Mon.	1.6, 5.4.1-.4.2 Magnetic Vector Potential	
Wed.	5.4.3 Multipole Expansion of the Vector Potential	
Thurs.		HW8
Fri.	Review	

Biot-Savart Law

$$\vec{B}(\vec{r}) = \frac{\mu_o}{4\pi}\int \frac{\vec{J}\times\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau'$$

It Follows that

$$\vec{\nabla}_r \cdot \vec{B}(\vec{r}) = 0$$

or equivalently

$$\int \vec{B}(\vec{r}) \cdot d\vec{a} = 0$$

Ampere's

$$\vec{\nabla}_r \times \vec{B}(\vec{r}) = \mu_o \vec{J}(\vec{r})$$

or equivalently

$$\oint \vec{B}(\vec{r}) \cdot d\vec{l} = \mu_o I$$

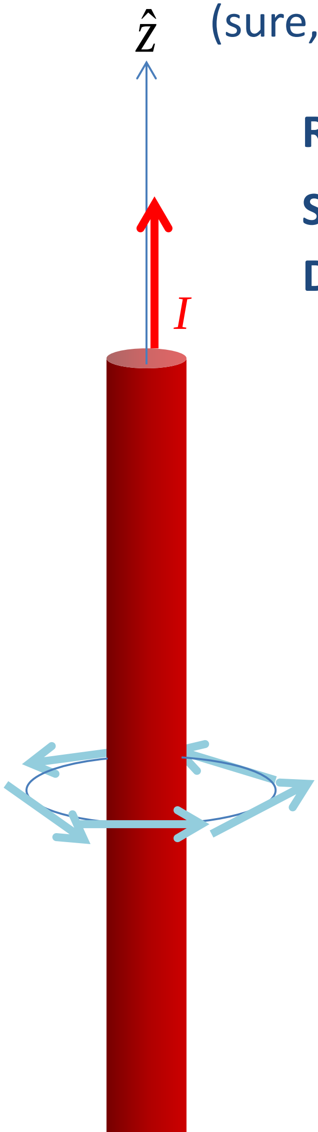
Shortcut to finding field if symmetry is right

Using Ampere's Law

$$\oint \vec{B}(\vec{r}) \cdot d\vec{l} = \mu_o I$$

Simple Example: 'very long', straight wire of uniform current

(sure, we already know the answer, but just to see how it's done)



