Fri.	10.2 Continuous Distributions	
Mon. Wed.	10.3 Point Charges (C 14) 4.1 Polarization	 HW9

Maxwell's Laws

Relating Fields and Sources

Relativistically Correct since instantaneous and local

$$\vec{\nabla}\times\vec{B}-\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}=\mu_0\vec{J}$$ Maxwell – Ampere's Law $$\oint\vec{B}\cdot d\vec{\ell}-\mu_0\varepsilon_0\int\frac{\partial\vec{E}}{\partial t}\cdot d\vec{a}=\mu_0\int\vec{J}\cdot d\vec{a}$$

$$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\varepsilon_0}$$ Gauss's Law $$\oint\vec{E}\cdot d\vec{a}=\frac{Q_{enc}}{\varepsilon_0}$$

$$\vec{\nabla}\cdot\vec{B}=0$$ Gauss's Law for Magnetism $$\oint\vec{B}\cdot d\vec{a}=0$$

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$ Faraday's Law $$\oint\vec{E}\cdot d\vec{\ell}=-\left.\frac{\partial\Phi_B}{\partial t}\right|_a=-\int\frac{\partial\vec{B}}{\partial t}\cdot d\vec{a}$$

Not necessarily Relativistically Correct

Helmholtz Theorem: if you know a vector field's curl and divergence (and time derivative), you know everything

Corresponding Relations between Potentials

(on the road to general solutions for E and B)

Combine

Maxwell's Relations Between Fields & Sources | with | Potentials' Relations to Fields | to Relate Potentials & Sources

~~$-\vec{\nabla}V \equiv \vec{E}$~~ $\quad \vec{\nabla}\times\vec{A} \equiv \vec{B}$

Redefine $\quad -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t} \equiv \vec{E}$

No effect on electro*statics.* In electro *dynamics,* work associated with V and dA/dt.

Faraday's Law

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \longrightarrow 0 = \vec{\nabla}\times\left(\vec{\nabla}V\right) = \frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{A}\right)$$

Doh! $\vec{\nabla}\times\left(\vec{\nabla}f\right) = 0$ For any scalar field *f*.

Maxwell – Ampere's Law

$$\vec{\nabla}\times\vec{B} - \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{J} \longrightarrow \vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right) + \mu_0\varepsilon_0\frac{\partial}{\partial t}\left(\vec{\nabla}V + \frac{\partial}{\partial t}\vec{A}\right) = \mu_0\vec{J}$$

Gauss's Law for Magnetism

$$\vec{\nabla}\cdot\vec{B} = 0 \longrightarrow \vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}\right) = 0$$

Gauss's Law

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \longrightarrow \vec{\nabla}\cdot\left(\vec{\nabla}V + \frac{\partial\vec{A}}{\partial t}\right) = -\frac{\rho}{\varepsilon_0}$$

Corresponding Relations between Potentials

(on the road to general solutions for E and B)

We want to solve for V and A given

$$\vec{\nabla}\cdot\left(\vec{\nabla}V+\frac{\partial\vec{A}}{\partial t}\right)=-\frac{\rho}{\varepsilon_0}$$

and

$$\left(\nabla^2\vec{A}-\mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2}\right)-\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}+\mu_0\varepsilon_0\frac{\partial V}{\partial t}\right)=-\mu_0\vec{J}$$

rephrase

$$\left(\nabla^2 V-\mu_0\varepsilon_0\frac{\partial^2 V}{\partial t^2}\right)+\frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{A}+\mu_0\varepsilon_0\frac{\partial V}{\partial t}\right)=-\frac{\rho}{\varepsilon_0}$$

Second, mixed term vanishes if

$$\vec{\nabla}\cdot\vec{A}=-\mu_0\varepsilon_0\frac{\partial V}{\partial t}$$

Lorentz Gauge

$$\vec{\nabla}\cdot\vec{A}_L\equiv-\mu_0\varepsilon_0\frac{\partial V_L}{\partial t}$$

Sort of Simple

$$\left(\nabla^2 V_L-\mu_0\varepsilon_0\frac{\partial^2 V_L}{\partial t^2}\right)=-\frac{\rho}{\varepsilon_0}$$

Sort of Simple

$$\left(\nabla^2\vec{A}-\mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2}\right)=-\mu_0\vec{J}$$

