

Wed., 11/13 Fri., 11/15	10.1 - .2.1 Potential Formulation 10.2 Continuous Distributions	
Mon., 11/18 Wed., 11/20 Fri., 11/22	10.3 Point Charges (C 14) 4.1 Polarization (C 14) 4.2 Field of Polarized Object	HW9

Transition. We ended last time with a complete set of Maxwell's Equations.

Maxwell's Equations (Complete!) – in both differential and integral forms

$$\text{Gauss's law} \quad \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad \oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$$

$$\text{Gauss for B.} \quad \vec{\nabla} \cdot \vec{B} = 0 \quad \oint \vec{B} \cdot d\vec{a} = 0$$

$$\text{Faraday's law} \quad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \quad \oint \vec{E} \cdot d\vec{\ell} + \frac{\partial \Phi_B}{\partial t} = 0$$

$$\text{Ampere-Maxwell's law} \quad \vec{\nabla} \times \vec{B} - \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J} \quad \oint \vec{B} \cdot d\vec{\ell} - \mu_0 \epsilon_0 \frac{\partial \Phi_E}{\partial t} = \mu_0 I_{enc}$$

The four laws concisely say how fields are related to each other and to sources. Perhaps a better description would be that these are *conditions* that the fields must satisfy. Helmholtz theorem asserts that, if you know a static vector field's divergence and its curl (and that it dies off at infinity) then, you know all there is to know about the field. Of course, if the field *isn't* static, then you also have to know how to get from one time to another – the time derivative. We've got all that information here. That said, it takes a bit of work to actually extract general expressions for the fields. The work is worth it since, while these four conditions are *concise* they are rather circumspect on the question of 'what *produces*' these fields, and they're quite often misread in that regard. In point of fact, *none* of these four laws (individually) addresses that question since they are all instantaneous relations and you can't read cause/effect out of that; in a sense, these are *kinematic* rather than *dynamic* relations. The general solutions that we'll develop, known variously as Jefimenko's Equations or as the Generalized Coulomb's Law and Biot-Savart law, do quite clearly relate fields to sources. Naturally, these relations are conceptually significant, but they are also of quantitative use – if you know how the sources are evolving through time, then you can say how the fields are evolving.

Surprisingly, the first step to building general expressions for E and B looks like a step in the wrong direction – rephrasing these four conditions on E & B in terms of conditions on A and V, the potentials.

Potential Formulation

Recall Vector Potential

Recall that, because $\vec{\nabla} \cdot \vec{B} = 0$ and Vector identity 9 says that the divergence of a curl is 0, we have free license to define a "vector potential" such that

$$\vec{\nabla} \times \vec{A} \equiv \vec{B}$$

Since we then have

$$\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A} \right) = 0$$

Gauss's Law for magnetism

Which is merely a mathematical truism (vector identity 9)

Where, what I mean by using the triple bar here is that the curl of A has been strictly defined, but other aspects have not, so there's some ambiguity to which we'll return.

Scalar Potential

Now, let's turn our eye to the scalar potential. If you recall, we'd defined it back when we *thought* that the curl of E was always 0. Running on that hunch, and vector identity 10, we said that that gave us license to define a scalar potential such that

$$\vec{\nabla} V \equiv -\vec{E} \text{ (no longer good)}$$

But wait a minute! We've got Faraday's Law now, and it tells us that if there's a time varying magnetic field, then the curl of E *isn't* 0. Conveniently, rewriting Faraday's Law in terms of the vector potential (rather than B), does give us something whose curl *is* 0.

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial}{\partial t} \left(\vec{\nabla} \times \vec{A} \right) = 0$$

$$\vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0$$

So, Vector Identity 10 tells us that we're free to define a scalar whose gradient equals *this* term in brackets. So, let's *redefine* V thus

$$\vec{\nabla} V \equiv - \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right)$$

That way, we get

$$\vec{\nabla} \times \left(\vec{\nabla} V \right) = 0$$

Faraday's Law

Which is merely a mathematical Truism (Vector identity 10)

Note that, if there's a constant A, then this reduces to our old definition, so everything we learned about this in *electrostatics* still holds but relations will have to change in *electrodynamics*. For example, you can't just slap a q on this thing and call it 'work' done on the charge.