Reason direction $\hat{B} = \hat{\phi}$

Select Loop accordingly $d\vec{l} = s d\varphi \hat{\varphi}$

Do math

$$\oint \vec{B}(\vec{r}) \cdot d\vec{l} = \mu_o I$$

$$B(\vec{r}) 2\pi s = \mu_o I$$

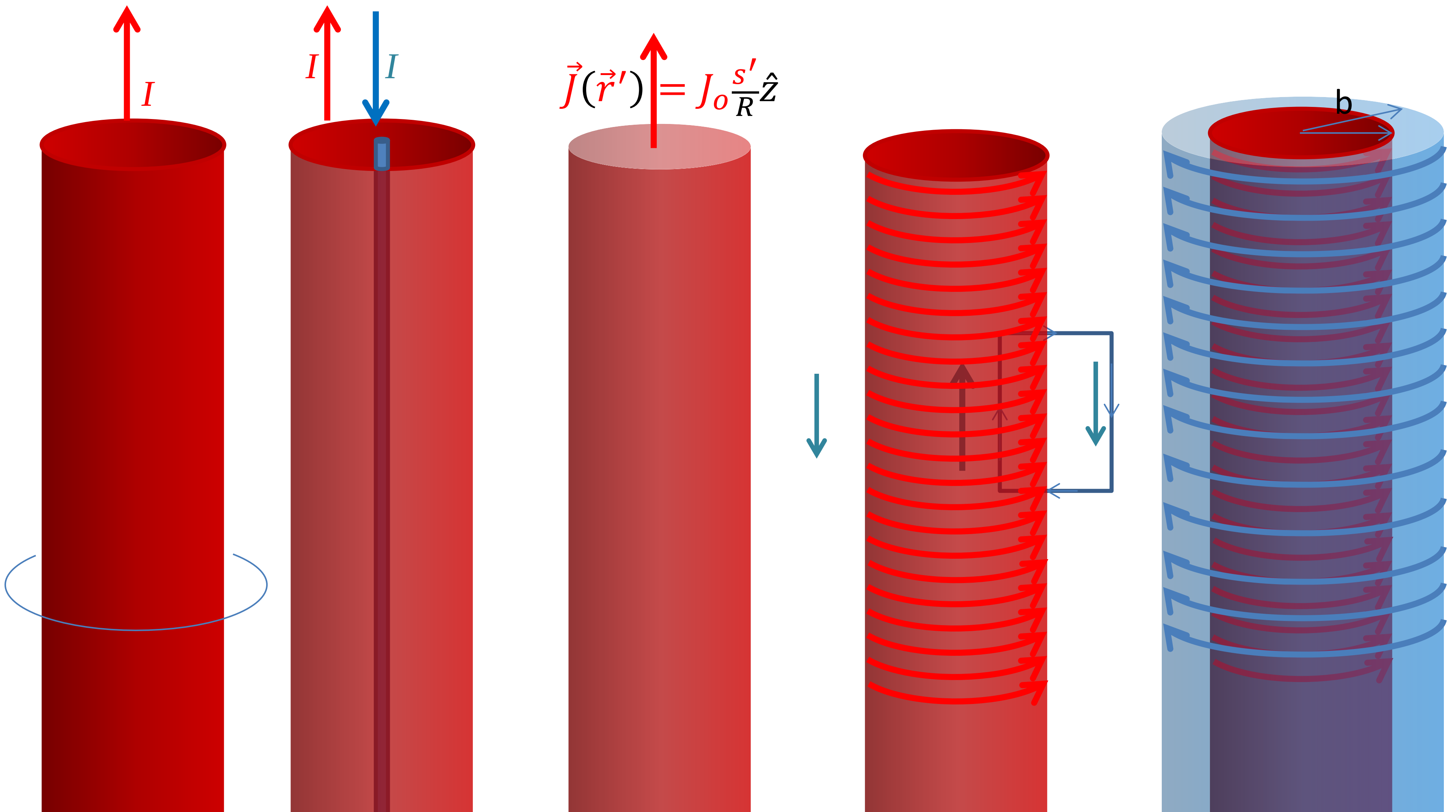
$$B(\vec{r}) = \frac{\mu_o I}{2\pi s}$$

$$\vec{B}(\vec{r}) = \frac{\mu_o I}{2\pi s} \hat{\varphi}$$

Using Ampere's Law

$$\oint \vec{B}(\vec{r}) \cdot d\vec{l} = \mu_o I$$

Examples

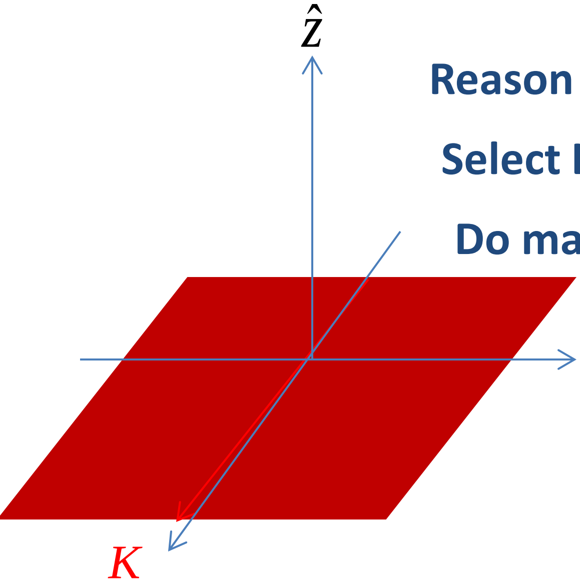


Using Ampere's Law for Boundary Condition

$$\oint \vec{B}(\vec{r}) \cdot d\vec{l} = \mu_o I$$

Example: 'very long' sheet with $\vec{K} = K\hat{x}$

What's B above and below?



Reason direction

Select Loop accordingly

Do math

Using Ampere's Law

$$\oint \vec{B}(\vec{r}) \cdot d\vec{l} = \mu_o I_{enc}$$

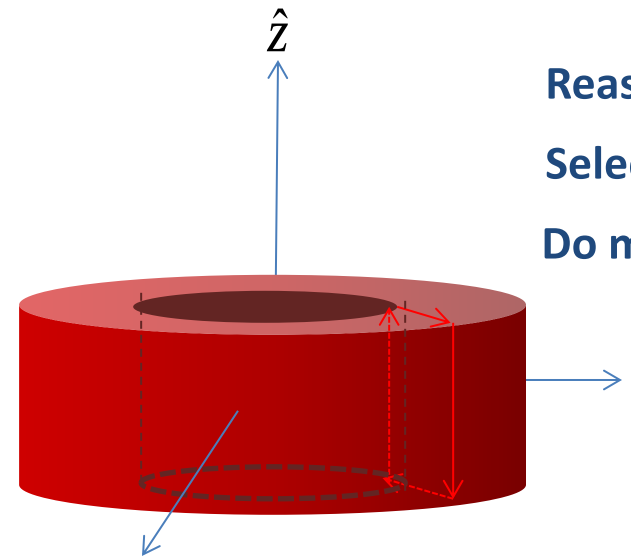
Example: torus

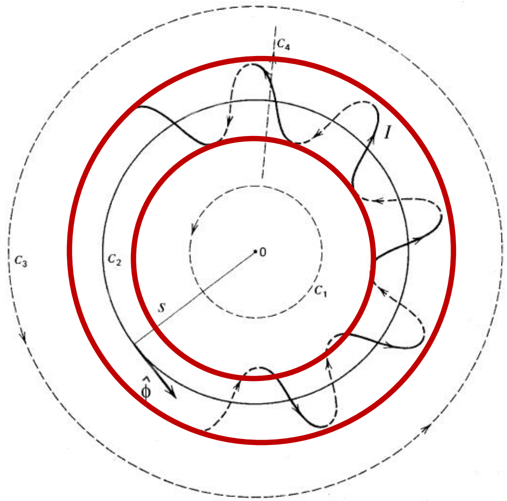
What's B above and below?

Reason direction

Select Loop accordingly

Do math





Introducing the Vector Potential

$$\vec{\nabla}_r \cdot \vec{B}(\vec{r}) = 0$$

or equivalently

$$\int \vec{B}(\vec{r}) \cdot d\vec{a} = 0$$

$$\vec{\nabla}_r \times \vec{B}(\vec{r}) = \mu_o \vec{J}(\vec{r})$$

or equivalently

$$\oint \vec{B}(\vec{r}) \cdot d\vec{l} = \mu_o I$$

But first, recall…

Motivating *Scalar* Potential, Mathematically

In mathland, say you have *scalar* field *f.*

What's $\vec{\nabla}\times\left(\vec{\nabla}f\right)=?$ Well, $\vec{\nabla}f=\left(\frac{\partial}{\partial x}f\right)\hat{x}+\left(\frac{\partial}{\partial y}f\right)\hat{y}+\left(\frac{\partial}{\partial z}f\right)\hat{z}$

So, $\vec{\nabla}\times\left(\vec{\nabla}f\right)=\left(\frac{\partial}{\partial y}\left(\frac{\partial}{\partial z}f\right)-\frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}f\right)\right)\hat{x}+(\ldots)\hat{y}+(\ldots)\hat{z}$

= 0

Free to define

$$\vec{F}\equiv\vec{\nabla}f$$

For whom,

$$\vec{\nabla}\times\left(\vec{F}\right)=0$$

Phrased the other way around:

For any vector field $\vec{F}$ for which $\vec{\nabla}\times\vec{F}=0$
There is a scalar field f such that $\vec{F}\equiv\vec{\nabla}f$

$\vec{\nabla}\times\vec{E}=0$ so we can define a scalar field, call it $-V$, such that $\vec{E}=-\vec{\nabla}V$

Motivating *Vector* Potential, Mathematically

In mathland, say you have *vector* field $\vec{f}$.

What's $\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{f}\right)=?$ Well, $\vec{\nabla}\times\vec{f}=\left(\frac{\partial}{\partial y}f_z-\frac{\partial}{\partial z}f_y\right)\hat{x}+\left(\frac{\partial}{\partial z}f_x-\frac{\partial}{\partial x}f_z\right)\hat{y}+\left(\frac{\partial}{\partial x}f_y-\frac{\partial}{\partial y}f_x\right)\hat{z}$

So,

$$\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{f}\right)=\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}f_z-\frac{\partial}{\partial z}f_y\right)+\frac{\partial}{\partial y}\left(\frac{\partial}{\partial z}f_x-\frac{\partial}{\partial x}f_z\right)+\frac{\partial}{\partial z}\left(\frac{\partial}{\partial x}f_y-\frac{\partial}{\partial y}f_x\right)$$

$$\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{f}\right)=\left(\frac{\partial^2}{\partial x\partial y}f_z-\frac{\partial^2}{\partial x\partial z}f_y\right)+\left(\frac{\partial^2}{\partial y\partial z}f_x-\frac{\partial^2}{\partial y\partial x}f_z\right)+\left(\frac{\partial^2}{\partial z\partial x}f_y-\frac{\partial^2}{\partial z\partial y}f_x\right)$$

$$\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{f}\right)=0$$

Free to define

$$\vec{F}\equiv\vec{\nabla}\times\vec{f}$$

For whom,

$$\vec{\nabla}\cdot\vec{F}=0$$

Phrased the other way around:

For any vector field $\vec{F}$ for which $\vec{\nabla}\cdot\vec{F}=0$
There is another vector field $\vec{f}$ such that $\vec{F}\equiv\vec{\nabla}\times\vec{f}$

$\vec{\nabla}\cdot\vec{B}=0$ so we can define a vector field, call it $\vec{A}$,such that $\vec{B}=\vec{\nabla}\times\vec{A}$

Re-Relating field and Potential

$$\vec{B} = \vec{\nabla} \times \vec{A}$$ Analogous to $$\vec{\nabla} \times \vec{B} = \mu_o \vec{J}$$

so

$$\vec{A} = \tfrac{1}{4\pi} \int \frac{\vec{B} \times \hat{\boldsymbol{\mathscr{r}}} d\tau'}{\boldsymbol{\mathscr{r}}^2}$$

as

$$\vec{B} = \tfrac{1}{4\pi} \mu_o \int \frac{\vec{J} \times \hat{\boldsymbol{\mathscr{r}}} d\tau'}{\boldsymbol{\mathscr{r}}^2}$$

$$\int \vec{B} \cdot d\vec{a} = \oint \left(\vec{\nabla} \times \vec{A}\right) \cdot d\vec{a}$$

By Stokes'

$$\Phi_B = \oint \vec{A} \cdot d\vec{l}$$

Magnetic flux sources vector potential

Analogous to

$$\mu_o I = \oint \vec{B} \cdot d\vec{l}$$

Relating Current and Potential

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

meanwhile

$$\vec{\nabla} \times \vec{B} = \mu_o \vec{J}$$

so

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \mu_o \vec{J}$$

by vector Identity (11)

$$\vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right) - \vec{\nabla}^2 \vec{A} = \mu_o \vec{J}$$

where

$$\vec{\nabla}^2 \vec{A} = \vec{\nabla}^2 \left(A_x \hat{x} + A_y \hat{y} + A_z \hat{z}\right)$$

Pause for motivating analogy

With the *scalar* potential, only the *differences* between two values are physically significant since the gradient relates to E, $\vec{E} = -\vec{\nabla} V$. V_o and $V_o + C$ would correspond to the same actual field.

The curl, not divergence, of A is physically meaningful; if it *had* a divergence, that term of A could be described as a gradient of a scalar field, which itself can have no curl, and thus must not be physically significant. So we're free to specify $\vec{\nabla} \cdot \vec{A} = 0$ without constraining A's possible curls.

$$\vec{\nabla}^2 \vec{A} = -\mu_o \vec{J}$$

This *choice* defines the "Coulomb Gauge"

(In Ch. 10, it will be mathematically convenient to make other choices)

Relating Current and Potential

$$\vec{\nabla}^2 \vec{A} = -\mu_o \vec{J} \quad \text{or} \quad \begin{cases} \nabla^2 A_x = -\mu_o J_x \\ \nabla^2 A_y = -\mu_o J_y \\ \nabla^2 A_z = -\mu_o J_z \end{cases}$$

Individually, these are same form as

$$\nabla^2 V = -\tfrac{1}{\varepsilon_o} \rho$$

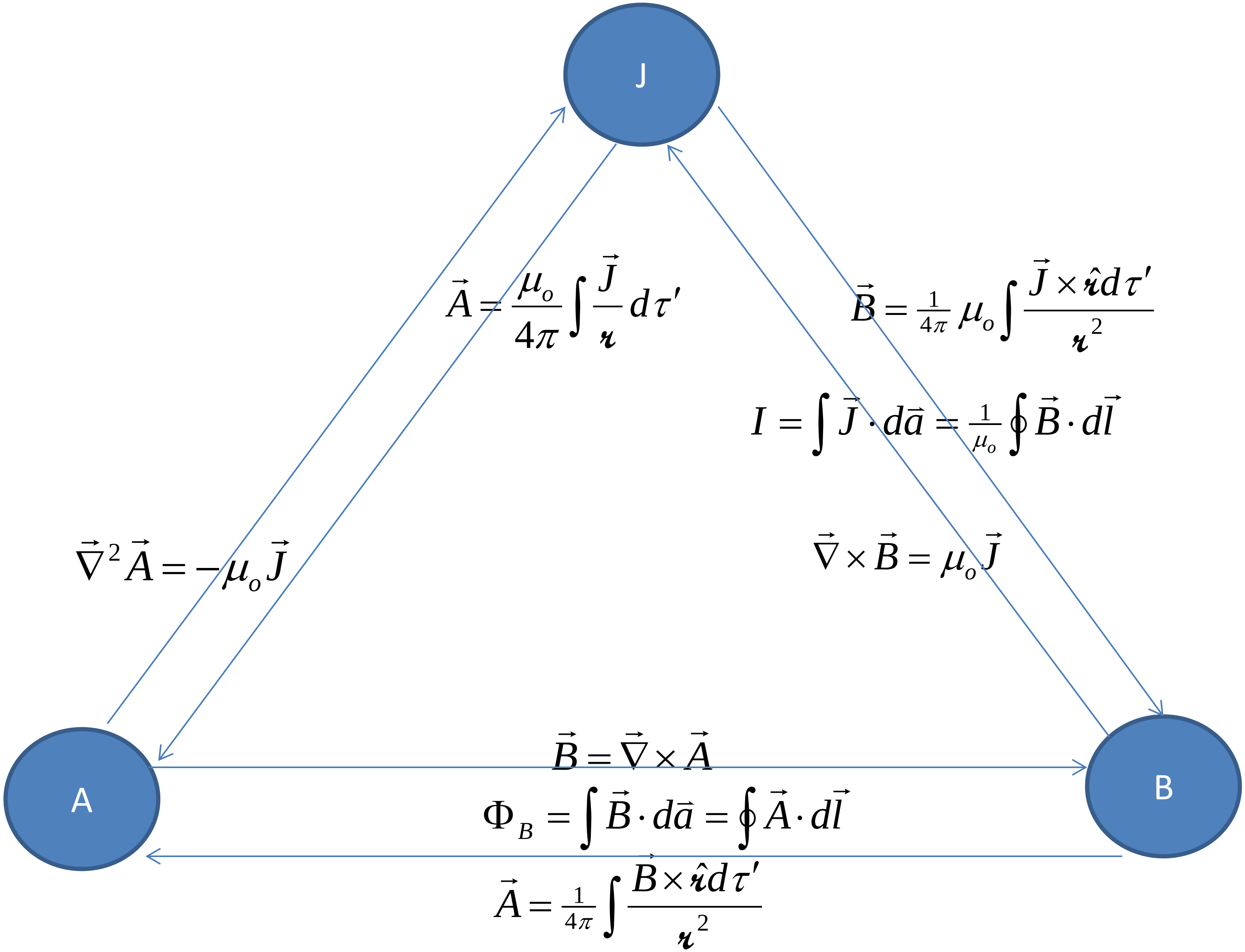
Which we've shown pairs with

$$V = \frac{1}{4\pi\varepsilon_o} \int \frac{\rho}{\mathscr{r}} d\tau'$$

So apparently $\vec{\nabla}^2 \vec{A} = -\mu_o \vec{J}$ pairs with

$$\vec{A} = \frac{\mu_o}{4\pi} \int \frac{\vec{J}}{\mathscr{r}} d\tau' \quad \text{or} \quad \begin{cases} A_x = \dfrac{\mu_o}{4\pi} \displaystyle\int \frac{J_x}{\mathscr{r}} d\tau' \\ A_y = \dfrac{\mu_o}{4\pi} \displaystyle\int \frac{J_y}{\mathscr{r}} d\tau' \\ A_z = \dfrac{\mu_o}{4\pi} \displaystyle\int \frac{J_z}{\mathscr{r}} d\tau' \end{cases}$$