To relate back to Coulomb's Gauge

$$\vec{A}_L=\vec{A}_C+\vec{\nabla}\lambda_L$$

$$\vec{\nabla}\cdot\left|\vec{A}_C+\vec{\nabla}\lambda_L\right|=0-\mu_0\varepsilon_0\frac{\partial V_L}{\partial t}$$

$$\vec{\nabla}^2\lambda_L=-\mu_0\varepsilon_0\frac{\partial V_L}{\partial t}$$

Quoting the form of sol'n to Poisson's Eq'n

$$\lambda_L=-\frac{\mu_0\varepsilon_0}{4\pi}\int\frac{\left|\frac{\partial V_L}{\partial t}\right|}{\mathscr{r}}d\tau'$$

Corresponding Relations between Potentials

(on the road to general solutions for E and B)

We want to solve for V and A given

*L*orentz Gauge

$$\vec{\nabla}\cdot\vec{A}_L \equiv -\mu_0\varepsilon_0 \frac{\partial V_L}{\partial t}$$

$$\left(\nabla^2 - \mu_0\varepsilon_0 \frac{\partial^2}{\partial t^2}\right)V_L = -\frac{\rho}{\varepsilon_0}$$

$$\left(\nabla^2 - \mu_0\varepsilon_0 \frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu_0\vec{J}$$

Minor Digression

$$\mu_0\varepsilon_0 = \left(4\pi\times10^{-7}\ {}^{N}\!/_{A^2}\right)\left(8.85\times10^{-12}\ {}^{C^2}\!/_{Nm^2}\right)$$

$$\mu_0\varepsilon_0 = \left(1.112\times10^{-17}\ {}^{s^2}\!/_{m^2}\right)$$

$$\mu_0\varepsilon_0 = \left(3.33\times10^{-9}\ {}^{s}\!/_{m}\right)^2$$

$$\mu_0\varepsilon_0 = \frac{1}{\left(2.9986\times10^{8}\ {}^{m}\!/_{s}\right)^2}$$

$$\mu_0\varepsilon_0 = \frac{1}{c^2}$$

$$\left(\nabla^2 - \tfrac{1}{c^2} \frac{\partial^2}{\partial t^2}\right)V_L = -\frac{\rho}{\varepsilon_0}$$

$$\left(\nabla^2 - \tfrac{1}{c^2} \frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu_0\vec{J}$$

D'Alembertian

$$\Box^2 \equiv \left(\nabla^2 - \tfrac{1}{c^2} \frac{\partial^2}{\partial t^2}\right)$$

$$\Box^2 V_L = -\frac{\rho}{\varepsilon_0}$$

$$\Box^2 \vec{A} = -\mu_0\vec{J}$$

Continuous Source Distribution

Solve

$$\left(\nabla^2 - \tfrac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)V_L = -\frac{\rho}{\varepsilon_0} \qquad \left(\nabla^2 - \tfrac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu_0 \vec{J}$$

As with solving any differential equation, "inspired guess" is a valid solution method

a) We already know for static charge or current distributions

$$\nabla^2 V_L = -\frac{\rho}{\varepsilon_0} \quad \text{and} \quad \nabla^2 \vec{A} = -\mu_0 \vec{J}$$

Are solved by

$$V(\vec{r}) = -\tfrac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,')}{\mathscr{r}}\,d\tau' \quad \text{and} \quad \vec{A}(\vec{r}) = -\tfrac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')}{\mathscr{r}}\,d\tau'$$

b) Without sources, we have the classic wave equation, so variations in V and A propagate

$$\left(\nabla^2 - \tfrac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)V_L = 0$$

$$\nabla^2 V_L = \tfrac{1}{c^2}\frac{\partial^2}{\partial t^2}V_L$$

$$V_L(\vec{r},t) \propto e^{i\vec{k}\cdot(\vec{r}-\vec{c}t)}$$

Apparently Maxwell's Laws *require* time separation, but don't dictate precede or follow.

Note: $V_L(\vec{r},t) \propto e^{i\vec{k}\cdot(\vec{r}+\vec{c}t)}$ would have worked too.

To be continued…

So a variation in V observed by an observer at time t was generated at a distance r away at previous time

$$t_r \equiv t - \frac{\mathscr{r}}{c}$$

$t_a \equiv t + \frac{\mathscr{r}}{c}$ would have worked too.