Rearranging this relation,

$$\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}$$

Now that we know how B and E can be represented in terms of vector and scalar potentials, we'll rewrite Maxwell's relations in terms of the potentials.

Gauss's law

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\vec{\nabla} \cdot \left(-\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t} \right) = \frac{\rho}{\epsilon_0}$$

$$\nabla^2 V + \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{A}) = -\frac{\rho}{\epsilon_0}$$

Gauss for B

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$$

Faraday's law

$$\vec{\nabla} \times (\vec{\nabla}V) = 0$$

Ampere-Maxwell's law

$$\vec{\nabla} \times \vec{B} - \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) - \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(-\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t} \right) = \mu_0 \vec{J} \quad \oint \vec{B} \cdot d\vec{\ell} - \mu_0 \epsilon_0 \frac{\partial \Phi_E}{\partial t} = \mu_0 I_{enc}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) - \mu_0 \epsilon_0 \left(\vec{\nabla} \frac{\partial V}{\partial t} + \frac{\partial^2 \vec{A}}{\partial t^2} \right) = \mu_0 \vec{J}$$

What do we make of the curl-curl term? Vector Identity 11 tells us that

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$$

So

$$\vec{\nabla} (\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} + \mu_0 \epsilon_0 \left(\vec{\nabla} \frac{\partial V}{\partial t} + \frac{\partial^2 \vec{A}}{\partial t^2} \right) = \mu_0 \vec{J}$$

$$\left(\nabla^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} \right) - \vec{\nabla} \left(\vec{\nabla} \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t} \right) = -\mu_0 \vec{J}$$

By rewriting Maxwell's laws in terms of potentials, we've switched from having *four* substantive equations (even Gauss's Law for B told us something important about B) to having just *two* (the other equations are mere mathematical truisms.) Actually, there are two more substantive equations – those that relate A and V back to E and B.

Unfortunately, the two we've ended up with are a bit messier than the four we started with; however, we've got the freedom to clean these up a little.

Gauge Transformations

That Freedom comes from the “ambiguity” I mentioned with A. In particular, Vector Identity 10 tells us that

$$\vec{\nabla} \times (\vec{\nabla} \lambda) = 0$$

So, if you’ve got an \vec{A} that satisfies $\vec{\nabla} \times \vec{A} \equiv \vec{B}$, then

$$\vec{A}' = \vec{A} + \vec{\nabla} \lambda$$

Also satisfies it. This ambiguity allows for some freedom in defining the vector potential.

But $\vec{\nabla} \times \vec{A} \equiv \vec{B}$ isn’t the only thing that links A to something measurable, $\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$ does too, and if we go adding a gradient of a scalar to A, that’s definitely going to mess up this relation, *unless* we compensatingly add something to V. Let’s see what.

Say we want to be able to describe the same E in terms of either V and A or a new V’ and a new A’ where $\vec{A}' = \vec{A} + \vec{\nabla} \lambda$, then what does V’ equal?

$$\begin{aligned} -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} &= \vec{E} = -\vec{\nabla} V' - \frac{\partial \vec{A}'}{\partial t} \\ -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} &= -\vec{\nabla} V' - \frac{\partial (\vec{A} + \vec{\nabla} \lambda)}{\partial t} \\ -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} &= -\vec{\nabla} V' - \frac{\partial \vec{A}}{\partial t} - \frac{\partial (\vec{\nabla} \lambda)}{\partial t} \\ -\vec{\nabla} V &= -\vec{\nabla} V' - \frac{\partial (\vec{\nabla} \lambda)}{\partial t} \\ \vec{\nabla} V' &= \vec{\nabla} V - \frac{\partial (\vec{\nabla} \lambda)}{\partial t} \\ \vec{\nabla} V' &= \vec{\nabla} \left(V - \frac{\partial \lambda}{\partial t} \right) \\ V' &= V - \frac{\partial \lambda}{\partial t} \end{aligned}$$

(Note: the last step is saying that if the gradients of the arguments are the equal, then so must be the arguments – that’s not completely accurate, they could differ by a constant, but we could imagine wrapping that into our definition of lambda.)

