec{E}\cdot\left(\vec{\nabla}V\right)$$ Product Rule

$$\vec{\nabla}V=-\vec{E}$$

$$W=\frac{\varepsilon_0}{2}\left[\int\vec{\nabla}\cdot\left(\vec{E}V\right)d\tau+\int E^2 d\tau\right]$$

$$\int_V\left(\vec{\nabla}\cdot\left(\vec{E}V\right)\right)d\tau=\oint_S\left(\vec{E}V\right)\cdot d\vec{a}$$ Divergence Theorem

$$\oint_S\left(\vec{E}V\right)\cdot d\vec{a}\Rightarrow 0$$ Sending Volume & Area out to infinity
To contain all of charge distribution's field

$$W=\frac{\varepsilon_0}{2}\int_{all.space}E^2 d\tau$$

Not for point charges

Recall in derivation of

$$W = \frac{1}{2}\int \rho V \, d\tau$$

that we explicitly required

$$dq(r') = \lim_{d\tau \to 0} \rho(r') d\tau' \to 0$$

to avoid accidentally including

$$\tfrac{1}{4\pi\varepsilon_o} \frac{q_i^2}{r_{ii}} = \tfrac{1}{4\pi\varepsilon_o} \frac{q_i^2}{0} = \infty$$

in our sum.

So,

$$W = \frac{\varepsilon_0}{2} \int_{all.space} E^2 d\tau$$

doesn't apply for a point charge.

Example: Work of assembling a charged, solid sphere