Combining what we know about these two special cases (constant or free space), we can guess

$$V(\vec{r},t) = -\tfrac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,',t_r)}{\mathscr{r}}\,d\tau' \quad \text{and} \quad \vec{A}(\vec{r},t) = -\tfrac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,',t_r)}{\mathscr{r}}\,d\tau'$$

Continuous Source Distribution

Solve

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)V_L = -\frac{\rho}{\varepsilon_0} \qquad \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu_0 \vec{J}$$

Our guess

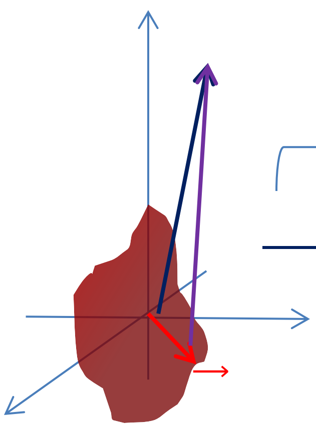
$$V(\vec{r},t) = -\frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,',t_r)}{\mathscr{r}}\,d\tau' \qquad \vec{A}(\vec{r},t) = -\frac{\mu_o}{4\pi}\int \frac{\vec{J}(\vec{r}\,',t_r)}{\mathscr{r}}\,d\tau'$$

where

–

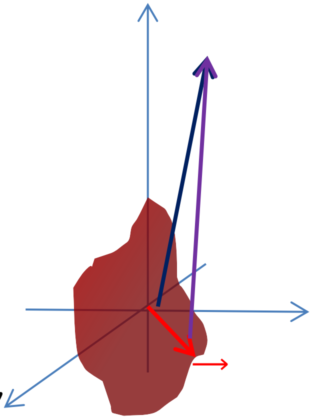
Plug in to test

$$\nabla^2 V =$$



Del asks how detected voltage changes as we change observation locations *not* source locations.

Continuous Source Distribution



Solve

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)V_L = -\frac{\rho}{\varepsilon_0} \qquad \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu_0 \vec{J}$$

Our guess

$$V(\vec{r},t) = -\frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,',t_r)}{\mathscr{r}}\,d\tau' \qquad \vec{A}(\vec{r},t) = -\frac{\mu_o}{4\pi}\int \frac{\vec{J}(\vec{r}\,',t_r)}{\mathscr{r}}\,d\tau'$$

where

Plug in to test

$\nabla^2 V =$

Product Rule 5 $\quad \vec{\nabla}\cdot(f\vec{A}) = (\vec{\nabla}f)\cdot\vec{A} + f(\vec{\nabla}\cdot\vec{A})$

$= -$

$=4\pi\delta^3(\;)$

Continuous Source Distribution

Solve

$$\left(\nabla^2 - \tfrac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)V_L = -\frac{\rho}{\varepsilon_0} \qquad \left(\nabla^2 - \tfrac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu_0\vec{J}$$

Our guess

$$V(\vec{r},t) = -\tfrac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{r}',t_r)}{\mathscr{r}}d\tau' \qquad \vec{A}(\vec{r},t) = -\tfrac{\mu_o}{4\pi}\int\frac{\vec{J}(\vec{r}',t_r)}{\mathscr{r}}d\tau'$$

where

$$t_r \equiv t - \frac{\mathscr{r}}{c}$$

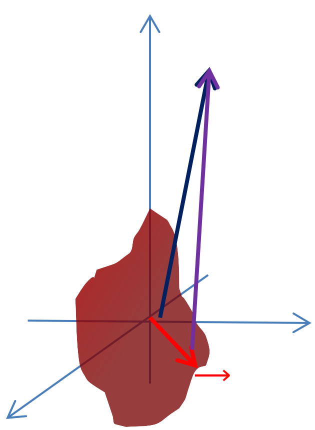
Plug in to test

$$\nabla^2 V = \vec{\nabla}\cdot\left(\vec{\nabla}V\right)$$

$$\nabla^2 V = \frac{1}{4\pi\varepsilon_o}\left(\int -\left(-\frac{\ddot{\rho}(\vec{r}',t_r)}{\mathscr{r}c^2} + \frac{\dot{\rho}(\vec{r}',t_r)}{c\mathscr{r}^2}\right) - \left(-\frac{\dot{\rho}(\vec{r}',t_r)}{\mathscr{r}^2 c} + \rho(\vec{r}',t_r)4\pi\delta^3(\mathscr{r})\right)d\tau'\right)$$

$$\nabla^2 V = \frac{1}{4\pi\varepsilon_o}\int\frac{\ddot{\rho}(\vec{r}',t_r)}{\mathscr{r}c^2}d\tau' - \frac{\rho(r,t)}{\varepsilon_o}$$

Of course $\ddot{\rho}(\vec{r}',t_r) = \dfrac{\partial^2\rho(\vec{r}',t_r)}{\partial t^2}$



$$\nabla^2 V = \tfrac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\frac{1}{4\pi\varepsilon_o}\int\frac{\rho(\vec{r}',t_r)}{\mathscr{r}c^2}d\tau'\right) - \frac{\rho(r,t)}{\varepsilon_o}$$

$$\nabla^2 V = \tfrac{1}{c^2}\frac{\partial^2}{\partial t^2}V - \frac{\rho(r,t)}{\varepsilon_o}$$

Continuous Source Distribution

$$\vec{A}\left(\vec{r},t\right) = -\tfrac{\mu_o}{4\pi}\int \frac{\vec{J}(\vec{r}\,',t_r)}{\mathscr{r}}\,d\tau' \quad \text{where} \quad t_r \equiv t - \frac{\mathscr{r}}{c}$$

Example: find the Vector potential for a wire carrying a linearly growing current.