So, imposing on our potentials that

$$\begin{aligned} \vec{\nabla} \times \vec{A} &\equiv \vec{B} \\ \vec{\nabla} V &\equiv - \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) \end{aligned}$$

Leaves a little bit of wiggle room when it comes to defining the potentials themselves

$$\vec{A}' = \vec{A} + \vec{\nabla} \lambda$$

$$V' = V - \frac{\partial \lambda}{\partial t}$$

We say that choosing different scalar lambdas is choosing different “gauge’s.” When it comes to calculating anything measurable, like forces (using fields), these lambdas mathematically drop out by the end, but a clever choice of lambda may make the work of *getting* to the end a bit easier. This freedom of choice is akin to the freedom to choose convenient reference frames or even coordinate systems for solving problems – you’re free to place the origin and orient the axes any way that is convenient; when it comes to the measurable predictions, such choices don’t matter.

Gauges and Div A

Another way to phrase this Gauge freedom is that you’ve constrained the *curl* of A how it’s time derivative is related to V, but left its *divergence* completely unconstrained. So, you can always choose to force it to have *any* divergence you want with the knowledge that different choices are going to be made up for by different V’s. For example, we’d previously chosen the **Coulomb Gauge**, in which

$$\vec{\nabla} \cdot \vec{A}_{Coul} = 0$$

For a given scenario, there will be a corresponding scalar potential, V_{Coul} .

Now what if we want to make some other choice, A_{other} such that

$$\vec{\nabla} \cdot \vec{A}' \neq 0$$

These two are related by

$$\begin{aligned}\vec{A}_O &= \vec{A}_{Coul} + \vec{\nabla} \lambda_{C \rightarrow O} \\ \vec{\nabla} \cdot \vec{A}_O &= \vec{\nabla} \cdot \vec{A}_{Coul} + \nabla^2 \lambda_{C \rightarrow O} \\ \vec{\nabla} \cdot \vec{A}_O &= 0 + \nabla^2 \lambda_{C \rightarrow O}\end{aligned}$$

Now, as far as lambda is concerned, this relation has the form of Poisson’s equation, and we know that it is solved with

$$\lambda_{C \rightarrow O}(r, t) = \frac{1}{4\pi} \int \frac{\vec{\nabla} \cdot \vec{A}_O(r', t)}{r} d\tau$$

Thus, the new and old *scalar* potential would be related by

$$\begin{aligned}V_{other} &= V_{Coul} - \frac{\partial \lambda_{C \rightarrow O}}{\partial t} \\ V_{other} &= V_{Coul} - \frac{\partial}{\partial t} \frac{1}{4\pi} \int \frac{\vec{\nabla} \cdot \vec{A}_{other}(r', t)}{r} d\tau \\ V_{other} &= V_{Coul} - \frac{1}{4\pi} \int \frac{\vec{\nabla} \cdot \frac{\partial}{\partial t} \vec{A}_{other}(r', t)}{r} d\tau\end{aligned}$$

This is all to demonstrate that you're free to choose any divergence of \vec{A} that you want, but, for better or for worse, it affects your V as a result (in the interest of maintaining the same correct E .)

Lorentz Gauge

Back to the task at hand, we want to solve the potential versions of Ampere-Maxwell Law

$$\left(\nabla^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} \right) - \vec{\nabla} \left(\vec{\nabla} \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t} \right) = -\mu_0 \vec{J}$$

And Gauss's Law

$$\nabla^2 V + \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{A}) = -\frac{\rho}{\epsilon_0}$$

Now, it may look like an unwanted complication, but we could rewrite the latter as

$$\left(\nabla^2 V - \mu_0 \epsilon_0 \frac{\partial^2 V}{\partial t^2} \right) + \frac{\partial}{\partial t} \left(\vec{\nabla} \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t} \right) = -\frac{\rho}{\epsilon_0}$$

(you'll demonstrate that in the homework)