Relating Current, Potential, and Field

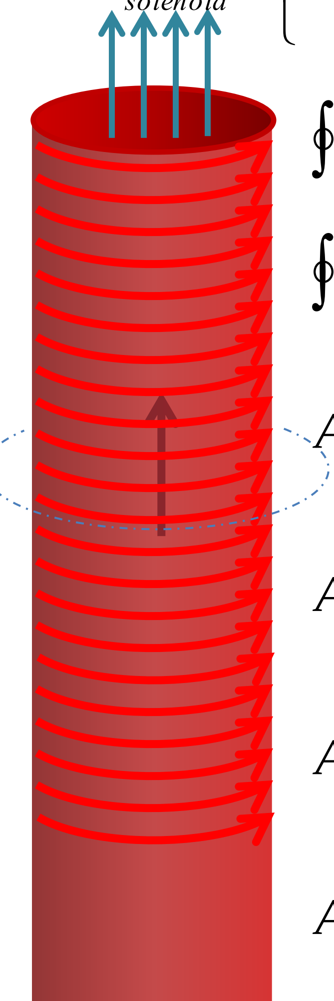


Finding Vector Potential from Field

$$\oint \vec{A}\cdot d\vec{\ell} = \int \vec{B}\cdot d\vec{a} = \Phi$$

Solenoid: Find the vector potential of an infinite solenoid with *n* turns per length, radius *R*, and current *I*.

$$\vec{B}_{solenoid} = \begin{cases} (\mu_0 n I)\hat{z} & s < R, \\ 0 & s > R. \end{cases}$$



$$\oint \vec{A}_{out}\cdot d\vec{\ell} = \int \vec{B}\cdot d\vec{a}$$

$$\oint \left(A_{out}\hat{\phi}\right)\cdot\left(s\,d\phi\hat{\phi}\right) = \int_0^{2\pi}\int_0^{s} B\left(s'ds'd\phi'\right)$$

$$A_{out}\oint s\,d\phi = \int_0^{2\pi}\int_0^{R} B_{in}s'ds'd\phi' + \int_0^{2\pi}\int_R^{s} B_{out}s'ds'd\phi'$$

$$A_{out}\,2\pi s = \int_0^{2\pi}\int_0^{R}\left(\mu_o nI\right)s'ds'd\phi' + \int_0^{2\pi}\int_R^{s}(0)s'ds'd\phi'$$

$$A_{out}\,2\pi s = \mu_o nI\,2\pi\int_0^{R} s'ds'd\phi' = \mu_o nI\,2\pi\left(\tfrac{1}{2}R^2\right)$$

$$A_{out} = \frac{\mu_o nIR^2}{2s}$$

$$\oint \vec{A}_{in}\cdot d\vec{\ell} = \int \vec{B}_{in}\cdot d\vec{a}$$

$$\oint \left(A_{in}\hat{\phi}\right)\cdot\left(s\,d\phi\hat{\phi}\right) = \int_0^{2\pi}\int_0^{s} B_{in}\left(s'ds'd\phi'\right)$$

$$A_{in}\,2\pi s = \int_0^{2\pi}\int_0^{s}\left(\mu_o nI\right)s'ds'd\phi'$$

$$A_{in}\,2\pi s = \mu_o nI\,2\pi\int_0^{s} s'ds'd\phi'$$

$$A_{in} = \mu_o nI\,2\pi\left(\tfrac{1}{2}s^2\right)$$

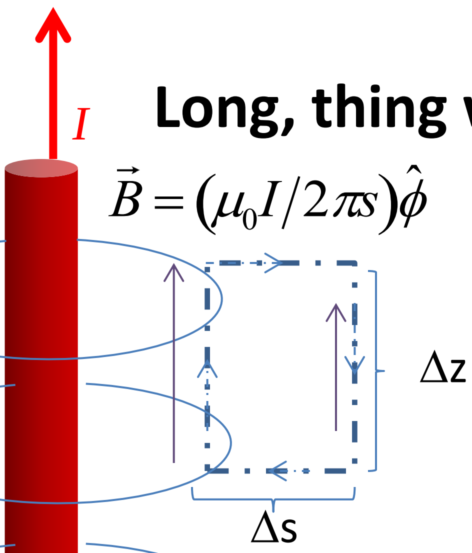
$$A_{in} = \frac{\mu_o nIs}{2}$$

Finding Vector Potential from Field

$$\oint \vec{A}\cdot d\vec{\ell} = \int \vec{B}\cdot d\vec{a} = \Phi$$

Long, thing wire: Find the vector potential of a thin wire carrying current *I*.

$$\vec{B} = (\mu_0 I/2\pi s)\hat{\phi}$$



$$\oint \vec{A}\cdot d\vec{\ell} = \int \vec{B}\cdot d\vec{a}$$

$$\int_{z}^{z+\Delta z}\left(A_{left}\hat{z}\right)\cdot d\vec{z} + \int_{s}^{s+\Delta s}\left(A_{top}\hat{z}\right)\cdot d\vec{s} + \int_{z+\Delta z}^{z}\left(A_{right}\hat{z}\right)\cdot d\vec{z} + \int_{s+\Delta s}^{s}\left(A_{bottom}\hat{z}\right)\cdot d\vec{s} = \int_{z}^{z+\Delta z}\int_{s}^{s+\Delta s} B\left(ds'dz'\right)$$

$$\int_{z}^{z+\Delta z}\left(A_{left}\hat{z}\right)\cdot d\vec{z} + \int_{z+\Delta z}^{z}\left(A_{right}\hat{z}\right)\cdot d\vec{z} = \int_{z}^{z+\Delta z}\int_{s}^{s+\Delta s} \frac{\mu_o I}{2\pi s'}\left(ds'dz'\right)$$

$$A_{left}\Delta z - A_{right}\Delta z = \frac{\mu_o I}{2\pi}\int_{s}^{s+\Delta s}\frac{1}{s'}ds'\Delta z$$

$$\Delta A = \frac{\mu_o I}{2\pi}\ln\left(\frac{s+\Delta s}{s}\right) = \frac{\mu_o I}{2\pi}\ln\left(1+\frac{\Delta s}{s}\right)$$