Defined piecewise through time

$$I(t) = \begin{cases} 0 & for \quad t < 0 \\ kt & for \quad t > 0 \end{cases}$$

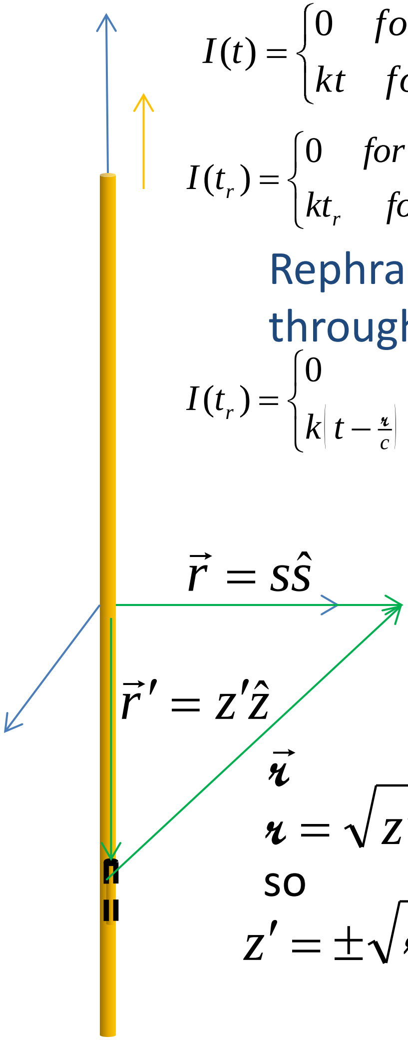
$$I(t_r) = \begin{cases} 0 & for \quad t_r < 0 \\ kt_r & for \quad t_r > 0 \end{cases}$$

$$\vec{A}\left(\vec{r},t\right) = -\tfrac{\mu_o}{4\pi}\int_{-\infty}^{\infty} \frac{I(\vec{r}\,',t_r)}{\mathscr{r}}\,d\vec{l}\,' = -\tfrac{\mu_o}{4\pi}\int_{-\infty}^{\infty} \frac{I(\vec{r}\,',t-\tfrac{\mathscr{r}}{c})}{\mathscr{r}}\,dz'\hat{z}$$

As time goes on, observer becomes aware of more and more of wire starting to carry current. At any time, some morsels are just too far away to contribute. Limits should reflect that.

Rephrase as piecewise through space

$$I(t_r) = \begin{cases} 0 & \text{for} \quad t - \tfrac{\mathscr{r}}{c} < 0 \\ k\left(t - \tfrac{\mathscr{r}}{c}\right) & \text{for} \quad t - \tfrac{\mathscr{r}}{c} > 0 \end{cases} \quad \text{or} \quad \begin{matrix} t < \tfrac{\mathscr{r}}{c} \\ t > \tfrac{\mathscr{r}}{c} \end{matrix} \quad \text{or} \quad \begin{matrix} ct < \mathscr{r} \\ ct > \mathscr{r} \end{matrix} \quad \text{or} \quad \begin{matrix} \left|z'\right| < \sqrt{\left(ct\right)^2 - s^2} \\ \left|z'\right| > \sqrt{\left(ct\right)^2 - s^2} \end{matrix}$$



$$\mathscr{r} = \sqrt{z'^2 + s^2}$$

so

$$z' = \pm\sqrt{\mathscr{r}^2 - s^2}$$

$$\vec{A}(\vec{r},t) = -\tfrac{\mu_o}{4\pi}\int_{z'-\sqrt{\left(ct\right)^2 - s^2}}^{z'=\sqrt{\left(ct\right)^2 - s^2}} \frac{k\left(t - \tfrac{\mathscr{r}}{c}\right)}{\mathscr{r}}\,dz'\hat{z}$$

$$= -\tfrac{\mu_o}{4\pi}k\left(t\int_{z'=-\sqrt{\left(ct\right)^2 - s^2}}^{z'=\sqrt{\left(ct\right)^2 - s^2}} \frac{dz'}{\sqrt{z'^2 + s^2}} - \frac{1}{c}\int_{z'=-\sqrt{\left(ct\right)^2 - s^2}}^{z'=\sqrt{\left(ct\right)^2 - s^2}} dz' \right)$$