Writing the two equations *this* way just screams out for a particular gauge, i.e., choice of $\vec{\nabla} \cdot \vec{A}$. Let's choose

$$\vec{\nabla} \cdot \vec{A}_L = -\mu_0 \epsilon_0 \frac{\partial V_L}{\partial t}$$

Which is known as the Lorentz (though better "Lorenz" since Lorenz, not Lorentz proposed it) Gauge. That kills the second term in both Gauss's and Ampere-Maxwell's laws.

$$\left(\nabla^2 \vec{A}_L - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}_L}{\partial t^2} \right) = -\mu_0 \vec{J}$$

$$\left(\nabla^2 V_L - \mu_0 \epsilon_0 \frac{\partial^2 V_L}{\partial t^2} \right) = -\frac{\rho}{\epsilon_0}$$

Perhaps now is as good a time as any to point out that $\mu_0 \epsilon_0 = \frac{1}{c^2}$ (yes, *that* c).

Now, recall that

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

What we're now doing to \vec{A} and V is operating with

$$\square = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}$$

Mathematically speaking, this is the Laplacian for a 4-D space with coordinates $\mathcal{R} = (x, y, z, ict)$. This operator is called the d'Alembertian

$$\square \bar{A}_L = -\mu_0 \bar{J} \quad \square \bar{V}_L = -\frac{\rho}{\epsilon_0}$$

10.2 Continuous Distribution.

These are solved by

$$V_L(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}', t_r)}{r} d\tau' \quad \bar{A}_L(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\bar{J}(\vec{r}', t_r)}{r} d\tau'$$

$$\text{Where } t_r \equiv t - \frac{r}{c}$$

Griffiths doesn't *derive* these results, rather he argues for their plausibility and then proves that they work. Of course, this is the form of the familiar solution in 3-D space without any time dependence. The need for evaluating at the retarded time thus defined looks plausible since there's a factor of c between the position and time coordinates in the 4-D space, and it sounds plausible since this effectively enforces that changes in the potentials propagate at the speed of light. Just as the light hitting my eye when I look up at the sky tonight is that which was emitted by the stars at their individual retardation times.

Then again, if we gave full marks for plausibility, then similar forms of Coulomb's law and the Biot-Savart Law would recommend themselves to us too. The difference is that we have simple differential equations to test these potential solutions against. So let's do that. Let's plug the scalar-potential expression into our equation

$$\square \bar{V}_L = -\frac{\rho}{\epsilon_0}$$

$$\left(\nabla^2 \bar{V}_L - \mu_0 \epsilon_0 \frac{\partial^2 \bar{V}_L}{\partial t^2} \right) = -\frac{\rho}{\epsilon_0}$$

Let's go after that first term, the Laplacian

$$\vec{\nabla} \cdot \left(\nabla \frac{1}{4\pi\epsilon_0} \left(\int \frac{\rho(\vec{r}', t_r)}{r} d\tau' \right) \right)$$

Where I've broken out the two del's in the del squared operator. Before I actually operate with it, I should remind you that the del's are with respect to r , not r' ; that is, I'm asking how the potential varies from one observation location to another, not how the density varies from one source location to another. The two places that r appears in the integral are the two places that r does – in the denominator *and* in t_r .

Taking one at a time,

$$\nabla \frac{1}{4\pi\epsilon_0} \left(\int \frac{\rho(\vec{r}', t_r)}{r} d\tau' \right) = \frac{1}{4\pi\epsilon_0} \left(\int \frac{\nabla \rho(\vec{r}', t_r)}{r} + \rho(\vec{r}', t_r) \nabla \left(\frac{1}{r} \right) d\tau' \right)$$

$$\frac{1}{4\pi\epsilon_0} \left(\int \frac{\left(\frac{\partial \rho(\vec{r}', t_r)}{\partial t_r} \frac{\partial t_r}{\partial r} \nabla r \right)}{r} + \rho(\vec{r}', t_r) \nabla \left(\frac{1}{r} \right) d\tau' \right)$$

Let's take a look at each of these derivatives in turn.