Finding J from Vector Potential

What current density would produce the vector potential $\vec{A} = k\,\hat{\phi}$ (where *k* is a constant) in cylindrical coordinates?

$$\vec{\nabla}^2 \vec{A} = -\mu_o \vec{J} \quad \text{where} \quad \vec{\nabla}^2 \vec{A} = \vec{\nabla}^2 A_x \hat{x} + \vec{\nabla}^2 A_y \hat{y} + \vec{\nabla}^2 A_z \hat{z}$$

So, convert to Cartesian $\vec{A} = k\,\langle -\sin\phi, \cos\phi, 0\rangle$

One component at a time

$$\vec{\nabla}^2 A_x = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial(-k\sin\phi)}{\partial s}\right) + \frac{1}{s^2}\frac{\partial(-k\sin\phi)}{\partial\phi^2} + \frac{\partial^2(-k\sin\phi)}{\partial z^2} = \ldots = k\frac{\sin\phi}{s^2}$$

$$\vec{\nabla}^2 A_y = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial(k\cos\phi)}{\partial s}\right) + \frac{1}{s^2}\frac{\partial(k\cos\phi)}{\partial\phi^2} + \frac{\partial^2(k\cos\phi)}{\partial z^2} = \ldots = -k\frac{\cos\phi}{s^2}$$

$$\vec{\nabla}^2 \vec{A} = \left\langle k\frac{\sin\phi}{s^2}, -k\frac{\cos\phi}{s^2}, 0\right\rangle$$

$$\vec{\nabla}^2 \vec{A} = -\frac{k}{s^2}\langle -\sin\phi, \cos\phi, 0\rangle = -\frac{k}{s^2}\hat{\phi} = -\mu_o \vec{J} \quad \text{so} \quad \vec{J} = \frac{1}{\mu_o}\frac{k}{s^2}\hat{\phi}$$

Finding J from Vector Potential

What current density would produce the vector potential $\vec{A} = k\,\hat{\phi}$ (where *k* is a constant) in cylindrical coordinates?

$$\vec{\nabla}^2 \vec{A} = -\mu_o \vec{J}$$

Alternatively,

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{1}{s}\frac{\partial}{\partial s}\left(sA_\phi\right)\hat{z} = \frac{1}{s}\frac{\partial}{\partial s}\left(ks\right)\hat{z} = \frac{k}{s}\,\hat{z}$$

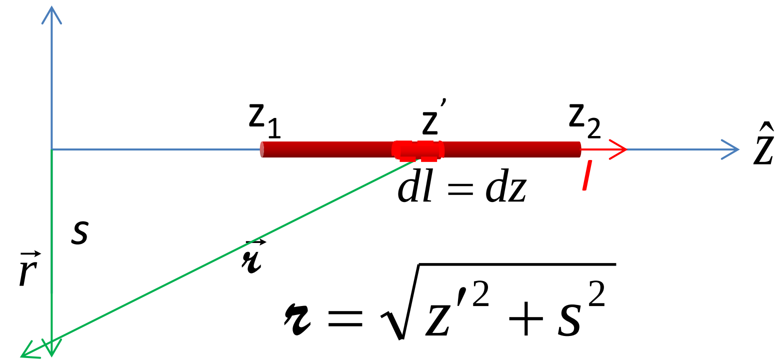
and then

$$\vec{J} = \tfrac{1}{\mu_o}\vec{\nabla} \times \vec{B} = \frac{1}{\mu_0}\left(-\frac{\partial B_z}{\partial s}\right)\hat{\phi} = \frac{1}{\mu_0}\left[-\frac{\partial}{\partial s}\left(\frac{k}{s}\right)\right]\hat{\phi} = \frac{k}{\mu_0 s^2}\,\hat{\phi}$$

Finding A from J

$$\vec{A} = \frac{\mu_o}{4\pi}\int \frac{\vec{J}}{\mathscr{r}} d\tau' = \frac{\mu_o}{4\pi}\int \frac{\vec{I}}{\mathscr{r}} dl'$$

Find the vector potential for a current *I* along the *z* axis from z_1 to z_2.



$$\mathscr{r} = \sqrt{z'^2 + s^2}$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{z_1}^{z_2} \frac{Idz}{\sqrt{z^2+s^2}}\,\hat{z}$$

$$\vec{A} = \frac{\mu_0 I}{4\pi}\left[\ln\left(z+\sqrt{z^2+s^2}\right)\right]_{z_1}^{z_2}\hat{z}$$

$$\vec{A} = \frac{\mu_0 I}{4\pi}\ln\left[\frac{z_2+\sqrt{z_2^2+s^2}}{z_1+\sqrt{z_1^2+s^2}}\right]\hat{z}.$$

Motivating Electric Potential, Physically

Akin to Potential *Energy*

Generally

$$W_{1\to 2} \equiv \int_a^b \vec{F}_{1\to 2} \cdot d\vec{\ell}$$

Object 2 is the "system", 1 is "external." Work done *by* object 1 when exerting force *on* object 2 which moves from *a* to *b*

$$\Delta P.E._{1,2} \equiv -\int_a^b \vec{F}_{1\to 2} \cdot d\vec{\ell}$$

Objects 1 and 2 are the "system". Change in their potential as they interact while separating from *a* to *b*

Electrically

$$\vec{F}_{1\to 2} = q_2 \vec{E}_1(\vec{r}_2)$$

Combining:

$$\Delta P.E._{1,2} = -\int_a^b q_2 \vec{E}_1(\vec{r}_2) \cdot d\vec{\ell} = -q_2 \int_a^b \vec{E}_1(\vec{r}_2) \cdot d\vec{\ell}$$

thus

$$\Delta V_1 \equiv \frac{\Delta P.E._{1,2}}{q_2} = -\int_a^b \vec{E}_1(\vec{r}_2) \cdot d\vec{\ell}$$

Physical Meaning of Vector Potential

Akin to potential *momentum*

From __future__ import time-varying electric

$$\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}$$

Consider your "system" a particle interacting with electric and magnetic fields
(*really* interacting with other charges via their electric and magnetic fields)

$$\frac{d}{dt}\vec{p} = \vec{F}_{net} = q\vec{v}\times\vec{B} + q\vec{E} = q\vec{v}\times\left(\vec{\nabla}\times\vec{A}\right) + q\left(-\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}\right) = q\vec{v}\times\left(\vec{\nabla}\times\vec{A}\right) + q\left(-\vec{\nabla}V - \frac{d}{dt}\vec{A} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{A}\right)$$

for any vector

$$\frac{d}{dt}\vec{A} = \frac{\partial \vec{A}}{\partial t} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{A}$$

$$\frac{d}{dt}\vec{p} = q\left(\vec{\nabla}\left(\vec{v}\cdot\vec{A}\right) - \frac{d}{dt}\vec{A}\right) + q\left(-\vec{\nabla}V\right)$$