For first integral $\displaystyle\int_{z'_{\min}}^{z'_{\max}} \frac{dz'}{\sqrt{s^2 + z'^2}} = \ln\left(\sqrt{s^2 + z'^2} + z'\right)\Big|_{z'_{\min}}^{z'_{\max}}$

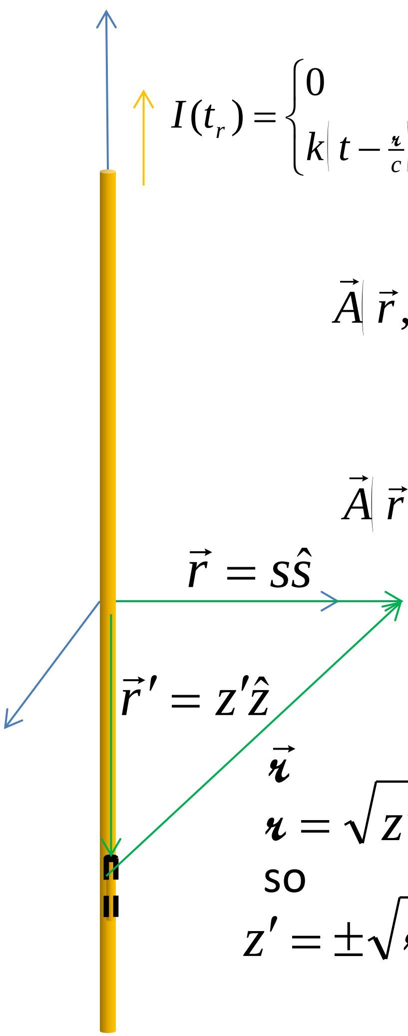
Continuous Source Distribution

$$\vec{A}(\vec{r},t) = -\frac{\mu_o}{4\pi}\int \frac{\vec{J}(\vec{r}\,',t_r)}{\mathscr{r}}\,d\tau' \quad \text{where} \quad t_r \equiv t - \frac{\mathscr{r}}{c}$$

Example: find the Vector potential for a wire carrying a linearly growing current.

Defined piecewise through time

$$\vec{A}(\vec{r},t) = -\frac{\mu_o}{4\pi}\int_{-\infty}^{\infty} \frac{I(\vec{r}\,',t_r)}{\mathscr{r}}\,d\vec{l}\,'$$



$$I(t_r) = \begin{cases} 0 & \text{for } \; t - \frac{\mathscr{r}}{c} < 0 \\ k\left(t - \frac{\mathscr{r}}{c}\right) & \text{for } \; t - \frac{\mathscr{r}}{c} > 0 \end{cases} \quad \text{or} \quad \begin{matrix} z' < \sqrt{(ct)^2 - s^2} \\ z' > -\sqrt{(ct)^2 - s^2} \end{matrix}$$

$$\mathscr{r} = \sqrt{z'^2 + s^2}$$

so

$$z' = \pm\sqrt{\mathscr{r}^2 - s^2}$$

$$\vec{A}(\vec{r},t) = -\frac{\mu_o}{4\pi}k\left(t\ln\left(\frac{\sqrt{(ct)^2 - s^2 + s^2} + \sqrt{(ct)^2 - s^2}}{\sqrt{(ct)^2 - s^2 + s^2} - \sqrt{(ct)^2 - s^2}}\right) - \frac{1}{c}\left(\sqrt{(ct)^2 - s^2} - -\sqrt{(ct)^2 - s^2}\right)\right)\hat{z}$$