Employing the Chain Rule,

$$\nabla \rho(\vec{r}', t_r) = \frac{\partial \rho(\vec{r}', t_r)}{\partial t_r} \frac{\partial t_r}{\partial r} \nabla r$$

Now, since $t_r \equiv t - \frac{r}{c}$, taking the derivative with respect to it is the same as taking the derivative with respect to t . So, we can just call

$$\frac{\partial \rho(\vec{r}', t_r)}{\partial t_r} = \frac{\partial \rho(\vec{r}', t_r)}{\partial t} = \dot{\rho}(\vec{r}', t_r)$$

Again, appealing to $t_r \equiv t - \frac{r}{c}$

$$\frac{\partial t_r}{\partial r} = -\frac{1}{c}$$

Next, you may recall from Ch 1, problem 1.13 (I believe we did a couple of these in class)

$$\nabla r = \hat{r} \text{ and for that matter } \nabla \left(\frac{1}{r} \right) = -\frac{\hat{r}}{r^2}$$

Putting them all together,

$$\nabla \rho(\vec{r}', t_r) = -\frac{\dot{\rho}(\vec{r}', t_r)}{c} \hat{r}$$

So, putting it all back together, we have

$$\nabla \frac{1}{4\pi\epsilon_0} \left(\int \frac{\rho(\vec{r}', t_r)}{r} d\tau' \right) = \frac{1}{4\pi\epsilon_0} \left(\int -\frac{\dot{\rho}(\vec{r}', t_r) \hat{r}}{cr} - \frac{\rho(\vec{r}', t_r) \hat{r}}{r^2} d\tau' \right)$$

Now for the second del,

$$\begin{aligned} \nabla^2 \frac{1}{4\pi\epsilon_o} \left(\int \frac{\rho(\vec{r}', t_r)}{u} d\tau' \right) &= \nabla \cdot \frac{1}{4\pi\epsilon_o} \left(\int -\frac{\dot{\rho}(\vec{r}', t_r)\hat{u}}{cu} - \frac{\rho(\vec{r}', t_r)\hat{u}}{u^2} d\tau' \right) \\ &\frac{1}{4\pi\epsilon_o} \left(\int -\nabla \cdot \left(\frac{\dot{\rho}(\vec{r}', t_r)\hat{u}}{c} \right) - \nabla \cdot \left(\rho(\vec{r}', t_r) \frac{\hat{u}}{u^2} \right) d\tau' \right) \end{aligned}$$

By vector identity 5, the product rule for divergence of a scalar times a vector is as simple as you'd hope, so

$$\frac{1}{4\pi\epsilon_o} \left(\int -\left(\frac{\hat{u}}{u} \nabla \dot{\rho}(\vec{r}', t_r) + \frac{\dot{\rho}(\vec{r}', t_r)}{c} \nabla \cdot \left(\frac{\hat{u}}{u} \right) \right) - \left(\frac{\hat{u}}{u^2} \nabla \rho(\vec{r}', t_r) + \rho(\vec{r}', t_r) \nabla \cdot \left(\frac{\hat{u}}{u^2} \right) \right) d\tau' \right)$$

$$\text{Now, just as } \nabla \cdot \left(\frac{\hat{u}}{u} \right) = -\frac{\dot{\rho}(\vec{r}', t_r)}{c} \hat{u}, \quad \nabla \cdot \left(\frac{\hat{u}}{u^2} \right) = -\frac{\dot{\rho}(\vec{r}', t_r)}{c} \hat{u}$$

$$\nabla \cdot \left(\frac{\hat{u}}{u} \right) = \frac{\mathbf{1}}{u^2} \quad \text{and} \quad \nabla \cdot \left(\frac{\hat{u}}{u^2} \right) = 4\pi\delta^3(\vec{r})$$

So, the integral simplifies to

$$\begin{aligned} &\frac{1}{4\pi\epsilon_o} \left(\int -\left(-\frac{\ddot{\rho}(\vec{r}', t_r)}{uc^2} + \frac{\dot{\rho}(\vec{r}', t_r)}{cu^2} \right) - \left(-\frac{\dot{\rho}(\vec{r}', t_r)}{u^2 c} + \rho(\vec{r}', t_r) 4\pi\delta^3(\vec{r}) \right) d\tau' \right) \\ &\frac{1}{4\pi\epsilon_o} \int \frac{\ddot{\rho}(\vec{r}', t_r)}{uc^2} d\tau' - \frac{\rho(r, t)}{\epsilon_o} \end{aligned}$$

(the delta function killed the integral everywhere but $r'=r$, of course there $t_r=t$.)