By Product rule (4)

$$\vec{\nabla}\left(\vec{v}\cdot\vec{A}\right) = \vec{v}\times\left(\vec{\nabla}\times\vec{A}\right) + \vec{A}\times\left(\vec{\nabla}\times\vec{v}\right) + \left(\vec{A}\cdot\vec{\nabla}\right)\vec{v} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{A}$$

Derivative with respect to potential not source velocity

$$\frac{d}{dt}\underbrace{\left(\vec{p} + q\vec{A}\right)}_{"p"} = -\vec{\nabla}\underbrace{q\left(V - \vec{v}\cdot\vec{A}\right)}_{"U"}$$

Consider your "system" a particle *and* the fields.
The force is negative gradient the potential energy

if $-\vec{\nabla}q\left(V - \vec{v}\cdot\vec{A}\right) = 0$ then $\vec{p}_i + q\vec{A}_i = \vec{p}_f + q\vec{A}_f = const$

'potential momentum'

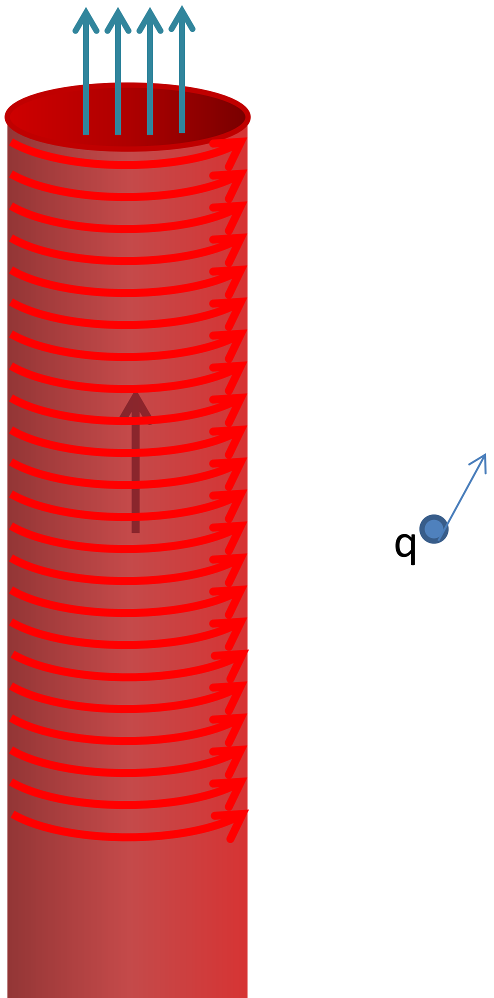
Finding Vector Potential

$$\oint \vec{A}\cdot d\vec{\ell} = \int \vec{B}\cdot d\vec{a} = \Phi$$

Charged particle outside a disappearing solenoid

$$\vec{A}_{initially} = \begin{cases} (\mu_0 n I s/2)\hat{\phi} & s < R, \\ \left(\mu_0 n I R^2/2s\right)\hat{\phi} & s > R. \end{cases}$$

$$\vec{A}_{finally} = 0$$



initially

$$\frac{d}{dt}\left(m\vec{v} + q\vec{A}\right) = -\vec{\nabla} q\left(V - \vec{v}\cdot\vec{A}\right) = 0$$

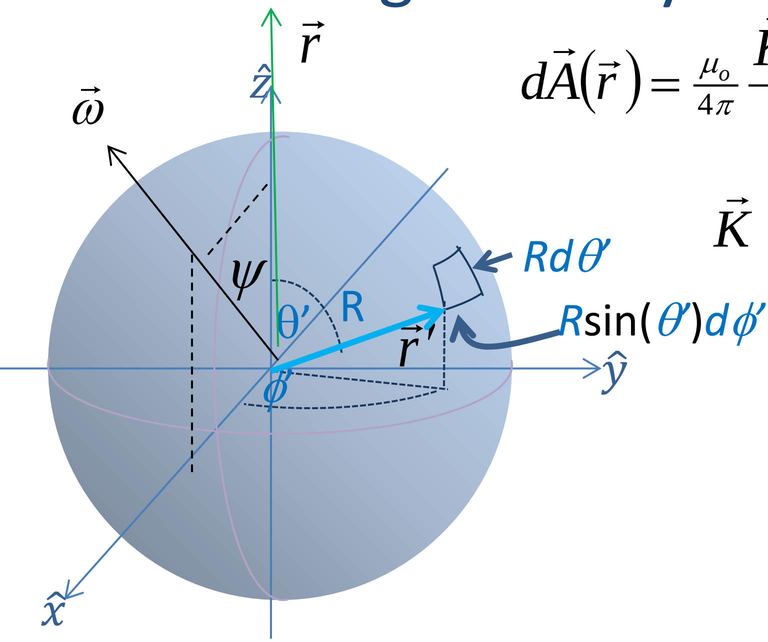
$$m\vec{v}_i + q\vec{A}_i = m\vec{v}_f + q\vec{A}_f$$

$$q\left(\mu_0 n I R^2/2s\right)\hat{\phi} = m\vec{v}_f$$

$$\frac{q}{m}\left(\mu_0 n I R^2/2s\right)\hat{\phi} = \vec{v}_f$$

$-\hat{x}$

Ex. 5.11: What's the magnetic potential of a sphere with surface charge density constant σ rotating at ω.



$$d\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi}\frac{\vec{K}da'}{\mathscr{r}} \qquad da' = R^2 d\varphi' \sin\theta' d\theta'$$

$$\mathscr{r} = \sqrt{R^2 + r^2 - 2Rr\cos\theta'}$$

$$\vec{K} = \sigma\vec{v}$$ What is $\vec{v}$?

If rotating about z, it would simply be $R\sin\theta'\omega\hat{\phi}$.

If $\vec{\omega} = \omega\hat{z}$ this would have been $\vec{\omega}\times\vec{r}\,' = R\sin\theta'\omega\hat{\phi} = \vec{v}$

Generally, $\vec{\omega}\times\vec{r}\,' = \vec{v}$

$$\vec{\omega} = \omega(\sin\psi\hat{x} + \cos\psi\hat{z})$$

$$\vec{r}\,' = R(\sin\theta'\cos\phi'\hat{x} + \sin\theta'\sin\phi'\hat{y} + \cos\theta'\hat{z})$$

$$\vec{v} = \vec{\omega}\times\vec{r}\,' = \omega(\sin\psi\hat{x} + \cos\psi\hat{z})\times R(\sin\theta'\cos\phi'\hat{x} + \sin\theta'\cos\phi'\hat{y} + \cos\theta'\hat{z})$$