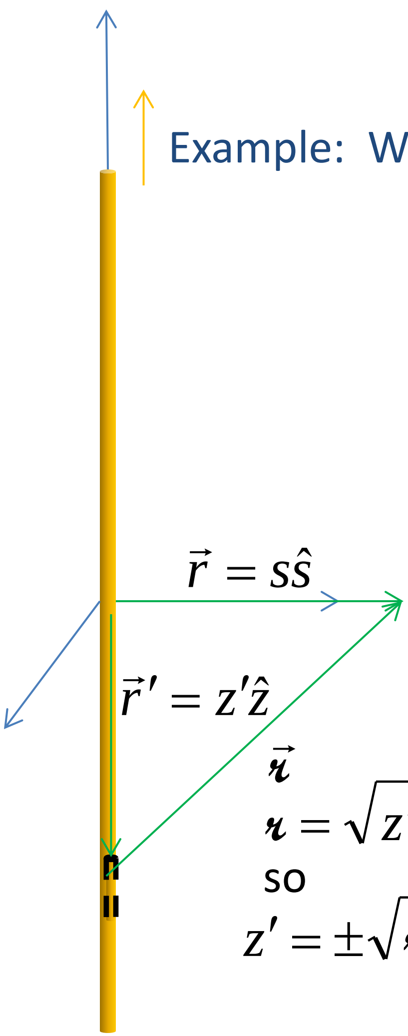
$$\vec{A}(\vec{r},t) = -\frac{\mu_o}{4\pi}k\left(t\ln\left(\frac{ct + \sqrt{(ct)^2 - s^2}}{ct - \sqrt{(ct)^2 - s^2}}\right) - \frac{2\sqrt{(ct)^2 - s^2}}{c}\right)\hat{z}$$

$$\vec{A}(\vec{r},t) = \begin{cases} -\frac{\mu_o}{4\pi}kt\left(\ln\left(\frac{1+\sqrt{1-\left(\frac{s}{ct}\right)^2}}{1-\sqrt{1-\left(\frac{s}{ct}\right)^2}}\right) - 2\sqrt{1-\left(\frac{s}{ct}\right)^2}\right)\hat{z} & \text{for } \; s < ct \\ 0 & \text{for } \; s > ct \end{cases}$$

Continuous Source Distribution

$$\vec{A}(\vec{r},t) = -\frac{\mu_o}{4\pi}\int \frac{\vec{J}(\vec{r}\,',t_r)}{\mathscr{r}}\,d\tau' \quad \text{where} \quad t_r \equiv t - \frac{\mathscr{r}}{c}$$

$$\vec{A}(\vec{r},t) = \begin{cases} -\frac{\mu_o}{4\pi}kt\left(\ln\left(\frac{1+\sqrt{1-\left(\frac{s}{ct}\right)^2}}{1-\sqrt{1-\left(\frac{s}{ct}\right)^2}}\right) - 2\sqrt{1-\left(\frac{s}{ct}\right)^2}\right)\hat{z} & \text{for } s < ct \\ 0 & \text{for } s > ct \end{cases}$$



Example: What are B and E?

$$\vec{\nabla}\times\vec{A} \equiv \vec{B}$$

$$\vec{B}(\vec{r},t) = \begin{cases} -\frac{\partial}{\partial s}\left(-\frac{\mu_o}{4\pi}kt\left(\ln\left(\frac{1+\sqrt{1-\left(\frac{s}{ct}\right)^2}}{1-\sqrt{1-\left(\frac{s}{ct}\right)^2}}\right) - 2\sqrt{1-\left(\frac{s}{ct}\right)^2}\right)\right)\hat{\phi} & \text{for } s < ct \\ 0 & \text{for } s > ct \end{cases}$$

A bit of math later:

$$\vec{B}(\vec{r},t) = \begin{cases} -\frac{\mu_o}{4\pi c}2k\sqrt{\left(\frac{ct}{s}\right)^2 - 1}\,\hat{\phi} & \text{for } s < ct \\ 0 & \text{for } s > ct \end{cases}$$

$$\mathscr{r} = \sqrt{z'^2 + s^2}$$

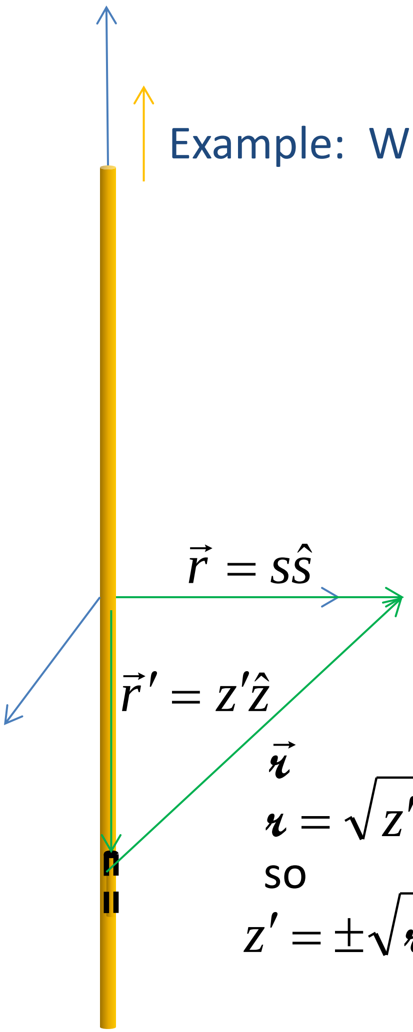
so

$$z' = \pm\sqrt{\mathscr{r}^2 - s^2}$$

Continuous Source Distribution

$$\vec{A}(\vec{r},t) = -\frac{\mu_o}{4\pi}\int \frac{\vec{J}(\vec{r}\,',t_r)}{\mathscr{r}}\,d\tau' \quad \text{where} \quad t_r \equiv t - \frac{\mathscr{r}}{c}$$