So,

$$\nabla^2 V_L = \frac{1}{4\pi\epsilon_o} \int \frac{\ddot{\rho}(\vec{r}', t_r)}{uc^2} d\tau' - \frac{\rho(r, t)}{\epsilon_o}$$

Invoking, the first term can be rewritten as

$$\begin{aligned} \nabla^2 V_L &= \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \frac{1}{4\pi\epsilon_o} \int \frac{\rho(\vec{r}', t_r)}{u} d\tau' - \frac{\rho(r, t)}{\epsilon_o} \\ \nabla^2 V_L &= \frac{1}{c^2} \frac{\partial^2}{\partial t^2} V_L - \frac{\rho(r, t)}{\epsilon_o} \end{aligned}$$

Tada.

Similarly, the same thing can be done for each component of a \vec{A} , proving that both

$$V_L(\vec{r}, t) = \frac{1}{4\pi\epsilon_o} \int \frac{\rho(\vec{r}', t_r)}{u} d\tau' \quad \vec{A}_L(\vec{r}, t) = \frac{\mu_o}{4\pi} \int \frac{\vec{J}(\vec{r}', t_r)}{u} d\tau'$$

Are correct.

Examples

Now that we've got them, let's use them.

Pr. 10.9 a) Suppose an infinite straight wire carries a linearly increasing current $I(t) = kt$ for $t > 0$. Find the electric and magnetic fields generated.

It's worth emphasizing that this current is piecewise defined

$$I(t) = \begin{cases} 0 & \text{for } t < 0 \\ kt & \text{for } t > 0 \end{cases}$$

While that may just look like a *time* issue, we'll see that it's also a *position* issue.

$$\text{Now,} \quad \vec{\nabla} \times \vec{A} \equiv \vec{B} \quad \text{and} \quad \vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}$$

Where

$$V_L(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}', t_r)}{r} d\tau' = 0, \quad \vec{A}_L(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}', t_r)}{r} d\tau' = \frac{\mu_0}{4\pi} \int \frac{I(\vec{r}', t_r)}{r} dz' \hat{z}$$

We can say

$$\vec{A}_L(\vec{r}, t) = \frac{\mu_0}{4\pi} \int_{-\infty}^{\infty} \frac{I(\vec{r}', t_r)}{r} dz' \hat{z}$$

But that's a little deceptive since, our observer won't see segments of the current turned on yet. We can see that a little better if we note

$$I(t_r) = \begin{cases} 0 & \text{for } t_r < 0 \\ kt_r & \text{for } t_r > 0 \end{cases}$$

Where $t_r = t - \frac{r}{c}$ rephrasing this back in terms of plain old t ,

$$kt_r = k \left(t - \frac{r}{c} \right)$$

and the condition for turning this on is

$$t - \frac{r}{c} > 0$$

$$t > \frac{r}{c}$$

$$tc > r$$

Of course, $r = \sqrt{s^2 + z'^2}$ so, putting this threshold in terms of the integration variable,

$$\begin{aligned}
tc &> \sqrt{s^2 + z'^2} \\
\left(\frac{c}{s}\right)^2 - s^2 &> z'^2 \\
z'^2 &> \left(\frac{c}{s}\right)^2 - s^2 \\
\sqrt{\left(\frac{c}{s}\right)^2 - s^2} &> |z'|
\end{aligned}$$

Now, we can rephrase our piece-wise defined current in terms of our variable of integration

$$I(t, z') = \begin{cases} 0 & \text{for } t < \frac{c}{s}, \quad |z'| < \sqrt{\left(\frac{c}{s}\right)^2 - s^2} \\ k \left(t - \frac{\sqrt{s^2 + z'^2}}{c} \right) & \text{for } t > \frac{c}{s}, \quad |z'| > \sqrt{\left(\frac{c}{s}\right)^2 - s^2} \end{cases}$$

The time threshold is simply a) saying that before a certain time, not even a message from the nearest point will reach the observer and b) we don't want imaginary terms in the spatial limits!