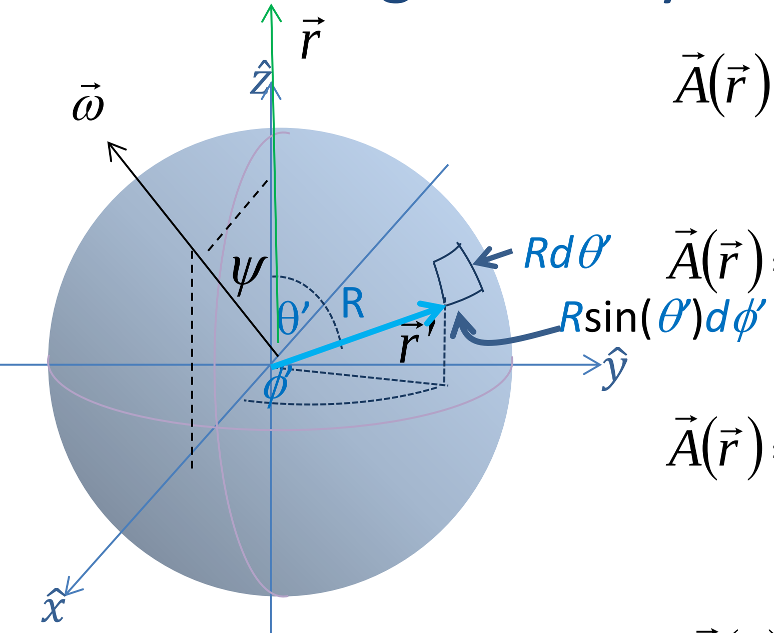
$$\vec{v} = \omega R((-\sin\theta'\cos\phi'\cos\psi)\hat{x} + (\cos\psi\sin\theta'\cos\phi' - \sin\psi\cos\theta')\hat{y} + \sin\psi\sin\theta'\sin\phi'\hat{z})$$

For the four terms to v, there will be four integrals. All but one has a factor of

$$\int_0^{2\pi}\cos\phi' d\varphi' = 0 \quad \text{or} \quad \int_0^{2\pi}\sin\phi' d\varphi' = 0$$

leaving $$\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi}\int_0^{\phi'=2\pi}\int_0^{\theta'=\pi}\frac{\sigma\omega R(-\sin\psi\cos\theta')R^2 d\phi'\sin\theta' d\theta'}{\sqrt{R^2 + r^2 - 2Rr\cos\theta'}}\hat{y}$$

Ex. 5.11: What's the magnetic potential of a sphere with surface charge density constant σ rotating at ω.



$$\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi}\int_0^{\phi'=2\pi}\int_0^{\theta'=\pi}\frac{\sigma\omega R(-\sin\psi\cos\theta')R^2 d\phi'\sin\theta' d\theta'}{\sqrt{R^2+r^2-2Rr\cos\theta'}}\hat{y}$$

$$\vec{A}(\vec{r}) = -\frac{\mu_o}{4\pi}2\pi\sigma\omega R^3\sin\psi\int_0^{\theta'=\pi}\frac{\cos\theta'\sin\theta' d\theta'}{\sqrt{R^2+r^2-2Rr\cos\theta'}}\hat{y}$$

$$\vec{A}(\vec{r}) = -\frac{\mu_o}{2}\sigma\omega R^3\sin\psi\int_1^{\cos\theta'=0}\frac{\cos\theta' d(\cos\theta')}{\sqrt{R^2+r^2-2Rr\cos\theta'}}\hat{y}$$

$$\vec{A}(\vec{r}) = -\frac{\mu_o}{2}\sigma\omega R^3\sin\psi\int_1^{\zeta=-1}\frac{\zeta d\zeta}{\sqrt{R^2+r^2-2Rr\zeta}}\hat{y}$$

$$\vec{A}(\vec{r}) = \frac{\mu_o}{2}\sigma\omega R^3\sin\psi\left(\frac{R^2+r^2+Rr\zeta}{3R^2r^2}\sqrt{R^2+r^2-2Rr\zeta}\right)\Bigg|_1^{-1}\hat{y}$$

$$\vec{A}(\vec{r}) = \frac{\mu_o\sigma}{6r^2}\omega R\sin\psi\left((R^2+r^2+Rr\zeta)\sqrt{R^2+r^2-2Rr\zeta}\right)\Bigg|_1^{-1}\hat{y}$$

$$\vec{A}(\vec{r}) = \frac{\mu_o\sigma}{6r^2}\omega R\sin\psi\left((R^2+r^2-Rr)\sqrt{(R+r)^2}-(R^2+r^2+Rr)\sqrt{(R-r)^2}\right)\hat{y}$$

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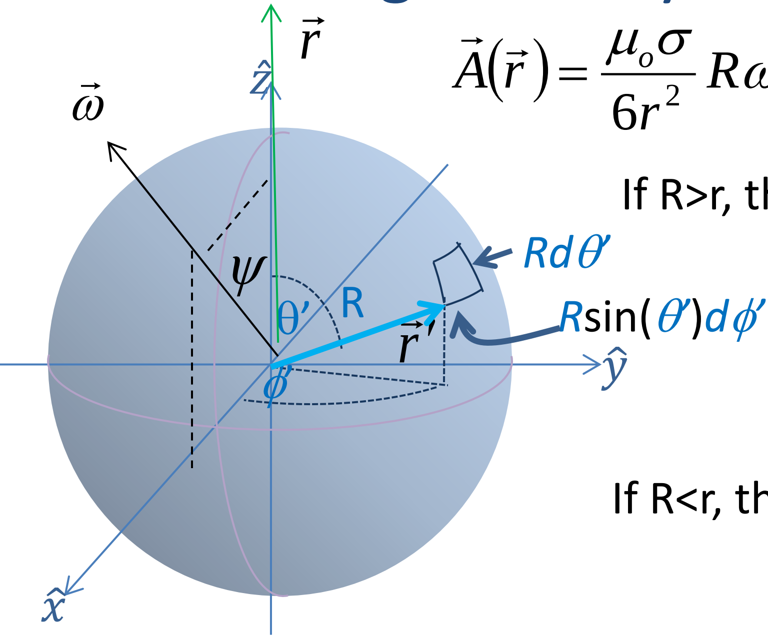


$$\vec{A}(\vec{r}) = \frac{\mu_o \sigma}{6r^2} R\omega \sin\psi \left(\left(R^2 + r^2 - Rr\right)\left(R + r\right) - \left(R^2 + r^2 + Rr\right)\left|R - r\right| \right)\hat{y}$$