$$\vec{A}(\vec{r},t) = \begin{cases} -\frac{\mu_o}{4\pi}kt\left(\ln\left(\dfrac{1+\sqrt{1-\left(\frac{s}{ct}\right)^2}}{1-\sqrt{1-\left(\frac{s}{ct}\right)^2}}\right) - 2\sqrt{1-\left(\frac{s}{ct}\right)^2}\right)\hat{z} & \text{for } s < ct \\ 0 & \text{for } s > ct \end{cases}$$



Example: What are B and E?

$$\mathscr{r} = \sqrt{z'^2 + s^2}$$

so

$$z' = \pm\sqrt{\mathscr{r}^2 - s^2}$$

$$\vec{B}(s,t) = \begin{cases} -\frac{\mu_o}{4\pi c}2k\sqrt{\left(\frac{ct}{s}\right)^2 - 1}\,\hat{\phi} & \text{for } s < ct \\ 0 & \text{for } s > ct \end{cases}$$

$$\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}$$

All but one factor of t is bound up in (s/ct), so same thing, times –(s/t), in z direction, and a term for the one lone t

$$\vec{E}(\vec{r},t) = \left(\frac{\mu_o}{4\pi}k\left(\ln\left(\dfrac{1+\sqrt{1-\left(\frac{s}{ct}\right)^2}}{1-\sqrt{1-\left(\frac{s}{ct}\right)^2}}\right) + 2\sqrt{1-\left(\frac{s}{ct}\right)^2}\right) + \frac{\mu_o}{4\pi}2k\sqrt{1-\left(\frac{s}{ct}\right)^2}\right)\hat{z}$$

$$\vec{E}(\vec{r},t) = \begin{cases} \frac{\mu_o}{4\pi}k\ln\left(\dfrac{1+\sqrt{1-\left(\frac{s}{ct}\right)^2}}{1-\sqrt{1-\left(\frac{s}{ct}\right)^2}}\right)\hat{z} & \text{for } s < ct \\ 0 & \text{for } s > ct \end{cases}$$

Continuous Source Distribution

$$\vec{A}(\vec{r},t) = -\frac{\mu_o}{4\pi}\int \frac{\vec{J}(\vec{r}\,',t_r)}{\mathscr{r}}\,d\tau' \quad \text{where} \quad t_r \equiv t - \frac{\mathscr{r}}{c}$$

Exercise: find the Vector potential for a wire that momentarily had a burst of current.

Defined piecewise through time

$$I(t) = q_o\delta(t - t_b)$$

$$\vec{A}(\vec{r},t) = -\frac{\mu_o}{4\pi}\int_{-\infty}^{\infty} \frac{I(\vec{r}\,',t_r)}{\mathscr{r}}\,d\vec{l}\,'$$

So, at some time, t_b, the current will blink on and off again. The observer will first notice the middle blink, then just either side of the middle, then a little further out,…

$$\vec{A}(\vec{r},t) = -\frac{\mu_o}{4\pi}\int_{-\infty}^{\infty} \frac{q_o\delta(t_r - t_b)}{\mathscr{r}}\,dz'\hat{z}$$

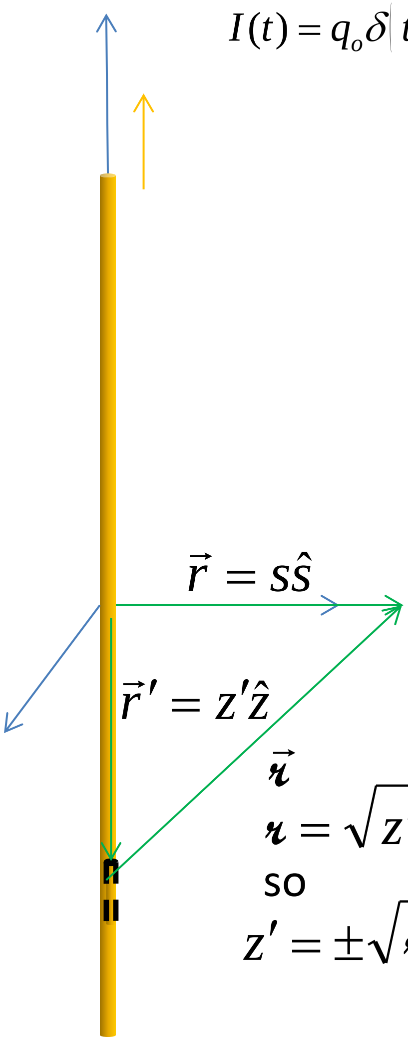
So, we get contribution to our integral only when