Now we've done all the new stuff thinking, from here out, it's old hat.

$$\begin{aligned}
\vec{A}_L(\vec{r}, t) &= \frac{\mu_o}{4\pi} \int_{-\infty}^{\infty} \frac{I(\vec{r}', t_r)}{r} dz' \hat{z} = \frac{\mu_o}{4\pi} \int_{z'_{\min} = -\sqrt{\left(\frac{c}{s}\right)^2 - s^2}}^{z'_{\max} = +\sqrt{\left(\frac{c}{s}\right)^2 - s^2}} \frac{k \left(t - \frac{\sqrt{s^2 + z'^2}}{c} \right)}{\sqrt{s^2 + z'^2}} dz' \hat{z} \\
\vec{A}_L(\vec{r}, t) &= \frac{\mu_o k}{4\pi} \left(t \int_{z'_{\min}}^{z'_{\max}} \frac{1}{\sqrt{s^2 + z'^2}} dz' - \frac{1}{c} \int_{z'_{\min}}^{z'_{\max}} dz' \right) \hat{z}
\end{aligned}$$

Now, we've seen integrals of this sort umpteen times.

$$\int_{z'_{\min}}^{z'_{\max}} \frac{1}{\sqrt{s^2 + z'^2}} dz' = \ln \left(\sqrt{s^2 + z'^2} + z' \right) \Big|_{z'_{\min}}^{z'_{\max}}$$

After developing E & B,

<http://web.mit.edu/viz/spin/> choose slow spin up – time evolving magnetic field for a sphere of charge spinning up

Pr. 10.11 & 10.12 to demonstrate when they reduce to old forms

Should Assign 10.10

"Are equations 10.4 and 10.5 important, or was he just trying to prove a point? It seemed to me by the wording of the sentence following 10.5 that these should be important, but he doesn't put a box around them, like he does for 10.2 and 10.3"

[Casey McGrath](#)

I think the purpose is to reduce the number of components we have to solve for. He talks about this a bit at the beginning of the gauge transformation section. Even though A is usually quite hard to calculate, it must make it easier, otherwise there would be no point.

Also, since the potential responds immediately to changes in charge or current distributions, it seems like this formulation essentially transforms out time. Maybe that will be quite useful later on.

[Freeman](#),

"Can we go over gauge transformations because the whole process/purpose was a little confusing?"

[Jessica Admin](#) [Post a response](#)

"Can we talk about the differential operator that is introduced, the one that looks like a square. I understand what it does but not why it is convenient for us."

[Casey P](#), AHoN swag 4 liphe

"Conceptually I'm still having a bit of trouble wrapping my mind around the notion of "choosing a gauge." For instance, how does one get Equation 10.12 (the Lorenz gauge) from equation 10.7? What would be the scalar function λ in this case?"

[Casey McGrath](#)

"Can we talk about the Lorenz gauge and its implications in more detail? Can we also go over the canonical momentum and U_{el} ?" [Spencer](#)

"I had a hard time understanding the section on the coulomb gauge and lorenz gauge. I don't think I understood those sections conceptually at all, so can we talk about those?" [Sam](#)

"Similar to Spencer's question I was wondering if you could give us a little more context to help us understand where writing the Lorentz force law in potential form is leading us and why it is illuminating as Griffith's claims at the start of 10.1.4" [Ben Kid](#)

"I agree with all of the statements thus far. When I started the reading I felt as if I was beginning the middle of an unfamiliar chapter. Are these all just methods for solving for time-dependent quantities? Do you think you give give a brief intro?" [Rachael Hach](#)

"Can we do some examples of calculating the advanced/retarded potential, esp. the limits of the integrals, which I found confusing (in example 10.2)"

[Sam Admin](#) [Post a response](#)

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"Can we go over examples like 10.1 together and the process involved in solving those problems? I found those confusing."

[Sam](#)