If R>r, then $\left|R - r\right| = R - r$

$$\left(R^2 + r^2 - Rr\right)\left(R + r\right) - \left(R^2 + r^2 + Rr\right)\left(R - r\right) = 2r^3$$

$$\vec{A}(\vec{r}) = \frac{\mu_o \sigma}{3} R\omega r \sin\psi\hat{y}$$

If R<r, then $\left|R - r\right| = r - R$

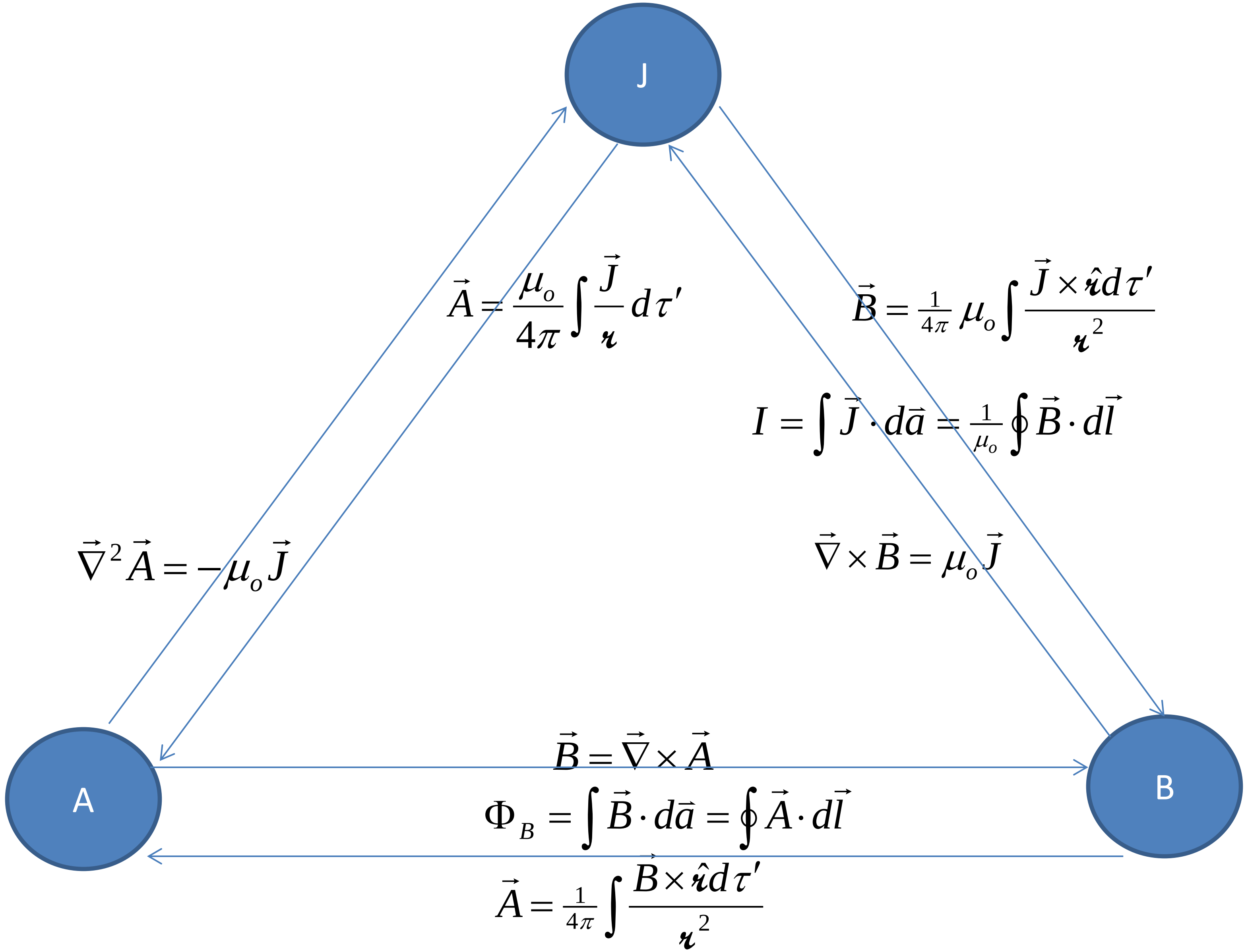
$$\left(R^2 + r^2 - Rr\right)\left(R + r\right) - \left(R^2 + r^2 + Rr\right)\left(r - R\right) = 2R^3$$

$$\vec{A}(\vec{r}) = \frac{\mu_o \sigma}{3r^2} R^4 \omega \sin\psi\hat{y}$$

Recognizing that $\vec{\omega} \times \vec{r} = \omega r \sin\psi\hat{y}$ these can be written generally

$$\vec{A}(\vec{r}) = \begin{cases} \dfrac{\mu_o \sigma}{3} R\vec{\omega} \times \vec{r} & r < R \\ \dfrac{\mu_o \sigma}{3r^3} R^4 \vec{\omega} \times \vec{r} & r > R \end{cases}$$

Relating Current, Potential, and Field



Fri.	1.6, 5.4.1-.4.2 Magnetic Vector Potential	
Mon. Wed. Thurs.	5.4.3 Multipole Expansion of the Vector Potential 7.1.1-7.1.3 Ohm’s Law & Emf	 HW7

Finding J from Vector Potential

What current density would produce the vector potential $\vec{A} = k\,\hat{\phi}$ (where *k* is a constant) in cylindrical coordinates?

$$\vec{\nabla}^2 \vec{A} = -\mu_o \vec{J}$$ where $$\vec{\nabla}^2 \vec{A} = \vec{\nabla}^2 A_x \hat{x} + \vec{\nabla}^2 A_y \hat{y} + \vec{\nabla}^2 A_z \hat{z}$$

So, convert to Cartesian $\vec{A} = k\left\langle -\sin\phi, \cos\phi, 0\right\rangle = k\left\langle -\frac{y}{\left(x^2+y^2\right)^{1/2}}, \frac{x}{\left(x^2+y^2\right)^{1/2}}, 0\right\rangle$

One component at a time

$$\vec{\nabla}^2 A_x = -k\left(\frac{\partial^2}{\partial x^2}\frac{y}{\sqrt{x^2+y^2}} + \frac{\partial^2}{\partial y^2}\frac{y}{\sqrt{x^2+y^2}}\right)$$

$$\vec{\nabla}^2 A_x = -k\left(\frac{\partial}{\partial x}\frac{-xy}{\left(x^2+y^2\right)^{3/2}} + \frac{\partial}{\partial y}\left(\frac{1}{\left(x^2+y^2\right)^{1/2}} + \frac{-y^2}{\left(x^2+y^2\right)^{3/2}}\right)\right) = \ldots = k\frac{y}{\left(x^2+y^2\right)^{3/2}}$$

similarly

$$\vec{\nabla}^2 A_y = -k\frac{x}{\left(x^2+y^2\right)^{3/2}}$$

$$\vec{\nabla}^2 \vec{A} = -k\left\langle -\frac{y}{\left(x^2+y^2\right)^{3/2}}, \frac{x}{\left(x^2+y^2\right)^{3/2}}, 0\right\rangle$$

$$\vec{\nabla}^2 \vec{A} = -\frac{k}{s^2}\left\langle -\frac{y}{\left(x^2+y^2\right)^{1/2}}, \frac{x}{\left(x^2+y^2\right)^{1/2}}, 0\right\rangle = -\frac{k}{s^2}\hat{\phi} = -\mu_o \vec{J}$$

so

$$\vec{J} = \frac{1}{\mu_o}\frac{k}{s^2}\hat{\phi}$$