$$t_b = t_r = t - \frac{\mathscr{r}}{c}$$

$$\mathscr{r} = c(t - t_b)$$

Which is true at *two* locations at any moment t:

$$z' = \pm\sqrt{(c(t - t_b))^2 - s^2}$$



$$\mathscr{r} = \sqrt{z'^2 + s^2}$$

so

$$z' = \pm\sqrt{\mathscr{r}^2 - s^2}$$

We could rephrase the delta function as being a spike at these two locations, or we could observe the integral is ‘even’ and then wave our hands

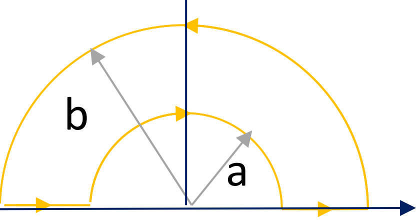
$$\vec{A}(\vec{r},t) = -\frac{\mu_o}{4\pi}2\int_{0}^{\infty} \frac{q_o\delta(t_r - t_b)}{\mathscr{r}}\,dz'\hat{z} = -\frac{\mu_o}{2\pi}\frac{q_o}{c(t - t_b)}\hat{z}$$

Continuous Source Distribution

$$\vec{A}(\vec{r},t) = -\frac{\mu_o}{4\pi}\int \frac{\vec{J}(\vec{r}',t_r)}{\mathscr{r}}\,d\tau' \quad \text{where} \quad t_r \equiv t - \frac{\mathscr{r}}{c}$$

HW Exercise: A neutral current loop made of two concentric arcs. The current rises with time as I(t) = kt (presumably just since t=0, but we'll assume we're long enough out.) What are A, and E at the origin?

$$\vec{A}_L(\vec{r},t) = \frac{\mu_o}{4\pi}\int \frac{I(\vec{r}',t_r)}{\mathscr{r}}\,dl'\hat{l}$$

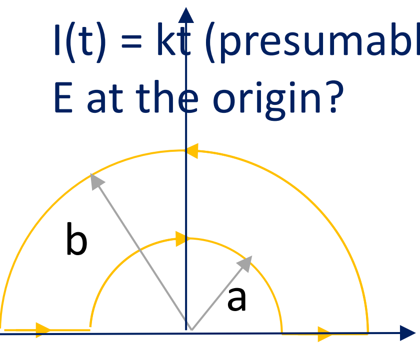


[redacted]

Continuous Source Distribution

$$\vec{A}(\vec{r},t) = -\frac{\mu_o}{4\pi}\int \frac{\vec{J}(\vec{r}\,',t_r)}{\mathscr{r}}\,d\tau' \quad \text{where} \quad t_r \equiv t - \frac{\mathscr{r}}{c}$$

Exercise: A neutral current loop made of two concentric arcs. The current rises with time as I(t) = kt (presumably just since t=0, but we'll assume we're long enough out.) What are A, and E at the origin?



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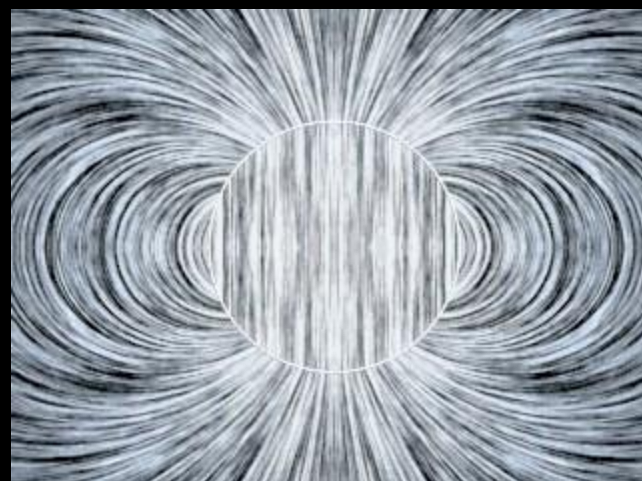
Charged sphere spinning up from rest



http://web.mit.edu/viz/spin/ choose slow spin up – time evolving magnetic field for a sphere of charge spinning up